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Comments on the Bosonnization Technique of Fermions*

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Abstract The conventional way for the bosonnization of fermions has been based on the Jordon and Wigner scheme. In this paper, we show that this scheme is not correct for $k=k'$. Therefore the bosonnization of fermions cannot be complete.

Key words Jordon and Wigner scheme, bosonnization technizne of fermion, canonnical commutation relation, completeness.

In reference [1], the bosonnization of fermions was traced back to the wrok by Jordon and Wigner [2]. Their theory can be summarized as following: if the boson satisfies the canonical commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'},\quad [a_k, a_{k'}] = 0,\quad [a_k^\dagger, a_{k'}^\dagger] = 0, \tag{1}$$

then by defining

$$d_k = \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q) a_k, \tag{2}$$

$$d_k^\dagger = a_k^\dagger \exp(-\mathrm{i}\pi \sum_{q=k}^{\infty} N_q), N_k = a_k^\dagger a_k,$$

it is easy to prove the anticommutation relations

$$\{d_k, d_{k'}^\dagger\} = \delta_{kk'},\quad \{d_k, d_{k'}\} = 0,\quad \{d_k^\dagger, d_{k'}^\dagger\} = 0. \tag{3}$$

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This bosonnization technique is usually called Jordon and Wigner scheme.

But as a matter of fact, this conclusion is not tenable. For example, multiplying d_k by $d_k^\dagger$ of Eq.(2) it gives

$$d_k d_k^\dagger = \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q) a_k a_k^\dagger \exp(-\mathrm{i}\pi \sum_{q=k}^{\infty} N_q);$$

while according to Eq.(1),

$$a_k a_k^\dagger = 1 + a_k^\dagger a_k = 1 + N_k \tag{4}$$

$$[N_q, N_k] = 0. \tag{5}$$

Thus

$$d_k d_k^\dagger = \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q)(1 + N_k) \exp(-\mathrm{i}\pi \sum_{q=k}^{\infty} N_q)$$
$$= 1 + N_k = a_k a_k^\dagger \tag{6}$$

or $d_k d_k^\dagger = a_k a_k^\dagger$; while multiplying $d_k^\dagger$ by d_k of Eq. (2) it gives

$$d_k^\dagger d_k = a_k^\dagger \exp(-\mathrm{i}\pi \sum_{q=k}^{\infty} N_q) \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q) a_k = a_k^\dagger a_k, \tag{7}$$

or $d_k^\dagger d_k = a_k^\dagger a_k$. Combining Eq. (6) with (7) yields

$$d_k d_k^\dagger + d_k^\dagger d_k = a_k a_k^\dagger + a_k^\dagger a_k = 1 + 2a_k^\dagger a_k, \tag{8}$$

so d_k and $d_k^\dagger$ do not satisfy Fermi statistics as Eq.(3).

However Eq. (6) and (7) can yield

$$d_k d_k^\dagger - d_k^\dagger d_k = a_k a_k^\dagger - a_k^\dagger a_k = 1 \tag{9}$$

which satisfies Bose statistics indeed. This statement contradicts with Eq. (3). Therefore, Jordon and Wigner scheme is not tenable.

On the other hand, multiplying d_k with d_k of Eq. (2) it yields

$$d_k^2 = \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q) a_k \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q) a_k =$$

$$a_k a_k \exp[\mathrm{i}\pi \sum_{q=k}^{\infty} (N_q - \delta_{q,k} - \delta_{q,k})] \exp[\mathrm{i}\pi \sum_{q=k}^{\infty} (N_q - \delta_{q,k})]$$

where

$$\sum_{q=k}^{\infty} \delta_{qk} = 1, \quad \exp(-\mathrm{i}\pi \sum_{q=k}^{\infty} \delta_{qk}) = e^{-\mathrm{i}\pi} = -1. \tag{10}$$

Then $d_k^2 = -a_k a_k \exp(2\mathrm{i}\pi \sum_{q=k}^{\infty} N_q)$.

