High Energy Physics and Nuclear Physics

Volume 14, Number 4

Analytical Treatment of SU(2)-Higgs System on Lattice at Finite Temperature

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The variational cumulant expansion treatment of lattice gauge theory at finite temperature is developed to the lattice $SU(2)$-Higgs system with the Higgs fields in the fundamental representation. The method is demonstrated by calculating the expectation value of Polyakov line $<L>$ to the 3rd order approximation. The result is in agreement with the Monte Carlo simulation result. The comparison with the mean field results is given.

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1. INTRODUCTION

The lattice gauge theory provides a powerful tool for the non-perturbative investigation of various models in field theories. Among these models the $SU(2)$-Higgs model at finite temperature has a special interest. Since it is one part of the unified electroweak model, it may be useful for understanding the Higgs mechanism at high temperature, which may play an important role in the evolution of the early universe. Besides, it is the simplest model of the non-abelian gauge system coupled with matter fields at finite temperature. The investigation of this system no doubt will be heuristic for the further treatment of QCD at finite temperature.

There are several works investigating the model at finite temperature by Monte Carlo simulations [1-5], but only a few investigations have been done by the analytic method, i.e. by the mean field calculation [2], in which by using the strong coupling expansion for the pure gauge part and the hopping parameter expansion for the coupling constant of $SU(2)$ with Higgs field, an effective action has been deduced. Then the mean field method has been applied.

Recently, an approximate analytical approach to the $SU(2)$ gauge system at finite temperature has been proposed [6]. It combines the variational method in the lagrangian formalism with the

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Supported by the National Natural Science Foundation of China.
Received on March 31, 1990.

cumulant expansion. It is the purpose of the present paper to develop this approach to the lattice $SU(2)$-Higgs system with the Higgs field in the fundamental representation. This case is more complicated than the case of the pure $SU(2)$ system. But after introducing the trial action with more variational parameters, the model is still analytically tractable. As an example, the expectation value of the Polyakov line is calculated to the 3rd order approximation. The result is in agreement with the Monte Carlo simulation result.

In section 2 the model at finite temperature and its treatment are given. In section 3 the statistical average of the Polyakov line is calculated. Finally, in the last section the results and discussions are given.

2. THE MODEL AT FINITE TEMPERATURE AND ITS TREATMENT

Let us consider a four-dimensional hypercubic lattice, on which the system of $SU(2)$ gauge field coupled with the Higgs field $\phi(x)$ in the fundamental representation is defined. To introduce the temperature the lattice is taken to be $\Lambda = (N_\sigma A)^3 \cdot N_\tau A$ where A is the lattice spacing, $N_\tau A$ is the finite time-like length with the periodic conditions on the boundaries. The temperature is then defined as $T = 1/N_\tau A$. In our treatment N_σ remains infinite. As usual [1,2] $U_l \in SU(2)$ are defined on links and the Higgs field $\Lambda(x)$ is defined on sites. The periodic conditions demand

$$\phi(\boldsymbol{x}, t = 0) = \phi(\boldsymbol{x}, t = N_\tau A), \quad U_l(\boldsymbol{x}, 0) = U_l(\boldsymbol{x}, N_\tau A). \tag{1}$$

In the fundamental representation $\phi(x)$ may be represented by the modular ρ_x and $\sigma(x) \in SU(2)$ defined on sites. After a suitable gauge transformation $\sigma(x)$ may be removed from the action. Therefore the action of the model is written as [7]

$$S = S_G + S_H + S_I. \tag{2}$$

$$S_G = \frac{1}{2}\beta\left(\sum_{P_\sigma} \operatorname{tr} U_{P_\sigma} + \sum_{P_\tau} \operatorname{tr} U_{P_\tau}\right), \tag{3}$$

$$S_H = -\lambda \sum_x (\rho_x^2 - 1)^2 - \sum_x \rho_x^2, \tag{4}$$

$$S_I = \kappa\left(\sum_{x,\sigma} \rho_x \rho_{x+\sigma} \operatorname{tr} U_{x,\sigma} + \sum_{x,\tau} \rho_x \rho_{x+\tau} \operatorname{tr} U_{x,\tau}\right). \tag{5}$$

