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螺线管透镜的三级李映射

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摘要 用李代数方法分析了带电粒子在螺线管透镜中的运动,得到六维相空间中粒子轨迹的三级近似解,并包含了运动的相对论效应. 当需要时,还可以扩展到更高级像差.

关键词 李映射 螺线管透镜 三级近似

螺线管透镜常常用作感应加速器、电子直线加速器、质子直线加速器、低能束流传输系统以及各种阴极射线管的聚焦. 由于螺线管透镜端部存在边缘场分布,要精确计算粒子运动的轨迹,需要利用常微分方程的数值解法. 若系统中包含大量元件,数值解法就显得非常不方便. 本文用李代数方法导出了六维相空间中相对论粒子在螺线管透镜中的三级轨迹. 为了把边缘场效应包含在内,把螺线管透镜的整个场区分成若干个小区间,在每个小区间上进行李代数分析.

1 李算符与李变换

当带电粒子在束流传输系统中运动时,它在六维相空间中的初始坐标与最终坐标可用一辛映射 $\boldsymbol{M}$ 相联系. 此映射又可以用一系列李变换的乘积表示. 将映射 $\boldsymbol{M}$ 作用于粒子的初始坐标,并取一定的近似,就可得到粒子轨迹的一级,二级和三级……近似表示. 这一过程就是所谓的李代数分析方法. 设 $\zeta=(q,p)$ 为六维相空间中的一组坐标变量. 当带电粒子在电磁场中从初始位置0运动到最终位置 f 时,粒子的初始坐标 ζ 与最终坐标 ζ^f 之间的关系可用下式表示[1]:

$$\zeta^f = \boldsymbol{M}\zeta, \tag{1}$$

其中 $\boldsymbol{M}$ 是一个辛映射,表示为

$$\boldsymbol{M} = \exp\left[-:\int_{q_1^0}^{q_1^f} H(\zeta,q_1)\mathrm{d}q_1 :\right], \tag{2}$$

这里 q_1 为 $(q_1,q_2,\cdots,q_n)$ 中的某一坐标 q_1,并取作独立变量. 令

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$$f = \int_{q_1^0}^{q_1} H(\zeta, q_1) \mathrm{d}q_1, \tag{3}$$

则

$$M = \exp(-:f:), \tag{4}$$

其中 $:f:$ 为李算符,当它作用于相空间中的任一函数 g 时,则执行以下泊松括号运算:

$$:f: g = [f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{5}$$

式(4)可写成级数的形式

$$M = \sum_{n=0}^{\infty} \frac{1}{n!} :f:^n \tag{6}$$

这里 $:f:$ 的幂定义为重泊松括号运算.

根据因式分解定理[2],式(4)还可以写成一系列因式相乘的形式

$$M = \cdots \exp(:f_4:) \exp(:f_3:) \exp(:f_2:), \tag{7}$$

其中

$$\begin{aligned} f_2 &= -lH_2, \\ f_3 &= -\int_0^l h_3^{\mathrm{int}}(\zeta, z_1) \mathrm{d}z_1, \\ f_4 &= -\int_0^l h_4^{\mathrm{int}}(\zeta, z_1) \mathrm{d}z_1 + \frac{1}{2}\int_0^l \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2 [-h_3^{\mathrm{int}}(\zeta, z_2), \; h_3^{\mathrm{int}}(\zeta, z_1)], \\ &\cdots\cdots \end{aligned} \tag{8}$$

l 为螺线管透镜各区间的纵向长度,而且

$$h_n^{\mathrm{int}}(\zeta, z) = H_n(M_2(z \leftarrow 0)\zeta, z) \tag{9}$$

利用式(6),可将式(8)改写为

$$M = \cdots M_4 M_3 M_2 = M_2 + :f_3: M_2 + \left(:f_4: + \frac{1}{2} :f_3:^2 \right) M_2 + \cdots, \tag{10}$$

其中第一项、第二项和第三项分别表示映射的一次项、二次项和三次项…….

