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# Integrability Conditions of the Three-dimensional Exactly Solved Model-Baxter-Bazhanov Model in Statistical Mechanics

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**From the chiral Potts model the "inversion" and "star-square" relations of the Baxter-Bazhanov model are obtained. The tetrahedron equation, which is a commutativity condition for the three-dimensional cubic lattice, is a consequence of the star-triangle relation of the chiral Potts model. The additional constraints in tetrahedron equation hold naturally when the Boltzmann weights are parameterized in terms of the Zamolodchikov angle variables. It is pointed out that the star-triangle relation of the three-dimensional model, which includes the result of Baxter-Bazhanov's, can be obtained by using the method given in this paper.**

**Key words: chiral Potts model, Baxter-Bazhanov model, "Star-Square" relation, Tetrahedron equation, Star-Triangle relation.**

## 1. INTRODUCTION

In comparison with the theory of two-dimensional exactly solved models, there are no good three-dimensional exactly solved models in statistical mechanics [1]. Recently, a nontrivial exactly solved model in statistical mechanics has been proposed by Bazhanov and Baxter [2], called the

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Baxter-Bazhanov model. And the integrability condition-tetrahedron equation [3], which ensures the commutativity of layer-to-layer transfer matrices, has been proved in [4] by using the "invention" and "star-square" relations. This model has been proposed by Bazhanov and Baxter in the study of the chiral Potts model [5]. The question of whether there any connections of the integrability conditions between the chiral Potts model and the Baxter-Bazhanov model will be dealt with in this paper. Our findings indicated that the "inversion" and "star-square" relations of the Baxter-Bazhanov model can be obtained from the star-triangle relation of the chiral Potts model, which means that the tetrahedron equation of the Baxter-Bazhanov model is also the consequence of the latter. When we parameterized the spectra of the Boltzmann weight as the angles exactly like those in the Zamolodchikov model [3], the additional constraints hold naturally. It should be noted that the star-triangle relation of the three-dimensional model, which contains the result given in [6], can be obtained following the method given in this paper.

The organization of the paper is as follows. In Sec. 2, we describe some formulas of the chiral Potts model and give the "inversion" relation of the Baxter-Bazhanov model. The connection between the "star-square" relation of the three-dimensional model and the star-triangle relation of the chiral Potts model is obtained in Sec. 3. We can prove the tetrahedron equation from the latter. In Sec. 4, the spectra of the Boltzmann weight are parameterized as the angle variables that are the dihedral angles between the rapidity planes connected with the elementary cube. And the additional constraints hold naturally with respect to the angle parameterization. The "symmetrical" point in [7] means that all angles have been determined. Finally, we point out that the star-triangle relation can be obtained by using the methods given in this paper.

## 2. FORMULAS OF CHIRAL POTTS MODEL AND "INVENTION" RELATION OF BAXTER-BAZHANOV MODEL

The star-triangle relation of the chiral Potts model has the form

$$\sum_{l=1}^{N} \overline{W}^{CP}_{qr}(m-l)W^{CP}_{pr}(n-l)\overline{W}^{CP}_{pq}(l-k)$$
$$= R_{pqr}W^{CP}_{pq}(n-m)\overline{W}^{CP}_{pr}(m-k)W^{CP}_{qr}(n-k) \tag{2.1}$$

where the Boltzmann weights are denoted by

$$\frac{W^{CP}_{pq}(n)}{W^{CP}_{pq}(0)} = \prod_{j=1}^{n}\frac{d_pb_q - a_pc_q\omega^j}{b_pd_q - c_pa_q\omega^j}, \quad \frac{\overline{W}^{CP}_{pq}(n)}{\overline{W}^{CP}_{pq}(0)} = \prod_{j=1}^{n}\frac{\omega a_pd_q - d_pa_q\omega^j}{c_pb_q - b_pc_q\omega^j}. \tag{2.2}$$

and the "rapidity" vector $(a_p, b_p, c_p, d_p)$ satisfies

$$a_p^N + k'b_p^N = kd_p^N, \quad k'a_p^N + b_p^N = kc_p^N, \tag{2.3}$$

with $k^2 + k'^2 = 1$. The factor $R_{pqr}$ is facterizable:

$$R_{pqr} = f_{pq}f_{qr}/f_{pr}, \quad f_{pq} = \left[\prod_{k=1}^{N}\left(\sum_{m=1}^{N}\omega^{mk}\overline{W}^{CP}_{pq}(m)\right)\Big/\prod_{k=1}^{N}W^{CP}_{pq}(k)\right]^{\frac{1}{N}}, \tag{2.4}$$

where $f_{qr}$ and $f_{pr}$ have the similar expressions.

