Spin Operator for the Relativistic Particle *

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Abstract A systematic theory of the appropriate spin operators for the relativistic states is developed. This paper discusses it in particle case, i.e., the quantum mechanics problem. For a massive relativistic particle with arbitrary nonzero spin, the spin operator should be replaced with the relativistic one. In the frame of irreducible representation of Poincaré group, this spin operator, which is named as moving spin and applied to all the canonical states of the particle, is constructed. Further discussion on the concept of moving spin in the quantum field theory will be followed.

Key words massive relativistic particle, canonical states, moving spin

1 Introduction

In high energy physics, the study on the kinematics problem on high energy procedures is the basis of the reliable analysis of all new phenomena. One may refer to the comprehensive works[1] made by S. U. Chung. On this subject, as is well known, the spin description for particles with nonzero spin is of pivotal importance. The canonical state scheme is the natural generalization of that of the nonrelativistic particle to the relativistic case, and it is the basis of partial wave analysis which is frequently utilized in experimental analysis. In this generalization, a relativistic spin operator must be employed. Pryce[2] first obtained an operator in similar form from the commutation relation requirement but with somewhat artificialness as stated by himself. Later work concerned this operator appeared in Refs. [3,4]. However, some ambiguity either on the argumentation or on the physical meaning still exists. For the importance of the topic, we performed a systematic study on this operator. We first clarified the problem of quantum mechanics and determined the appropriate spin operator for a relativistic particle. Hereby we further discussed it in the quantum field theory.

The discussion in this paper, however is merely on the quantum mechanics. Pertaining to the canonical states of the particle, which are the base of the irreducible representation of Poincaré group ($\mathscr{P}$), the spin operator is constructed. We name this operator as moving spin of a particle (PMS). Our argumentation is based on the basic equations of the particle states and the properties of the representation of $\mathscr{P}$, and surely this is a strict one. The extension to the relativistic quantum field theory is left to the subsequent paper[5].

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2 Relativistic Spin Operator for a Massive Particle with Nonzero Spin

For a particle with mass m and spin s, the static states $|\tilde{p},\nu)$ which form an irreducible representation of rotation group can be characterized as following

$$\hat{p}|\tilde{p},\nu)=\tilde{p}|\tilde{p},\nu), \tag{1}$$

$$\hat{s}_z|\tilde{p},\nu)=\nu|\tilde{p},\nu), \tag{2}$$

$$\hat{s}_\pm|\tilde{p},\nu)=\sqrt{s(s+1)-\nu(\nu\pm1)}|\tilde{p},\nu\pm1), \tag{3}$$

where $\tilde{p}=(m,\mathbf{0})$ and $\hat{s}_\pm=\hat{s}_x\pm i\hat{s}_y$. The notation for the operators is specified in the Appendix. Eq. (3) means that the Condon-Shortley convention is adopted for the states with different ν. We may rewrite Eqs. (2) and (3) into

$$\hat{s}^i|\tilde{p},\nu)=s^{(i)}_{\mu,\nu}|\tilde{p},\mu) \qquad i=1,2,3\ (\text{or } x,y,z), \tag{4}$$

where $s^{(i)}_{\mu,\nu}$ is the spin representation matrices for spin-s which reads $s^{(z)}_{\mu\nu}=\mu\delta_{\mu,\nu}$, $s^{(x)}_{\mu,\nu}=(s^{(+)}_{\mu,\nu}+s^{(-)}_{\mu,\nu})/2$, $s^{(y)}_{\mu,\nu}=(s^{(+)}_{\mu,\nu}-s^{(-)}_{\mu,\nu})/2i$, while $s^{(+)}_{\mu,\nu}=\sqrt{s(s+1)-\nu(\nu+1)}\,\delta_{\mu,\nu+1}$ and $s^{(-)}_{\mu,\nu}=\sqrt{s(s+1)-\nu(\nu-1)}\,\delta_{\mu,\nu-1}$. Owing to $\hat{L}|\tilde{p},\nu)=0$, $\hat{s}^i$ in Eq. (4) may be replaced with $\hat{J}^i$.