where U_{P_σ} is an ordered product of U_l around a plaquette formed by space-like links, where as U_{PT} is an ordered product of U_l around a plaquette formed by two space-like and two time-like links. $\sum_{P_\sigma}$ and $\sum_{P_\tau}$ are the sum over all these plaquettes. $\sum_x$, $\sum_{x,\sigma}$ and $\sum_{x,\tau}$ are sums over all sites, space-like links and time-like links respectively. The total number of these summations are $1/2M(d-1)(d-1)$, $N = M(d-1)$, M, $N_{P_T} = M(d-1)$, $N_{l\sigma} = M(d-1)$ and $N_{l_\tau} = M$ respectively. $\beta = 4/g^2$, λ and κ are constants.

The partition function of the system is

$$Z = \int [dU][d\rho]e^{S}. \tag{6}$$

$$[d\rho] = \prod_{x} \rho_x^3 d\rho_x. \tag{7}$$

In order to calculate by using the cumulant expansion, we introduce the trial action

$$S_0 = J\sum_{\sigma} \mathrm{tr}\, U_\sigma + K\sum_{\tau} \mathrm{tr}\, U_\tau - \alpha \sum_{x} \rho_x \tag{8}$$

It differs from the S_0 introduced for treating the pure $SU(2)$ system at finite temperature [6] by the presence of the last term, which is the simplest form introduced for treating the Higgs modular. J, K and α are real parameters to be determined. The partition function Z_0 of the auxiliary lattice system with the action S_0 can be calculated explicitly

$$Z_0 = \int [dU][d\rho]e^{S_0} = [f(J)]^{M(d-1)} \cdot [f(K)]^{M} \cdot [g_4(\alpha)]^{M}. \tag{9}$$

where $f(x) = I_1(2x)/x$ is the well-known single link integral [8] and

$$g_n(\alpha) = \int_0^\infty d\rho \cdot \rho^{n-1} e^{-\alpha\rho} = \alpha^{-n} \cdot (n-1)!. \tag{10}$$

Through the cumulant expansion the free energy per site can be expressed as [9]

$$F = F_0 + \frac{1}{M}\sum_{n=1}^{\infty} \frac{1}{n!} \mathscr{K}_n(\beta, \lambda, \kappa, J, K, \alpha). \tag{11}$$

where

$$\mathscr{K}_n(\beta, \lambda, \kappa, J, K, \alpha) = \langle (S_G + S_H + S_I - S_0)^n \rangle_c, \tag{12}$$

and $<x^n>_c$ is the n-th order cumulant expansion. It can be expressed through its lower order cumulant expansion and $<x^n>_0$, the statistical average with respect to S_0. Therefore all thermodynamic quantities can be calculated order by order.

Eq.(11) is exact. $F \equiv F(\beta, \lambda, \kappa)$ is independent of J, K, α. But in practice we always make the expansion to a finite order, which will then depend on J, K and α. In the first order expansion by the existence of the Jensen inequality, J, K and α may be determined by the variational condition of

$$\frac{\delta F_1}{\delta J} = 0, \quad \frac{\delta F_1}{\delta K} = 0, \quad \frac{\delta F_1}{\delta \alpha} = 0. \tag{13}$$