The Baxter-Bazhanov model is a three-dimensional exactly solved model in statistical mechanics with interactions round the cube. The Boltzmann weight elementary building blocks can be written as

$$W(x,y,z|k,l) = W(x,y,z|k-l)\Phi(l), \tag{2.5}$$

with the properties

$$W(x,y,z|k-l) = \prod_{j=1}^{k-l} \frac{y}{z - x\omega^j}, \quad x^N + y^N = z^N,$$
$$\omega = e^{2\pi i/N}, \quad \omega^{1/2} = e^{\pi i/N}, \quad \Phi(l) = (\omega^{1/2})^{l(N+l)}. \tag{2.6}$$

Considering Eq. (2.3), we get

$$a_p = c_p = d_q = 1, \quad b_q = \omega,$$
$$d_p = c_q = (k+k')^{1/N}, \quad b_p = a_q = (k-k')^{1/N}. \tag{2.7}$$

where we have made a choice of $\omega = \exp(2\pi i/N)$ in the formulas of the chiral Potts model. Then, from Eq. (2.2) we have

$$W_{pq}^{CP}(n)/W_{pq}^{CP}(0) = \delta_{n,0}, \quad \overline{W}_{pq}^{CP}(n)/\overline{W}_{pq}^{CP}(0) = 1. \tag{2.8}$$

where $\delta_{n,0}$ is the Kronecker symbol on $Z_N$ and $n \in Z_N$. In this case, we can write

$$f'_{pq} = \left[\prod_{m=1}^{N}\left(\sum_{l=1}^{N} \omega^{ml} \frac{\overline{W}_{pq}^{CP}(l)}{\overline{W}_{pq}^{CP}(0)}\right)\prod_{m=1}^{N}\frac{W_{pq}^{CP}(0)}{W_{pq}^{CP}(m)}\right]^{1/N} = N. \tag{2.9}$$

Furthermore, we have, by taking the limit of $k \to k'$,

$$W_{qr}^{CP}(n)/W_{qr}^{CP}(0) = \overline{W}_{pr}^{CP}(0)/\overline{W}_{pr}^{CP}(n) = W(k^{-1/N}a_r, b_r, \omega d_r|n),$$
$$\overline{W}_{qr}^{CP}(n)/\overline{W}_{qr}^{CP}(0) = 1/W(k^{-1/N}b_r, \omega a_r, \omega^2 c_r|-n),$$
$$W_{pr}^{CP}(n)/W_{pr}^{CP}(0) = W(k^{-1/N}b_r, \omega a_r, \omega c_r|-n). \tag{2.10}$$

So we get

$$\sum_{l=1}^{N} \frac{W(k^{-1/N}b_r, \omega a_r, \omega c_r|l,n)}{W(k^{-1/N}b_r, \omega a_r, \omega^2 c_r|l,m)} = R'_{pqr}\delta_{n,m}. \tag{2.11}$$

with $R'_{pqr} = f'_{pq}f'_{qr}f'_{pr}$ where we have used the star-triangle relation of the chiral Potts model. In fact, the factor $R'_{pqr}$ can be obtained easily from the preceding relation,

$$\sum_{l=1}^{N}(1 - z\omega^l)^{-1} = N(1 - z^N)^{-1}, \tag{2.12}$$

due to the relation

$$R'_{pqr} = N(1 - \omega k^{1/N}c_r/b_r)/(1 - kc_r^N/b_r^N). \tag{2.13}$$

Set $x = k^{-1/N}b_r$, $y = \omega a_r$, $z = \omega c_r$. Then the relation (2.11) has the forms

$$\sum_{l=1}^{N} \frac{W(x,y,z|l,n)}{W(x,y,\omega z|l,m)} = N\delta_{n,m}\frac{1 - z/x}{1 - z^N/x^N}. \tag{2.14}$$

This is just the "inversion" relation of the Boltzmann weight elementary building blocks of the three-dimensional Baxter-Bazhanov model.