2　Hamilton 函数及其展开

在圆柱坐标系(r, θ, z)中,螺线管透镜中的磁矢势 A 可表示为

$$A = e_\theta A, \quad A_x = 0, \quad A_y = 0, \tag{11}$$

其中 e_θ 为角方向上的单位矢量. 设束轴沿着 z 方向,粒子的静止质量为 m_0,电荷为 q. 根据电子光学的知识,轴对称磁场的矢势 A 可表示为

$$A = \sum_{n=0}^{\infty} (-1)^n \frac{1}{k!(k+1)!} \left(\frac{r}{2} \right)^{2n+1} B^{(2n)}(z), \tag{12}$$

其中 B 为沿轴线的磁场分布,上标$(2n)$表示求导次数. 将沿角方向的矢势 A 分解到直角坐标系 x 和 y 方向上,得

$$A_x = -y \sum_{n=0}^{\infty} (-1)^n \frac{(x^2 + y^2)^n}{2^{n+1} k!(k+1)!} B^{(2n)}(z), \tag{13}$$

$$A_y = -x \sum_{n=0}^{\infty} (-1)^n \frac{(x^2 + y^2)^n}{2^{n+1} k!(k+1)!} B^{(2n)}(z), \tag{14}$$

如果取 z 为独立变量,$\zeta=(\tau,x,y,\tau'_\tau,P_x,P_y)$为正则变量,则 Hamilton 函数 H 表示为

$$H=-\left[(\tau'+P_\tau^0)^2-(x'-qA_x/p_0)^2-(y'-qA_y/p_0)^2-m_0^2c^2/p_0^2\right]^{\frac{1}{2}}, \tag{15}$$

其中 $\tau=ct-z/\beta_0=z(1/\beta-1/\beta_0)$,它反映了任意粒子相对于参考粒子的时间偏离;$\beta=c/v$($v$ 为任意粒子的速度);$\beta_0=c/v_0$(v_0 为参考粒子的速度);c 为光速;x 为粒子轨迹的 x 偏离;y 为粒子轨迹的 y 偏离.

$$\begin{aligned}&\tau'=p_t-p_t^0;\\&x'=p_x/p_0(p_x\text{ 为粒子 }x\text{ 方向的动量}),P_0=m_0\gamma v_0(\text{参考粒子的动量});\\&y'=p_y/p_0(p_y\text{ 为粒子 }y\text{ 方向的动量});\\&p_t=-H_t,p_t^0\text{ 为参考轨道上 }p_t\text{ 的值}.\end{aligned} \tag{16}$$

将 Hamilton 函数在平衡轨道附近作展开,得

$$H=\sum_{i=0}^{\infty}H_i,$$

其中

$$\begin{aligned}
H_0&=-\frac{1}{\beta_0^2\gamma_0^2},\\
H_1&=0,\\
H_2&=\frac{k^2}{2}x^2+kxy'+\frac{1}{2}x'^2+kx'y+\frac{k^2}{2}y^2+\frac{1}{2}y'^2+\frac{1}{2\beta_0^2\gamma_0^2}\tau'^2,\\
H_3&=\frac{k^2}{2\beta_0}x^2\tau'-\frac{k}{\beta_0}xy'\tau'+\frac{1}{2\beta_0}x'^2\tau'+\frac{k}{\beta_0}x'y\tau'+\frac{k^2}{2\beta_0}y^2\tau'+\frac{1}{2\beta_0}y'^2\tau'+\frac{1}{2\beta_0^3\gamma_0^2}\tau'^3,\\
H_4&=\frac{1}{8}(k^4-2kk'')x^4-\frac{1}{2}\left(k^3-\frac{1}{2}k''\right)x^3y'+\frac{1}{4}k^2x^2x'^2+\frac{1}{2}\left(k^3-\frac{1}{2}k''\right)x^2x'y+\\
&\frac{1}{4}(k^4-2kk'')x^2y^2+\frac{3}{4}k^2x^2y'^2+\frac{1}{4}\left(\frac{3}{\beta_0^2}-1\right)k^2x^2\tau'^2-\frac{1}{2}kxx'^2y'-k^2xx'yy'-\\
&\frac{1}{2}\left(k^3-\frac{1}{2}k''\right)xy^2y'-\frac{1}{2}kxy'^3-\frac{1}{2}\left(\frac{3}{\beta_0^2}-1\right)kxy'\tau'^2+\frac{1}{8}x'^4+\frac{1}{2}kx'^3y+\\
&\frac{3}{4}k^2x'^2y^2+\frac{1}{4}x'^2y'^2+\frac{1}{4}\left(\frac{3}{\beta_0^2}-1\right)x'^2\tau'^2+\frac{1}{2}\left(k^3-\frac{1}{2}k''\right)x'y^3+\frac{1}{2}kx'yy'^2+\\
&\frac{1}{2}\left(\frac{3}{\beta_0^2}-1\right)kx'y\tau'^2+\frac{1}{8}(k^4-2kk'')y^4+\frac{1}{4}k^2y^2y'^2+\frac{1}{4}\left(\frac{3}{\beta_0^2}-1\right)k^2y^2\tau'^2+\\
&\frac{1}{8}y'^4+\frac{1}{4}\left(\frac{3}{\beta_0^2}-1\right)y'^2\tau'^2+\frac{1}{8}\left(\frac{5}{\beta_0^4\gamma_0^2}-\frac{1}{\beta_0^2\gamma_0^2}\right)\tau'^4
\end{aligned} \tag{18}$$