From the above static state $|\tilde{p},\nu)$, the canonical state, which describes a relativistic particle, is defined as follows

$$|p,\nu)_c=\mathrm{T}[Q(p)]|\tilde{p},\nu), \tag{5}$$

where $p=p^\mu=(E,\boldsymbol{p})$ is a 4-momentum, 'T' means the representation of space-time transformation on the space of one particle states, and $Q(p)$ is a boost which changes the momentum $\tilde{p}$ to p and reads as follow

$$Q(p)=\begin{pmatrix}\frac{E}{m} & \frac{\boldsymbol{p}}{m}\\ \frac{\boldsymbol{p}}{m} & \frac{\boldsymbol{pp}}{m(m+E)}+\overleftrightarrow{I}\end{pmatrix}, \tag{6}$$

which satisfies $Q^*=Q^{\mathrm{T}}=Q$ and $Q^{-1}(p)=Q(\bar{p})$, where $\bar{p}=\bar{p}^\mu=(E,-\boldsymbol{p})$.

States described by Eq. (5) are orthogonal for different p or ν, and as the base for an irreducible representation[6] of $\mathcal{P}$ possess the completeness. Owing to the Lorentz invariance, the orthonormality relation is adopted as ${}_c(p',\nu'|p,\nu)_c=\tilde{\delta}(p,p')\delta_{\nu,\nu'}$, and correspondingly, the completeness relation reads $\sum_\nu\int\tilde{\mathrm{d}}p\,|p,\nu)_{c\,c}(p,\nu|=1$. In the above $\tilde{\delta}(p,p')=2p_0\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$, while $\tilde{\mathrm{d}}p=\mathrm{d}^3p/2p_0$.

By $\mathrm{T}[Q^{-1}(p)]\hat{p}^\mu\mathrm{T}[Q(p)]=Q(p)\hat{p}^\mu$, first it is fairly simple to verify that $|p,\nu)_c$ is an eigenstate of momentum $\hat{p}^\mu$

$$\hat{p}^\mu|p,\nu)_c=p^\mu|p,\nu)_c. \tag{7}$$

As for the other symbol ν, a proper operator is needed. Let $\hat{s}(p)^i=\mathrm{T}[Q(p)]\hat{s}^i\mathrm{T}[Q^{-1}(p)]$, then from Eq. (4)

$$\hat{s}(p)^i|p,\nu)_c=s^{(i)}_{\mu,\nu}|p,\mu)_c. \tag{8}$$

It is found that $\hat{s}(p)^i$ acts on the moving state $|p,\nu)_c$ just in the same manner as $\hat{s}^i$ acts on the static state $|\tilde{p},\nu)$. What is remarkable is that in the following we can find that there exists an operator which acts on the all canonical states as defined in Eq. (5) exactly in the same manner as $\hat{s}^i$ acts on the static state $|\tilde{p},\nu)$. First we may calculate the explicit expression of $\hat{s}(p)^i$. Via the Lorentz tensor $\hat{s}^{\mu\nu}$, we may get after some computation

$$m\hat{\boldsymbol{s}}(p)=p_0\hat{\boldsymbol{J}}-(m+p_0)^{-1}\boldsymbol{pp}\cdot\hat{\boldsymbol{J}}-\boldsymbol{p}\times\hat{\boldsymbol{K}}. \tag{9}$$

Owing to Eq. (16) in the Appendix, we found that all the p in the above equation can be replaced with the operator $\hat{p}$ to get an alternate operator which keeps Eq. (8) unchanged. Or rather, we can define

$$m\hat{s}(\hat{p}) = \hat{p}_0\hat{\boldsymbol{J}} - (m + \hat{p}_0)^{-1}\hat{\boldsymbol{p}}(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{J}}) - \hat{\boldsymbol{p}}\times\hat{\boldsymbol{K}} , \tag{10}$$

which satisfies

$$\hat{s}(\hat{p})^{(i)} | p, \nu)_c = s^{(i)}_{\mu,\nu} | p, \mu)_c , \tag{11}$$

and the commutation rules

$$[\hat{s}(\hat{p})^i, \hat{s}(\hat{p})^j] = [\hat{J}^i, \hat{s}(\hat{p})^j] = i\epsilon^{ijk}\hat{s}(\hat{p})^k . \tag{12}$$

Eq. (12) exhibits the required property as a spin operator for $\hat{s}(\hat{p})^i$. Just as showed by Eq. (11), it is the spin operator for a moving particle with spin. We can fairly call it the moving spin of the particle.