## 3. "STAR-SQUARE" RELATION OF THE BAXTER-BAZHANOV MODEL

Ten years ago, as the generalization of the star-triangle relation in the eight-vertex model, the "star-square" relation was introduced [1]. For the three-dimensional Baxter-Bazhanov model, the "star-square" relation proposed by Kashaev *et al.* has the following form:

$$\left\{\sum_{\sigma=1}^{N}\frac{W(x_1,y_1,z_1|a+\sigma)W(x_2,y_2,z_2|b+\sigma)}{W(x_3,y_3,z_3|c+\sigma)W(x_4,y_4,z_4|d+\sigma)}\right\}_0$$

$$=\frac{1}{\Phi(a-b)\omega^{(a+b)/2}}\cdot\left(\frac{x_2y_1}{x_1z_2}\right)^a\left(\frac{x_1y_2}{x_2z_1}\right)^b\left(\frac{z_3}{y_3}\right)^c\left(\frac{z_4}{y_4}\right)^d$$

$$\cdot\frac{W\left(\frac{\omega x_3x_4z_1z_2}{x_1x_2z_3z_4}\middle|c+d-a-b\right)}{W\left(\frac{x_4z_1}{x_1z_4}\middle|d-a\right)W\left(\frac{x_3z_2}{x_2z_3}\middle|c-b\right)W\left(\frac{x_3z_1}{x_1z_3}\middle|c-a\right)W\left(\frac{x_4z_2}{x_2z_4}\middle|d-b\right)} \tag{3.1}$$

where the subscript "0" after the curly brackets indicates that the l. h. s. of the preceding equation is normalized to unity at zero exterior spins so as to keep the variables $x_i$, $y_i$, $z_i$ $(i = 1, 2, 3, 4)$ from changing. The function $W(x|l)$ is given by the relation

$$W(x,y,z|l)=(y/z)^lW(x/z|l),\ l\in Z_N. \tag{3.2}$$

Here the constraint $y_1y_2z_3z_4/y_3y_4z_1z_2=\omega$ should be satisfied. Now we derive this relation from the star-triangle of the chiral Potts model. For the "rapidity" vector $(a_p, b_p, c_p, d_p)$ and $(a_q, b_q, c_q, d_q)$ setting

$$y_{pq}=((d_pb_q)^N-(a_pc_q)^N)^{1/N}=((b_pd_q)^N-(c_pa_q)^N)^{1/N},$$
$$\bar{y}_{pq}=((\omega a_pd_q)^N-(d_pa_q)^N)^{1/N}=((c_pb_q)^N-(b_pc_q)^N)^{1/N}, \tag{3.3}$$

we have

$$\frac{W^{CP}_{pq}(n)}{W^{CP}_{pq}(0)}=\frac{W(c_pa_q,y_{pq},b_pd_q|n)}{W(a_pc_q,y_{pq},d_pb_q|n)},$$
$$\frac{\overline{W}^{CP}_{pq}(n)}{\overline{W}^{CP}_{pq}(0)}=\frac{W(b_pc_q,\bar{y}_{pq},c_pb_q|n)}{W(d_pa_q,\bar{y}_{pq},\omega a_pd_q|n)}. \tag{3.4}$$

where we have used the relation (2.2). When $c_r = 0$, the following relations hold:

$$W^{CP}_{qr}(n)/W^{CP}_{qr}(0)=W_{qR(r)}(n),\ \overline{W}^{CP}_{qr}(n)/\overline{W}^{CP}_{qr}(0)=1/W_{R(q)R(r)}(n); \tag{3.5}$$

$$W^{CP}_{pr}(n)/W^{CP}_{pr}(0)=W_{pR(r)}(n),\ \overline{W}^{CP}_{pr}(n)/\overline{W}^{CP}_{pr}(0)=1/W_{R(p)R(r)}(n). \tag{3.6}$$

where $R$ is defined by

$$R(a_p,b_p,c_p,d_p)=(b_p,\omega a_p,d_p,c_p)$$

and we have used the notation

$$W_{pq}(n) \equiv W(\omega^{-1}c_pb_q, d_pa_q, b_pc_q | n). \tag{3.7}$$

From the star-triangle relation of the chiral Potts model, and by taking account of the relations (3.4), (3.5) and (3.6), we get

$$\sum_{\sigma=1}^{N} \frac{W_{pR(r)}(a+\sigma)W(\omega^{1-b+d}a_pd_q, \omega^{1/2}\bar{y}_{pq}, \omega d_pa_q | b+\sigma)}{W_{R(q)R(r)}(c+\sigma)W(c_pb_q, \omega^{1/2}\bar{y}_{pq}, \omega b_pc_q | d+\sigma)}$$

$$= R'_{pqr}\frac{W_{qR(r)}(a-d)W(\omega^{1-b+d}a_pd_q, \omega^{1/2}\bar{y}_{pq}, \omega d_pa_q | b-d)W(c_pa_q, y_{pq}, b_pd_q | a-c)}{W_{R(p)R(r)}(c-d)W(a_pc_q, y_{pq}, d_pb_q | a-c)} \tag{3.8}$$

where the following property has been used.