……

3 一级近似

利用式(8)中的第一个表达式,$f_2=-lH_2$,得

$$\boldsymbol{M}_2=\exp(-:zH_2:)\big|_{z=l} \tag{19}$$

用下标“1”表示映射的一次项,不带脚标的量 ζ 表示初值向量,将 $\boldsymbol{M}_2$ 作用于 ζ,可得到轨迹的一级近似解,用矩阵表示为

$$\begin{bmatrix} x_1 \\ x'_1 \\ y_1 \\ y'_1 \\ \tau_1 \\ \tau'_1 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \frac{1}{k}\sin\theta\cos\theta & \sin\theta\cos\theta & \frac{1}{k}\sin^2\theta & 0 & 0 \\ -k\sin\theta\cos\theta & \cos^2\theta & -k\sin^2\theta & \sin\theta\cos\theta & 0 & 0 \\ -\sin\theta\cos\theta & -\frac{1}{k}\sin^2\theta & \cos^2\theta & \frac{1}{k}\sin\theta\cos\theta & 0 & 0 \\ k\sin^2\theta & -\sin\theta\cos\theta & -k\sin\theta\cos\theta & \cos^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{l}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \\ y \\ y' \\ \tau \\ \tau' \end{bmatrix}, \quad (20)$$

其中 $\theta = kl$, $k = \dfrac{9B}{2p_0}$

4 二级近似

利用式(8)中的第二个表达式和式(9),可求出函数 f_3,再利用式(10)可得轨迹的二次项. 结果为

$$\begin{aligned} x_2 = {} & x\tau'\frac{1}{\beta_0}\left(-\frac{1}{2}\sin^2\theta\cos^4\theta + \frac{1}{4}\sin^2\theta\cos^2\theta - \frac{7}{4}\theta\sin\theta\cos\theta\right) + \\ & x'\tau'\frac{1}{\beta_0 k}\left[\frac{1}{4}(\sin\theta\sin^5\theta - \sin^3\theta\cos^3\theta) + \frac{1}{8}(\sin^3\theta\cos\theta - \sin\theta\sin\theta^3\theta) + \frac{7}{8}\theta(\cos^2\theta - \sin^2\theta)\right] + \\ & y\tau'\frac{1}{\beta_0}\left[\frac{1}{4}(\sin\theta\sin^5\theta - \sin^3\theta\cos^3\theta) + \frac{1}{8}(\sin^3\theta\cos\theta - \sin\theta\sin\theta^3\theta) + \frac{7}{8}\theta(\cos^2\theta - \sin^2\theta)\right] + \\ & y'\tau'\frac{1}{\beta_0 k}\left(\frac{1}{2}\sin^2\theta\cos^4\theta - \frac{1}{4}\sin^2\theta\cos^2\theta + \frac{7}{4}\theta\sin\theta\cos\theta\right), \end{aligned}$$

$$\begin{aligned} x'_2 = {} & x\tau'\frac{k}{\beta_0}\left[\frac{1}{4}(\sin^3\theta\cos^3\theta - \sin\theta\sin\theta^5\theta) + \frac{1}{8}(\sin\theta\sin^3\theta - \sin^3\theta\cos\theta) + \right. \\ & \left.\frac{7}{8}\theta(\sin^2\theta - \cos^2\theta)\right] + x'\tau'\frac{1}{\beta_0}\left(-\frac{1}{2}\sin^2\theta\cos^4\theta + \frac{1}{4}\sin^2\theta\cos^2\theta - \frac{7}{4}\theta\sin\theta\cos\theta\right) + \\ & y\tau'\frac{k}{\beta_0}\left(\frac{1}{2}\sin^2\theta\cos^4\theta - \frac{1}{4}\sin^2\theta\cos^2\theta + \frac{7}{4}\theta\sin\theta\cos\theta\right) + \\ & y'\tau'\frac{1}{\beta_0}\left[\frac{1}{4}(\sin\theta\sin^5\theta - \sin^3\theta\cos^3\theta) + \frac{1}{8}(\sin^3\theta\cos\theta - \sin\theta\sin\theta^3\theta) + \frac{7}{8}\theta(\cos^2\theta - \sin^2\theta)\right] \end{aligned}$$