Under an arbitrary Lorentz transformation L, the canonical state transforms as follows, $T[L] | p,\nu)_c = T[LQ(p)] | \tilde{p},\nu) = \sum_{\mu} D^s_{\mu\nu}[\bar{R}] | Lp,\mu)_c$, where $\bar{R} = \bar{R}(L,p) = Q^{-1}(Lp)LQ(p)$ is the Wigner rotation. Considering the special case of an infinitesimal rotary transformation, we may obtain $\hat{\boldsymbol{J}} | p,\nu)_c = [-i\hat{\boldsymbol{p}}\times\partial/\partial\hat{\boldsymbol{p}} + \hat{\boldsymbol{s}}(\hat{p})] | p,\nu)_c$. Owing to the completeness relation of the canonical states, it is followed that

$$\hat{\boldsymbol{J}} = -i\hat{\boldsymbol{p}}\times\frac{\partial}{\partial\hat{\boldsymbol{p}}} + \hat{\boldsymbol{s}}(\hat{p}). \tag{13}$$

The above equation means that we have now redivide the angular momentum operator $\hat{\boldsymbol{J}}$ into $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}}(\hat{p}) + \hat{\boldsymbol{s}}(\hat{p})$ with $\hat{\boldsymbol{L}}(\hat{p}) = -i\hat{\boldsymbol{p}}\times\partial/\partial\hat{\boldsymbol{p}}$. We may also give $\hat{\boldsymbol{L}}(\hat{p})$ a name, moving orbital angular of the particle (PMOA).

3 Conclusion Remarks

We have constructed the appropriate spin operator PMS for the relativistic particle in the frame of irreducible representation of $\mathscr{P}$. PMS is applicable to all the canonical states of the particle. In the succedent paper[5], the discussion is extended to the quantum field theory which results in two new operators, field quanta spin and moving spin for the field system. All these results enable us a better understanding of the spin, and we regard them as the starting point for our endeavor to the spin crisis, the contemporary puzzle about the spin of a proton.

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Appendix

Momentum and Angular Momentum of a Particle as the Generators of $\mathscr{P}$

Discussions in this paper are intimately related to $\mathscr{P}$. A brief review of Poincaré algebra is provide here. Some notation of the paper is also comprised in this appendix.

Suppose a particle with mass m and spin s is described with the wavefunction $\psi(x)$, where $x = x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)$. We take the metric tensor for Minkowski space as $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Now perform a space-time transformation $\tau(a, L): x \mapsto x' = Lx + a$, then the wave function

$\psi(x)$ transforms as $\psi(x) \mapsto \psi'(x) = T^{-1}[a, L]\psi(x) = T[L^{-1}]T[P(-a)]\psi(x)$, where a is the parameter for the space-time translation. Suppose that the parameter for Lorentz transformation L is $\omega^{\mu\nu}$, we may write

$$\psi'(x) = e^{\frac{i}{2}\omega^{\mu\nu}\hat{J}_{\mu\nu}} e^{-ia^{\mu}\hat{p}_{\mu}}\psi(x), \tag{14}$$

where $\hat{p}_{\mu}$ and $\hat{J}_{\mu\nu}$ are momentum and angular momentum operators respectively, and the generators for space-time transformation. $\hat{J}_{\mu\nu}$ consists of two parts

$$\hat{J}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{s}_{\mu\nu} = x_{\mu}\hat{p}_{\nu} - x_{\nu}\hat{p}_{\mu} + \hat{s}_{\mu\nu}, \tag{15}$$

while $\hat{s}_{\mu\nu}$ is the spin tensor of this particle. As an example, for the Dirac particle, $\hat{s}_{\mu\nu}^{(+)} = \frac{1}{2}\frac{1}{2i}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu})$. The commutations in the following are familiar,