$$W(x,y,z|l)W(x\omega^l, y, z|k) = W(x,y,z|l+k). \tag{3.9}$$

Setting

$$x_1 = \frac{a_pd_q}{d_pa_q}, \quad x_2 = \frac{c_pa_r}{b_pd_r}, \quad x_3 = \frac{d_qa_r}{a_qd_r}, \quad x_4 = \frac{c_pb_q}{b_pc_q}. \tag{3.10}$$

we have

$$\sum_{\sigma=1}^{N} \omega^{\sigma}\frac{W(x_1|\sigma)W(x_2|a+\sigma)}{W(x_3/\omega|\sigma)\,W(x_4/\omega|\sigma)}$$

$$= R(x_1, x_2, x_3, x_4)\frac{W(x_2/x_3|a)\;W(x_2/x_4|a)}{W(x_1x_2/x_3x_4|a)}. \tag{3.11}$$

where $R'_{pqr} = R(x_1, x_2, x_3, x_4)$ and the relation (3.2) has been used. By making the transformation $x_1 \longleftrightarrow x_2$, from the preceding equation, we have the relation

$$\sum_{\sigma=1}^{N} \frac{W_{pR(r)}(\sigma)W(\omega^{1-b+d}a_pd_q, \omega^{1/2}\bar{y}_{pq}, \omega d_pa_q|\sigma)}{W_{R(q)R(r)}(\sigma)W(c_pb_q, \omega^{1/2}\bar{y}_{pq}, \omega b_pc_q|\sigma)}$$

$$= R'_{pqr}\left(\frac{c_pa_r}{b_pd_r}\right)^{d-b}\frac{W(\omega^{b-d-1}d_pa_r/a_pd_r|d-b)}{W(\omega^{b-d-1}d_pa_q/a_pd_q|d-b)}$$

$$\cdot\frac{W(\omega^{b-d-1}c_pd_pa_qb_q/a_pb_pc_qd_q|\,d-b)}{W(\omega^{b-d-1}d_pb_q/a_pc_q|d-b)}. \tag{3.12}$$

And the relation (3.8) can be changed into the form

$$\sum_{\sigma=1}^{N} \frac{W_{pR(r)}(a+\sigma)W(\omega^{1-b+d}a_pd_q, \omega^{1/2}\bar{y}_{pq}, \omega d_pa_q|b+\sigma)}{W_{R(q)R(r)}(c+\sigma)W(c_pb_q, \omega^{1/2}\bar{y}_{pq}, \omega b_pc_q|d+\sigma)}$$

$$= \frac{R'_{pqr}}{\Phi(a-d)\Phi(b-d)}\left(\frac{a_pd_qb_r}{\omega^{1/2}c_pa_qa_r}\right)^a\left(\frac{\omega^{b-d-1}\bar{y}_{pq}}{a_pd_q}\right)^b$$

$$\cdot\left(\frac{\omega a_qd_r}{c_qb_r}\right)^c\left(\frac{\omega^{\frac{1}{2}-b+d}c_pc_qa_r}{\bar{y}_{pq}d_r}\right)^d$$

$$
\cdot \frac{W\left(\frac{\omega^{b-d-1}d_p a_r}{a_p d_r}\Big|d-b\right)W\left(\frac{\omega^{b-d-1}d_p b_q}{a_p c_q}\Big|c+d-a-b\right)}{W\left(\frac{\omega^{b-d-1}d_p a_q}{a_p d_q}\Big|d-b\right)W\left(\frac{\omega^{b-d-1}d_p b_q}{a_p c_q}\Big|d-b\right)}
$$

$$
\cdot \frac{1}{W\left(\frac{b_q b_r}{\omega c_q a_r}\Big|d-a\right)W\left(\frac{\omega^{b-d-1}d_p a_r}{a_p d_r}\Big|c-b\right)W\left(\frac{b_p c_q}{\omega c_p a_q}\Big|c-a\right)}. \tag{3.13}
$$