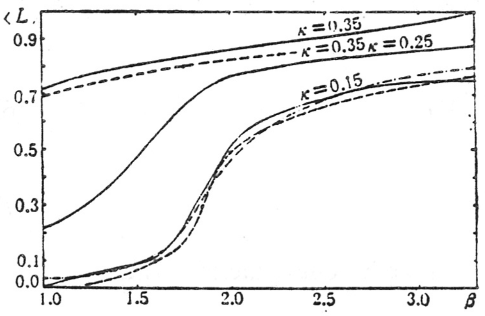


Fig. 1

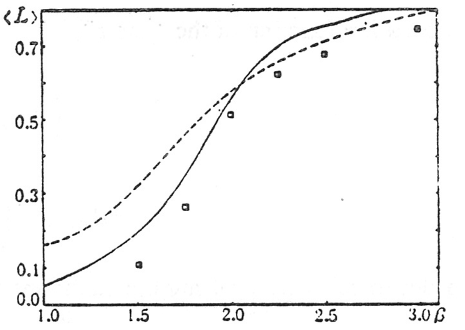


Fig. 2

where

$$
\begin{aligned}
F_1 &= F_0 + \frac{1}{M}\langle S_G + S_H + S_I - S_0\rangle_C \\
&= -(d-1)\ln f(J) - \ln f(K) - \ln g_4(\alpha) - \frac{\beta}{2}(d-1)[(d-2)\omega_1^4 + 2\omega_1^2\xi_1^2] \\
&\quad - 2\kappa\left(\frac{4}{\alpha}\right)[(d-1)\omega_1 + \xi_1] + \lambda\left[\frac{7!}{3!\alpha^4} - \frac{2\cdot 5!}{3!\alpha^2} + 1\right] \\
&\quad + \frac{5!}{3!\alpha^2} - 4 + 2(d-1)J\omega_1 + 2K\xi_1.
\end{aligned}
\tag{14}
$$

and $\omega_1 = I_2(2J)/I_1(2J)$, $\xi_1 = I_2(2K)/I_1(2K)$. To demonstrate the efficiency of the method, for simplicity, we take the symmetrical lattice. Then $J = K$, $\omega_1 = \xi_1$. Therefore the Eq.(13) is simplified to two coupled equations

$$
\begin{cases}
J = \beta(d-1)\omega_1^3 + \kappa\left(\frac{4}{\alpha}\right)^2, \\
1 = \frac{10}{\alpha^2} + \lambda\left(\frac{840}{\alpha^4} - \frac{20}{\alpha^2}\right) - d\cdot\kappa\left(\frac{4}{\alpha}\right)^2\omega_1.
\end{cases}
\tag{15}
$$

For given β, λ and κ, J and α can be obtained by solving Eqs.(15).

3. CALCULATION OF THE STATISTICAL AVERAGE OF THE POLYAKOV LINE

An effective approach to investigate the deconfinement phase transition of the pure gauge model at finite temperature is to inspect the statistical average of the Polyakov line $\langle L\rangle$ with

$$
L = \frac{1}{2}\operatorname{tr}\prod_{t=1}^{N_t} U_t(x, t) \tag{16}
$$

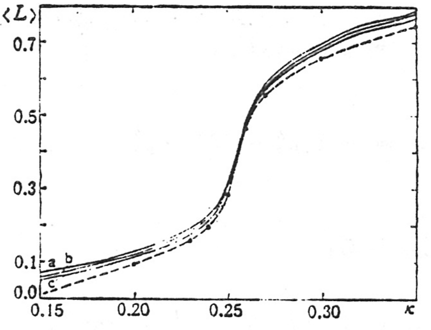


Fig. 3

When there is a matter field coupled with the pure gauge field, $<L>$ may still signal the deconfinement phase transition [2,4]. It can be calculated analytically order by order by using the general formula [6]

$$\langle L\rangle = \langle L\rangle_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\langle L(S-S_0)^n\rangle_C = \sum_{n=0}^{\infty}\frac{1}{n!}\langle L_n\rangle. \tag{17}$$

To the 3rd order approximation

$$\begin{aligned}
\langle L_0\rangle &= \langle L\rangle_0,\\
\langle L_1\rangle &= \langle LS\rangle_C - K\frac{\partial}{\partial K}\langle L\rangle_0,\\
\langle L_2\rangle &= \langle LS^2\rangle_C - 2\left(J\frac{\partial}{\partial J} + K\frac{\partial}{\partial K} + \alpha\frac{\partial}{\partial \alpha}\right)\langle LS\rangle_C + K^2\frac{\partial^2}{\partial K^2}\langle L\rangle_0,\\
\langle L_3\rangle &= \langle LS^3\rangle_C - 3\left(J\frac{\partial}{\partial J} + K\frac{\partial}{\partial K} + \alpha\frac{\partial}{\partial \alpha}\right)\langle LS^2\rangle_C\\
&\quad + 3\left(J^2\frac{\partial^2}{\partial J^2} + K^2\frac{\partial^2}{\partial K^2} + \alpha^2\frac{\partial^2}{\partial \alpha^2} + 2JK\frac{\partial}{\partial J}\frac{\partial}{\partial K} + 2J\alpha\frac{\partial}{\partial J}\frac{\partial}{\partial \alpha}\right.\\
&\quad \left. + 2K\alpha\frac{\partial}{\partial K}\frac{\partial}{\partial \alpha}\right)\langle LS\rangle_C - K^3\frac{\partial^3}{\partial K^3}\langle L\rangle_0.
\end{aligned} \tag{18}$$