$$\begin{aligned}
&[\hat{p}_{\mu}, \hat{p}_{\nu}] = 0, [\hat{p}_{\mu}, \hat{L}_{\sigma\rho}] = i(\eta_{\mu\sigma}\hat{p}_{\rho} - \eta_{\mu\rho}\hat{p}_{\sigma}), [\hat{L}_{\mu\nu}, \hat{L}_{\sigma\rho}] = -i(\eta_{\mu\sigma}\hat{L}_{\nu\rho} - \eta_{\nu\sigma}\hat{L}_{\mu\rho} + \eta_{\nu\rho}\hat{L}_{\mu\sigma} - \eta_{\mu\rho}\hat{L}_{\nu\sigma}),\\
&[\hat{p}_{\mu}, \hat{s}_{\sigma\rho}] = [\hat{L}_{\mu\nu}, \hat{s}_{\sigma\rho}] = 0, [\hat{s}_{\mu\nu}, \hat{s}_{\sigma\rho}] = -i(\eta_{\mu\sigma}\hat{s}_{\nu\rho} - \eta_{\nu\sigma}\hat{s}_{\mu\rho} + \eta_{\nu\rho}\hat{s}_{\mu\sigma} - \eta_{\mu\rho}\hat{s}_{\nu\sigma}),\\
&[\hat{p}_{\mu}, \hat{J}_{\sigma\rho}] = i(\eta_{\mu\sigma}\hat{p}_{\rho} - \eta_{\mu\rho}\hat{p}_{\sigma}), [\hat{J}_{\mu\nu}, \hat{T}_{\sigma\rho}] = -i(\eta_{\mu\sigma}\hat{T}_{\nu\rho} - \eta_{\nu\sigma}\hat{T}_{\mu\rho} + \eta_{\nu\rho}\hat{T}_{1\mu\sigma} - \eta_{\mu\rho}\hat{T}_{\nu\sigma})(T = L, s, J).
\end{aligned} \tag{16}$$

The last two lines of the above equation indicate that $\hat{p}_{\mu}$ is a Lorentz vector, and that $\hat{L}_{\mu\nu}$, $\hat{s}_{\mu\nu}$ and $\hat{J}_{\mu\nu}$ are all

The Pauli – Lubanski is defined as $\hat{W}_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}\hat{p}^{\nu}\hat{J}^{\sigma\rho}$, with the commutations

$$[\hat{W}_{\mu}, \hat{W}_{\nu}] = -i\epsilon_{\mu\nu\sigma\rho}\hat{p}^{\sigma}\hat{W}^{\rho}, \quad [\hat{W}_{\mu}, \hat{p}_{\nu}] = 0, \quad [\hat{W}_{\mu}, \hat{J}_{\sigma\rho}] = i(\eta_{\mu\sigma}\hat{W}_{\rho} - \eta_{\mu\rho}\hat{W}_{\sigma}). \tag{17}$$

Next, the two Casimir operators are defined as $\hat{C}_1 = \hat{p}_{\nu}\hat{p}^{\nu}$ and $\hat{C}_2 = \hat{W}_{\nu}\hat{W}^{\nu}$, which satisfy $[\hat{C}_i, \hat{p}_{\nu}] = [\hat{C}_i, \hat{J}_{\mu\nu}] = 0, (i = 1, 2)$.

From the Lorentz tensors $\hat{J}, \hat{L}, \hat{s}$, we may define the following space vector operators

$$\begin{aligned}
&\hat{L}_i = \epsilon_{ijk}\hat{L}^{jk} = \epsilon_{ijk}\hat{L}^{jk}, \quad \hat{s}_i = \epsilon_{ijk}\hat{s}^{jk} - \epsilon_{ijk}\hat{s}_{jk}, \quad \hat{J}_i = \hat{L}_i + \hat{s}_i;\\
&\hat{Z}_i = \hat{L}_{0i} = -\hat{L}^{0i}, \quad \hat{T}_i = \hat{s}_{0i} = -\hat{s}^{0i}, \quad \hat{K}_i = \hat{Z}_i + \hat{T}_i.
\end{aligned} \tag{18}$$

相对论粒子的自旋算符*

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摘要 发展了关于相对论态自旋算符的系统理论. 考虑了具有非零静质量的粒子情况. 对带自旋的相对论粒子,通常的自旋算符需换为相对论的自旋算符. 在 Poincaré 群不可约表示的框架里,构造了适用于粒子任意正则态的自旋算符,称为运动自旋. 本文的讨论限于量子力学. 随后将在量子场论中对此作进一步深入研究.

关键词 相对论粒子 粒子正则态 运动自旋

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