Because the eigenvalues of N_q are $0,1,2,\cdots$ i.e.

$$N_q = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & \ddots \end{pmatrix}, \tag{11}$$

so

$$\exp(2\mathrm{i}\pi N_q) = [1, e^{2\mathrm{i}\pi}, e^{4\mathrm{i}\pi}, \cdots] = 1. \tag{12}$$

and

$$(d_k)^2 = -(a_k)^2. \tag{13}$$

By taking Hermitian conjugate of Eq. (13), it provides

$$(d_k^\dagger)^2 = -(a_k^\dagger)^2. \tag{14}$$

The combination of Eq. (8), (13) and (14) gives

$$\begin{cases} d_k d_k^\dagger + d_k^\dagger d_k = 1 + 2a_k^\dagger a_k \\ d_k^2 = -a_k^2 \\ (d_k^\dagger)^2 = -(a_k^\dagger)^2 , \end{cases} \tag{15}$$

which is not Fermi statistic. In fact, Eq. (15) contradicts with Eq. (3).

$$\begin{cases} d_k d_k^\dagger + d_k^\dagger d_k = 1 \\ d_k^2 = 0 \\ (d_k^\dagger)^2 = 1 . \end{cases} \tag{16}$$

Therefore, Fermi operator defined by Eq. (2) is not complete.

In general, multiplying d_k by $d_{k'}^\dagger$ of Eq. (2) we get

$$d_k d_{k'}^\dagger = \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q) a_k a_k^\dagger \exp(-\mathrm{i}\pi \sum_{q=k}^{\infty} N_q) =$$

$$a_k a_{k'}^\dagger \exp[\mathrm{i}\pi \sum_{q=k}^{\infty} (N_q - \delta_{q,k} + \delta_{q,k'})] \exp(-\mathrm{i}\pi \sum_{q=k'}^{\infty} N_q) .$$

By Eq.(10) we obtain

$$d_k d_{k'}^\dagger = -a_k a_{k'}^\dagger \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q - \mathrm{i}\pi \sum_{q=k'}^{\infty} N_q + \mathrm{i}\pi \sum_{q=k}^{\infty} \delta_{q,k'}).$$

According to Eq. (1) we have

$$d_k d_{k'}^\dagger = \delta_{kk'} - a_{k'}^\dagger a_k \exp(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q - \mathrm{i}\pi \sum_{q=k'}^{\infty} N_q + \mathrm{i}\pi \sum_{q=k}^{\infty} \delta_{q,k'}), \tag{17}$$

Then multiplying $d_{k'}^\dagger$ by d_k of Eq. (2) we can also get

$$d_{k'}^\dagger d_k = a_{k'}^\dagger \exp\left(-\mathrm{i}\pi \sum_{q=k'}^{\infty} N_q\right) \exp\left(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q\right) a_k =$$

$$a_{k'}^\dagger a_k \exp\left[-\mathrm{i}\pi \sum_{q=k'}^{\infty} (N_q - \delta_{q,k}) + \mathrm{i}\pi \sum_{q=k}^{\infty} (N_q - \delta_{k,q})\right],$$

By applying Eq. (10) we yield

$$d_{k'}^\dagger d_k = -a_{k'}^\dagger a_k \exp\left(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q - \mathrm{i}\pi \sum_{q=k'}^{\infty} N_q + \mathrm{i}\pi \sum_{q=k'}^{\infty} \delta_{q,k}\right). \tag{18}$$