They are the generalization of Eq.(16) in Ref. [6]. $<L>_0$ only contains U_T on the time-like links, thus it is only the function of K. The main effort is in the calculation of $<LS^i>_c$. The calculation of other terms is made easier by differentiating the lower order results.

According to Ref. [6] it is convenient to introduce diagrammatical notations. In $<LS^i>_c$ only connected diagrams contribute. As shown in the Eqs.(20) in Ref. [6] a vertical bar will represent a group element on the time-like link, and a horizontal bar as well as an inclined bar will represent a group element on the space-like link. Dots and dotted lines represent the connection of the top and the bottom in the time-like direction of the diagram due to the periodic boundary condition. For a closed loop the trace is implied. Diagrams related to the newly introduced Higgs field are as follows:

$$\bigcirc = \rho_x, \qquad \circ\!\!-\!\!\circ = \rho_x \rho_{x+\hat{\sigma}} \mathrm{tr} U_\sigma, \qquad \begin{array}{c}\circ\\ | \\ \circ\end{array} = \rho_x \rho_{x+\hat{\tau}} \mathrm{tr} U_\tau,$$

$$\bullet = -\lambda(\rho_x^2 - 1)^2 - \rho_x^2. \tag{19}$$

Then we have

$$\langle L \rangle_0 = \frac{1}{2} \langle L_{0,1} \rangle_0 = \xi_1^2,$$

$$\langle LS \rangle_c = \frac{1}{2} \cdot \frac{1}{2} \beta \cdot \alpha_{1,1} \langle L_{1,1} \rangle_c + \frac{1}{2} \gamma_{1,1} \langle T_{1,1} \rangle_c,$$

$$\langle LS^2 \rangle_c = \frac{1}{2} \cdot \left(\frac{1}{2} \beta\right)^2 \cdot \sum_{i=1}^{6} \alpha_{2,i} \langle L_{2,i} \rangle_c + \frac{1}{2} \sum_{i=1}^{8} \gamma_{2,i} \langle T_{2,i} \rangle_c, \tag{20}$$

$$\langle LS^3 \rangle_c = \frac{1}{2} \left(\frac{1}{2} \beta\right)^3 \sum_{i=1}^{29} \alpha_{3,i} \langle L_{3,i} \rangle_c + \frac{1}{2} \sum_{i=1}^{60} \gamma_{3,i} \langle T_{3,i} \rangle_c.$$

Where the connected diagrams $L_{n,i}$ from pure gauge part S_G and their corresponding symmetric coefficients $\alpha_{n,i}$ are presented in the Table 1 in Ref. [6]. In the appendix, connected diagrams related to S_H and S_I to the 2nd order expansion $<T_{n,i}>_0$ and their corresponding symmetric coefficients $\gamma_{n,i}$ are given. There are 60 connected diagrams to the 3rd order expansion; we will not present all of them, but only give some results of $<T_{3,i}>_0$, since their calculations are more complicated. As for other connected diagrams, their contributions can be written out directly from the diagrammatic rules given in the appendix.