Let

$$
\begin{gathered}
x_1 = c_p a_r,\quad x_2 = \omega^{1-b+d} a_p d_q,\quad x_3 = d_q a_r,\quad x_4 = c_p b_q, \\
y_1 = d_p b_r,\quad y_2 = y_4 = \omega^{1/2}\bar{y}_{pq},\quad y_3 = c_q b_r, \\
z_1 = b_p d_r,\quad z_2 = \omega d_p a_q,\quad z_3 = \omega a_q d_r,\quad z_4 = \omega b_p c_q.
\end{gathered} \tag{3.14}
$$

The "star-square" relation (3.1) of the Baxter-Bazhanov model can be obtained from relations (3.12) and (3.13).

Then the connection is built between the "star-square" relation of the three-dimensional model and the star-triangle relation of the two-dimensional model. We know that the tetrahedron equation plays an important role in the interaction-round-a-cube model, exactly as in the Yang-Baxter equation in the two-dimensional integrable model, which ensures the commutativity of the layer-to-layer transfer matrices. And the tetrahedron equation of the Baxter-Bazhanov model is a consequence of the "inversion" relation and the "star-square" relation owing to the work by Kashaev *et al.* Therefore, our result means that the tetrahedron equation of the three-dimensional Baxter-Bazhanov model can be derived from the star-triangle relation of the chiral Potts model in two dimensions. Note that the constraints, $y_1 y_2 z_3 z_4 / y_3 y_4 z_1 z_2 = \omega$, should be satisfied because the right-hand-side term of the relation is independent on the spin $\sigma$ appearing in the sum of the left-hand-side terms of the relation.

## 4. ADDITIONAL CONDITIONS IN THE TETRAHEDRON EQUATION

In [4], Kashaev *et al.* found the additional conditions in the tetrahedron equation for the Baxter-Bazhanov model. These constraints are

$$
\begin{gathered}
\omega \frac{x_{23}}{x_3}\frac{x'_4}{x'_{24}}\frac{x''_{24}}{x''_2}\frac{x'''_2}{x'''_{24}} = 1,\quad \frac{x_{13}}{x_1}\frac{x'_1}{x'_{14}}\frac{x''_{14}}{x''_1}\frac{x'''_1}{x'''_{14}} = 1, \\
\frac{x_{14}}{x_4}\frac{x'_4}{x'_{14}}\frac{x''_{14}}{x''_4}\frac{x'''_4}{x'''_{24}} = 1,\quad \frac{x_{13}}{x_3}\frac{x'_3}{x'_{13}}\frac{x''_{13}}{x''_1}\frac{x'''_2}{x'''_{23}} = 1.
\end{gathered} \tag{4.1}
$$

From the viewpoint of spherical trigonometry, these conditions hold very naturally when we parameterize these variables as Zamolodchikov's angles. Following the method in [3], we introduce a large sphere (its radius is much larger than the size of the tetrahedra) with a point near the vertices as the center. Consider four great circles on the sphere corresponding to the four "would planes" connected with the cube. A fragment of the stereographic projection of this sphere is shown in Fig. 1, exactly as in Zamolodchikov's model. There are four spherical triangles: (123), (146), (345), (256). Denoted by $l$, $l'$, $l''$, and $l'''$, the sums of the three sides of them are divided by $2N$. In this way, the variables in the preceding equations can be parameterized as

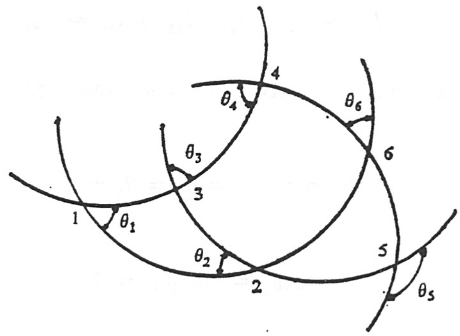


**Fig. 1**

$$
\begin{gathered}
x_1 = c_1/s_1,\ x_2 = \omega^{-1/2} s_1/c_1,\ x_3 = \exp(-il_3) s_2/c_2, \\
x_4 = \omega^{-1/2}\exp(-il_3) c_2/s_2,\ x_{12} = 1/c_1 s_1, \\
x_{13} = \exp[i(l - l_3)] s_3/c_2 s_1,\ x_{14} = \exp[i(l_2 - l)] c_3/s_1 s_2, \\
x_{23} = \omega^{-1/2}\exp[i(l_1 - l)] c_3/c_1 c_2,\ x_{24} = \exp(-il) s_3/s_2 c_1, \\
x_{34} = \exp(-il_3)/c_2 s_2.
\end{gathered}
\tag{4.2}
$$