The combination of Eq.(17) and (18) gives

$$d_k d_{k'}^\dagger + d_{k'}^\dagger d_k = \delta_{k,k'} - a_{k'}^\dagger a_k \exp\left(\mathrm{i}\pi\sum_{q=k}^{\infty} N_q - \mathrm{i}\pi\sum_{q=k'}^{\infty} N_q\right)$$
$$\left[\exp\left(\mathrm{i}\pi\sum_{q=k}^{\infty}\delta_{q,k'}\right) + \exp\left(\mathrm{i}\pi\sum_{q=k'}^{\infty}\delta_{q,k}\right)\right], \tag{19}$$

where

$$\sum_{q=k}^{\infty}\delta_{q,k'} = \begin{cases} 1, & k<k' \\ 0, & k>k' \\ 1, & k=k' \end{cases}, \quad \exp\left(\mathrm{i}\pi\sum_{q=k}^{\infty}\delta_{q,k'}\right) = \begin{cases} -1, & k<k' \\ 1, & k>k' \\ -1, & k=k' \end{cases},$$

$$\sum_{q=k'}^{\infty}\delta_{q,k} = \begin{cases} 0, & k<k' \\ 1, & k>k' \\ 1, & k=k' \end{cases}, \quad \exp\left(\mathrm{i}\pi\sum_{q=k'}^{\infty}\delta_{q,k}\right) = \begin{cases} 1, & k<k' \\ -1, & k>k' \\ 1, & k=k' \end{cases}.$$

Thus

$$\exp\left(\mathrm{i}\pi\sum_{q=k}^{\infty}\delta_{q,k'}\right) + \exp\left(\mathrm{i}\pi\sum_{q=k'}^{\infty}\delta_{q,k}\right) = \begin{cases} 0 & k<k' \\ 0 & k>k' \\ -2 & k=k' \end{cases}. \tag{20}$$

or

$$\exp\left(\mathrm{i}\pi\sum_{q=k}^{\infty}\delta_{q,k'}\right) + \exp\left(\mathrm{i}\pi\sum_{q=k'}^{\infty}\delta_{q,k}\right) = -2\delta_{kk'}. \tag{21}$$

Substituting Eq. (21) into (19), we obtain

$$d_k d_{k'}^\dagger + d_{k'}^\dagger d_k = (1 + 2a_k^\dagger a_k)\delta_{kk'}, \tag{22}$$

Then multiplying d_k by $d_{k'}$ of Eq. (2) we get

$$d_k d_{k'} = \exp\left(\mathrm{i}\pi\sum_{q=k}^{\infty} N_q\right) a_k \exp\left(\mathrm{i}\pi\sum_{q=k'}^{\infty} N_q\right) a_{k'} =$$
$$a_k a_{k'} \exp\left[\mathrm{i}\pi\sum_{q=k}^{\infty}(N_q - \delta_{q,k} - \delta_{q,k'})\right] \exp\left[\mathrm{i}\pi\sum_{q=k'}^{\infty}(N_q - \delta_{q,k'})\right],$$

By applying of Eq. (10), we yield

$$d_k d_{k'} = a_k a_{k'} \exp\left(\mathrm{i}\pi\sum_{q=k}^{\infty} N_q + \mathrm{i}\pi\sum_{q=k'}^{\infty} N_q - \mathrm{i}\pi\sum_{q=k}^{\infty}\delta_{q,k'}\right), \tag{23}$$

By interchanging k and k', we get

$$d_{k'} d_k = a_{k'} a_k \exp\left(\mathrm{i}\pi\sum_{q=k'}^{\infty} N_q + \mathrm{i}\pi\sum_{q=k}^{\infty} N_q - \mathrm{i}\pi\sum_{q=k'}^{\infty}\delta_{q,k}\right),$$

From Eq. (1) we obtain

$$d_{k'} d_k = a_k a_{k'} \exp\left(\mathrm{i}\pi\sum_{q=k'}^{\infty} N_q + \mathrm{i}\pi\sum_{q=k}^{\infty} N_q - \mathrm{i}\pi\sum_{q=k'}^{\infty}\delta_{q,k}\right). \tag{24}$$