4. RESULTS AND DISCUSSION

At fixed parameters β, λ and κ, J and α can be determined by solving Eq.(15). Then substituting them into Eqs.(20) and (18) with $\omega_i = \xi_i$, we obtain the approximate value of $<L>$ for $N_\tau = 2$ to the 3rd order of cumulant expansion. In the Fig. 1 $<L>$-β relation for $N_\tau = 2$, $\lambda = 0.10$ and several fixed κ are presented by solid lines. The heavy dots guided by the dashed lines represent Monte Carlo results [1], and the dot-dashed line is for the mean field result [2]. As it has been observed in the Monte Carlo simulation that the change of $<L>$ near the phase transition at $N_\tau = 4$ is sharper than the one at $N_\tau = 2$, thus in order to determine precisely the deconfinement phase transition, the calculation for larger N_τ is necessary. In the Fig. 2 the Monte Carlo simulation of $<L>$-β relation for $N_\tau = 2$, $\lambda = 5.0$ and $\kappa = 0.32$ is presented by small squares. The dashed line represents the mean field result, whereas the solid line represents the cumulant expansion result to the 3rd order.

In Fig. 3 the $<L>$-κ relation is given for $N_\tau = 2$, $\lambda = 0.10$ and $\beta = 1.25$. The heavy dots guided by the dashed line are the Monte Carlo results [1]. The solid lines a, b and c are calculated results from the cumulant expansion to the first, second and third order approximation respectively. Obviously, it shows that the calculated results converge order by order to the Monte Carlo result as its limit. It is hoped that use of the approach presented here to calculate the case with larger N_τ and higher order expansion will give us more accurate analytic results about deconfinement phase transition at finite temperature.

Table 1

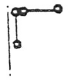
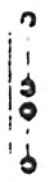
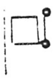
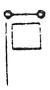
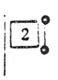
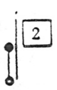
n	i	$T_{n,i}$	$\alpha_{n,i}$	$\eta_{n,i}$	$\langle T_{n,i}\rangle_0$
1	1		2	κ	$\xi_1\xi_2G_1^2$
2	1		$8r$	κ^2	$2\omega_1\xi_1\xi_2G_1^2G_2$
n	i	$T_{n,i}$	$\alpha_{n,i}$	$\eta_{n,i}$	$\langle T_{n,i}\rangle_0$
2	2		2	κ^2	$\xi_1\xi_3G_2^2$
	3		2	κ^2	$\frac{1}{2}\xi_2^2G_2^2$
	4		$4r$	$\frac{1}{2}\beta\kappa$	$\frac{1}{2}\omega_1^2\xi_1\xi_2^2G_1^2$
	5		$8r$	$\frac{1}{2}\beta\kappa$	$\frac{1}{2}\omega_1\xi_1^2\omega_2\xi_2G_1^2$
	6		$4r$	$\frac{1}{2}\beta\kappa$	$\omega_1^2\xi_1^2\xi_3G_1^2$
	7		$4r$	$\frac{1}{2}\beta\kappa$	$\frac{1}{2}\omega_1^2\xi_1\xi_2^2G_1$
	8		8	κ	$\xi_1\xi_2X_{11}G_1$
3	1		$12r$	$\left(\frac{1}{2}\beta\right)^2\kappa$	$\xi_1\left[4\omega_1\xi_1+\frac{1}{3}(\omega_3-2\omega_1)(\xi_3-2\xi_1)\Omega\Xi\right]G_1^2$
	2		$6r$	$\left(\frac{1}{2}\beta\right)^2\kappa$	$\xi_1[\xi_3+(\xi_4-\xi_2)\Omega^2\Xi]G_1^2$
	3		$5r$	$\left(\frac{1}{2}\beta\right)^2\kappa$	$\xi_1\left[4\xi_1^2+\frac{1}{3}(\xi_3-2\xi_1)^2\Omega^2\right]G_1^2$
	4		$6r$	$\left(\frac{1}{2}\beta\right)^2\kappa$	$\frac{1}{2}\xi_2[2\xi_1+(\xi_3-2\xi_1)\Omega^2\Xi]G_1^2$

Table 1 (cont.)