The variables $x_1', x_2', \ldots, x_{34}'; x_1'', x_2'', \ldots, x_{34}'', x_1''', x_2''', \ldots, x_{34}'''$ are expressed similarly by $s_i', c_i', l_i', l'$; $s_i'', c_i'', l_i'', l'', s_i''', c_i''', l_i''', l'''$, respectively, where

$$
\begin{gathered}
c_1' = c_1 = \cos^{1/N}\frac{\theta_1}{2},\quad s_1' = s_1 = \sin^{1/N}\frac{\theta_1}{2}, \\
c_1''' = c_2 = \cos^{1/N}\frac{\theta_2}{2},\quad s_1''' = s_2 = \sin^{1/N}\frac{\theta_2}{2}, \\
s_1'' = c_3 = \cos^{1/N}\frac{\theta_3}{2},\quad c_1'' = s_3 = \sin^{1/N}\frac{\theta_3}{2}, \\
s_3' = s_3'' = \cos^{1/N}\frac{\theta_4}{2},\quad c_3' = c_3'' = \sin^{1/N}\frac{\theta_4}{2}, \\
c_2'' = c_2''' = \cos^{1/N}\frac{\theta_5}{2},\quad s_2'' = s_2''' = \sin^{1/N}\frac{\theta_5}{2}, \\
s_2' = c_3''' = \cos^{1/N}\frac{\theta_6}{2},\quad c_2' = s_3''' = \sin^{1/N}\frac{\theta_6}{2};
\end{gathered}
\tag{4.3}
$$

and

$$
\begin{gathered}
l_1 = l_{23}/N,\ l_1' = l_{46}/N,\ l_1'' = l_{45}/N,\ l_1''' = l_{56}/N, \\
l_2 = l_{13}/N,\ l_2' = l_{14}/N,\ l_2'' = l_{34}/N,\ l_2''' = l_{26}/N, \\
l_3 = l_{12}/N,\ l_3' = l_{16}/N,\ l_3'' = l_{35}/N,\ l_3''' = l_{25}/N.
\end{gathered}
\tag{4.4}
$$

From the preceding relations, it is evident that the additional conditions (4.1) are equivalent to

$$
l_{12} + l_{26} = l_{16},\ l_{32} + l_{25} = l_{35},
$$

$$l_{45} + l_{65} = l_{45}, \; l_{13} + l_{34} = l_{14}. \tag{4.5}$$

Of course, these relations hold naturally (see Fig. 1). Furthermore, we get the "symmetrical point" [7]:

$$\theta_1 = \theta_5, \theta_2 = \theta_3 = \theta_4 = \theta_6.$$

It means that

$$\cos\theta_2 = \sin\theta_1/2 = 0. \tag{4.6}$$

That is, all of the angles have been determined at the "symmetrical point."

## 5. CONCLUSIONS AND REMARKS

The "inversion" and "star-square" relations of the three-dimensional Baxter-Bazhanov model can be obtained from the chiral Potts model, which is a two-dimensional integrable model. Then we can derive the tetrahedron equation of the three-dimensional model by using the star-triangle relation of the chiral Potts model. Note that we can get also the star-triangle relation

$$\sum_{l=1}^{N} \frac{W_{pR(r)}(n-l)}{W_{qr}(l-m)W_{pq}(k-l)} = R'_{pqr} \frac{W_{pR(q)}(n-m)W_{R^{-1}(q)r}(k-n)}{W_{pr}(k-m)} \tag{5.1}$$

with $a_p = d_r = 0$; or

$$\sum_{l=1}^{N} \frac{W_{pR(r)}(n+l)}{W_{R(q)R(r)}(m+l)W_{pq}(k+l)} = \bar{R}'_{pqr} \frac{W_{pR(q)}(n-m)W_{qR(r)}(n-k)}{W_{R(p)R(r)}(m-k)} \tag{5.2}$$

with $a_p = c_r = 0$. The restricted star-triangle relation introduced in [6] can be obtained from the preceding relations. Further details will be given elsewhere. We hope the method in this paper will be useful in constructing the solution of the tetrahedron equations and finding new three-dimensional integrable models, especially the elliptic case of the spectra.

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