$$\begin{aligned} y_2 = {} & x\tau'\frac{1}{\beta_0}\left[\frac{1}{4}(\sin^3\theta\cos^3\theta - \sin\theta\cos^5\theta) + \frac{1}{8}(\sin\theta\cos^3\theta - \sin^3\theta\cos\theta) + \right. \\ & \left.\frac{7}{8}\theta(\sin^2\theta - \cos^2\theta)\right] + x'\tau'\frac{1}{\beta_0 k}\left(-\frac{1}{2}\sin^2\theta\cos^4\theta + \frac{1}{4}\sin^2\theta\cos^2\theta - \frac{7}{4}\theta\sin\theta\cos\theta\right) + \\ & y\tau'\frac{1}{\beta_0}\left(-\frac{1}{2}\sin^2\theta\cos^4\theta + \frac{1}{4}\sin^2\theta\cos^2\theta - \frac{7}{4}\theta\sin\theta\cos\theta\right) + \\ & y'\tau'\frac{1}{\beta_0}\left[\frac{1}{4}(\sin\theta\cos^5\theta - \sin^3\theta\cos^3\theta) + \frac{1}{8}(\sin^3\theta\cos\theta - \sin\theta\cos^3\theta) + \frac{7}{8}\theta(\cos^2\theta - \sin^2\theta)\right], \end{aligned}$$

$$y'_2 = x\tau'\frac{k}{\beta_0}\left(\frac{1}{2}\sin^2\theta\cos^4\theta - \frac{1}{4}\sin^2\theta\cos^2\theta + \frac{7}{4}\theta\sin\theta\cos\theta\right) +$$
$$x'\tau'\frac{1}{\beta_0}\left[\frac{1}{4}(\sin^3\theta\cos^3\theta - \sin\theta\cos^5\theta) + \frac{1}{8}(\sin\theta\cos^3\theta - \sin^3\theta\cos\theta) + \frac{7}{8}\theta(\sin^2\theta - \cos^2\theta)\right] +$$
$$y\tau'\frac{k}{\beta_0}\left[\frac{1}{4}(\sin^3\theta\cos^3\theta - \sin\theta\cos^5\theta) + \frac{1}{8}(\sin\theta\cos^3\theta - \sin^3\theta\cos\theta) + \frac{7}{8}\theta(\sin^2\theta - \cos^2\theta)\right] +$$
$$y'\tau'\frac{1}{\beta_0}\left(\frac{1}{2}\sin^2\theta\cos^4\theta - \frac{1}{4}\sin^2\theta\cos^2\theta + \frac{7}{4}\theta\sin\theta\cos\theta\right),$$
$$\tau_2 = \frac{1}{\beta_0}\left(x^2\frac{k}{2} - xy' + x'^2\frac{1}{2k} + x'y + y^2\frac{k}{2} + y'^2\frac{1}{2k}\right)\cdot$$
$$\left(\frac{1}{4}\sin\theta\cos^3\theta - \frac{1}{8}\sin\theta\cos\theta + \frac{7}{8}\theta\right) + \tau'^2\frac{3}{2\beta_0^3\gamma_0^2},$$
$$\tau'_2 = 0$$

5 三级近似

根据式(8)可以算得函数 f_4,它包含 h_3^{int} 和 h_4^{int} 两部分的贡献. h_3^{int} 在第4节中算出, h_4^{int} 由下式算出:

$$h_4^{\text{int}}(\zeta, z) = H_4(\boldsymbol{M}_2(z \leftarrow 0)\zeta, z) = \boldsymbol{M}_2 H_4,$$

当 f_4 算得以后,就可以利用式(10)计算映射的三次项,表示为:

$$\zeta_4 = \left(:f_4: + \frac{1}{2}:f_3:^2\right)\boldsymbol{M}_2\zeta = \left(:f_4: + \frac{1}{2}:f_3:^2\right)\zeta_1.$$

限于篇幅,其最后结果不在这里一一列出.

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Third Order Lie Map for the Solenoidal Lenses

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Abstract Solenoidal lenses are usually used to focus charged particle beams moving in inductive accelerators, electron linear accelerators, proton linear accelerators, low energy beam transport systems as well as some cathode ray tubes. To understand the nonlinear optical properties of solenoidal lenses, Lie algebraic methods are used in the analysis of particle trajectories in this kind of lenses, and the results of third order approximation are obtained. Because of the existence of fringing fields, one should divide the fringing field into several small segments, and apply the Lie map to each of the segments.

Key words Lie map, solenoid lenses, third order approximation

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