5		$12r$	$\left(\frac{1}{2}\beta\right)^2\kappa$	$\xi_1^2\left\{\frac{1}{6}(\omega_3-2\omega_1)[\xi_2^2-(1+3\Xi^2)]\Omega+\omega_1(1+3\Xi)\right\}G_1^2$
6		$12r$	$\left(\frac{1}{2}\beta\right)^2\kappa$	$\frac{1}{2}\xi_1^2\{\xi_2\xi_3\Omega^2+(1-\Omega^2)[2\xi_1+(\xi_3-2\xi_1)\Xi]\}G_1^2$
7		$12r$	$\left(\frac{1}{2}\beta\right)^2\kappa$	$\frac{1}{4}\xi_1\xi_2[\xi_2^2\Omega^2+(1-\Omega^2)(1+3\Xi^2)]G_1^2$

APPENDIX

In the calculation of $<LS^n>_c = <L(S_G + S_H + S_I)^n>_c$, except $<LS_G^n>_c$ calculated to the 3rd order in Ref. [6], we now need to calculate the connected diagrams of all the terms from the above n-th power expansion. We use $\eta_{n,i}$ to denote the product of β and κ, the coefficients of expansion, and $\alpha_{n,i}$ to denote the symmetric coefficients of equivalent diagrams. In Eq.(20) $\gamma_{n,i} = \alpha_{n,i}\eta_{n,i}$ and the following notations are used:

$$\omega_n=\frac{1}{f(J)}\frac{\partial^n f(J)}{\partial J^n},\quad \xi_n=\frac{1}{f(K)}\frac{\partial^n f(K)}{\partial K^n},\quad \Omega=\frac{1}{3}(\omega_2-1),\quad \Xi=\frac{1}{3}(\xi_2-1),$$
$$r=2(d-1),\quad R_0=2d-3,\quad R=4R_0,\quad R_i=R_{i-1}-1, \tag{A.1}$$

$$G_n=\langle\rho_x^n\rangle_0=\frac{(n+3)!}{3!\,\alpha^n}, \tag{A.2}$$

$$\begin{aligned}
X_{11}&=\langle\rho_x[-\lambda(\rho_x^2-1)^2-\rho_x^2]\rangle_0=-\lambda\sum_{i=0}^{2}(-1)^iC_2^iG_{2i+1}-G_3,\\
X_{12}&=\langle\rho_x[-\lambda(\rho_x^2-1)^2-\rho_x^2]^2\rangle_0=\lambda^2\sum_{i=0}^{4}(-1)^iC_4^iG_{2i+1}+2\lambda\sum_{i=0}^{2}(-1)^iC_2^iG_{2i+3}+G_5,\\
X_{21}&=\langle\rho_x^2[-\lambda(\rho_x^2-1)^2-\rho_x^2]\rangle_0=-\lambda\sum_{i=0}^{2}(-1)^iC_2^iG_{2i+2}-G_4.
\end{aligned} \tag{A.3}$$

All the integrals of connected diagrams can be evaluated by the method described in Ref. [6] and [9]. From the calculation procedure some convenient diagrammatic rules can be deduced. The basic technique for calculating $<\cdots>_0$ is decomposing the closed loop (Polyakov line or Wilson loop), i.e., decomposing the trace of product of a set of group elements into a product of the traces of single group element. When one link is decomposed from a closed loop, the remaining part is again regarded as a closed one. This procedure is continued until all the links are decomposed. We call the n overlap links in a connected diagram an n-multiple link. An n-multiple link is called single connected, if it is the only common link of n diagrams. When calculating $<\cdots>_0$ of a connected diagram, we decompose O and • first, then we decompose links from the simple to complex ones. The total result of $<\cdots>_0$ of a given diagram is the product of contributions from all parts given by the following rules:

(1) n on a sit contributes G_n,

(2) ○• on a site contributes X_{11},

(3) ○•• on a site contributes X_{12},

(4) ○○• on a site contributes X_{21},

(5) Each space (time)-like link contributes $2\omega_1(2\xi_1)$,

(6) Each simple connected n-multiple space (time)-like link contributes $\omega_n(\xi_n)$,

(7) m simple connected multiple links in the connected diagram contributes $(1/2)^{m-1}$.

In Table 1 all connected diagrams of LS^i to the second order ($i = 2$), a part of connected diagram with non-simple connected multiple links to the 3rd order, their symmetrical coefficients and their $<\cdots>_0$ are presented. The contributions of other diagrams to the 3rd order are easy to write down by use of the above rules. For higher expansions the above rules have to be extended.

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