The combination of Eq. (23) and (24) yields

$$d_k d_{k'} + d_{k'} d_k = a_k a_{k'} \exp\left(\mathrm{i}\pi \sum_{q=k}^{\infty} N_q + \mathrm{i}\pi \sum_{q=k'}^{\infty} N_q\right)$$

$$\left[\exp\left(-\mathrm{i}\pi \sum_{q=k}^{\infty} \delta_{qk'}\right) + \exp\left(-\mathrm{i}\pi \sum_{q=k'}^{\infty} \delta_{qk}\right)\right], \tag{25}$$

Taking the Hermitian conjugate of Eq. (21) which is

$$\exp\left(-\mathrm{i}\pi \sum_{q=k}^{\infty} \delta_{q,k'}\right) + \exp\left(-\mathrm{i}\pi \sum_{q=k'}^{\infty} \delta_{q,k}\right) = -2\delta_{kk'} . \tag{26}$$

Eq. (25) can be simplified as

$$d_k d_{k'} + d_{k'} d_k = -2a_k^2 \exp\left(2\mathrm{i}\pi \sum_{q=k}^{\infty} N_q\right) \delta_{kk'} .$$

From Eq. (12) we get

$$d_k d_{k'} + d_{k'} d_k = -2a_k^2 \delta_{kk'} , \tag{27}$$

Again taking its Hermitian conjugate, we obtain

$$d_k^\dagger d_{k'}^\dagger + d_{k'}^\dagger d_k^\dagger = -2(a_k^\dagger)^2 \delta_{kk'} . \tag{28}$$

The combination of (22), (27) and (28) yields

$$\begin{cases} d_k d_{k'}^\dagger + d_{k'}^\dagger d_k = (1 + 2a_k^\dagger a_k)\delta_{kk'} \\ d_k d_{k'} + d_{k'} d_k = -2a_k^2 \delta_{kk'} \\ d_k^\dagger d_{k'}^\dagger + d_{k'}^\dagger d_k^\dagger = -2(a_k^\dagger)^2 \delta_{kk'} \end{cases} , \tag{29}$$

Let $k = k'$ in Eq. (29), we find

$$\begin{cases} d_k d_k^\dagger + d_k^\dagger d_k = 1 + 2a_k^\dagger a_k \\ d_k^2 = -a_k^2, \qquad (d_k^\dagger)^2 = -(a_k^\dagger)^2 \end{cases} , \tag{30}$$

which is exactly Eq. (15). In this case, Fermi statistic cannot be satisfied. In Eq. (29), by setting $k \neq k'$, we get

$$\begin{cases} d_k d_{k'}^\dagger + d_{k'}^\dagger d_k = 0 \\ d_k d_{k'} + d_{k'} d_k = 0 \\ d_k^\dagger d_{k'}^\dagger + d_{k'}^\dagger d_k^\dagger = 0 . \end{cases} \tag{31}$$

So $d_k, d_{k'}, d_k^\dagger$ and $d_{k'}^\dagger$ satisfy Fermi Statistics.

The Bosonnization technique is one of the theoretical bases of the two dimension quantum field theory. From above discussion, we conclude that Jordon and wigner scheme is tenable only when $k \neq k'$. However, it fails when $k = k'$. Therefore, our discussion provides at least such an indication, i.e., one should be careful and cautious when using Jordon and wigner's bosonnization formula in order to avoid the non-physical effects. At the same time, we point out that a rigorous theory for the bosonnization technique is the beginning of a new research field. It is, of course, not at all an easy job.

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关于费米子玻色化技术的注记*

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摘要 传统根据 Jordon 和 Wigner 理论,构造了费米子玻色化方案. 讨论证明了这种方案只在 $k \neq k'$ 时才成立,而在 $k = k'$ 时则不成立,从而说明了费米子的玻色化技术是不完备的.

关键词 Jordon and Wigner 方案 费米子的玻色化技术 正则对易关系 完备性

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