# Photon higher-order squeezing effects of the *q* analogue of a single-mode field interacting with a Ξ-type three-level atom

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**Abstract** The nonlinear theory of interaction between the $q$ analogue of a single-mode field and a Ξ-type three-level atom has been established. And the formal solution of the Schrödinger equation in the representation and its average number are obtained. Then, the photon squeezing effects are studied through numerical calculation. The results show that the $q$ deformation nonlinear action has a lot of influence on the quantum coherence and quantum properties. When $q$ approaches 1, the theory reduces to the common linear theory.

**Key words** quantum optics, quantum properties, $q$ deformation, higher-order squeezing effects

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## 1 Introduction

The Jaynes-Cummings model (JCM)[1] describing the interaction of a single two-level atom with a single quantized cavity mode of the radiation field plays a central role in quantum optics[2—5]. In spite of its mathematical simplicity, this model is physically realistic. The JCM describes many purely quantum mechanical phenomena such as Rabi oscillations, collapses and revivals of the atomic inversion or sub-Poissonian photon statistics and many nonclassical properties. Squeezing effects are purely a nonclassical phenomenon, which has been predicted and observed[6—12]. This phenomenon has attracted considerable attention due to its many applications such as gravity wave detection, high-resolution spectroscopy, quantum nondemolition experiments and optical communication networks. Field squeezing effects in the Jaynes-Cummings model were first studied by Mestre and Zubairy[13], and later generalized to the three-level atom case[14, 15].

The $q$ deformation of Lie algebra (the so-called quantum group) was discovered by Jimbo[16, 17]. Quantum groups have their origin in the quantum inverse problem and in physics they are related to the solutions of integrable systems, to particular problems of statistical physics and to a conformal field theory[18]. Attention has been paid to the realization of the quantum group in terms of the $q$ analogue of the quantum harmonic oscillator ($q$-QHO)[19—23]. The quantum harmonic oscillator (QHO) is one of the most fundamental objects in quantum physics. In quantum optics the QHO describes, in particular, the single mode of the quantized cavity field.

The Jaynes-Cummings model with a $q$ analogue of field has been studied[24, 25]. Buzek V utilized the JCM to analyse the influence of the $q$ deformation of the bosonic field (corresponding to the single-mode quantized cavity field) on the atomic inversion of the two-level atom[26]. The deformed photonic field interacting with matter presented many interesting properties[27, 28]. It claimed that the three-level atom would be more suitable than the two-level atom due to the possibility for obtaining more information[29, 30]. Here we aim to study the influence of the $q$ deformation on the photon squeezing effects of the $q$ analogue of a single mode of the bosonic field interacting with a Ξ-type three-level atom. The nonlinear theory of interaction between the $q$ analogue of a single-mode field and a Ξ-type three-level atom was

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first established, and the Hamiltonian of the system is obtained.

## 2 The $q$ analogue of the QHO and the $q$ analogue of the coherent state

Macfarlae[31] and Biedenharm[32] have introduced a $q$ deformation of the Heisenberg algebra generated by a $q$ creation operator $a_q^+$, $q$ annihilation operator $a_q$, identity operator 1 and a $q$ number operator $N_q$ in the $q$ analogue of the Hilbert space with the $q$ boson vacuum $|0\rangle_q$ defined as $a_q|0\rangle_q=0$. The algebraic relations for the operators $a_q$, $a_q^+$ and $N_q$ are

$$\begin{aligned} &a_q a_q^+ - q^{-1} a_q^+ a_q = q^N , \\ &[N_q, a_q^+] = a_q^+ , \\ &[N_q, a_q] = -a_q . \end{aligned} \tag{1}$$

The operator $a_q$, $a_q^+$ and $N_q$ act in Hilbert space with the basis $|n\rangle_q$ $(n=0,\ 1,\ 2,\ \cdots)$,

$$\begin{aligned} a_q|0\rangle_q &= 0 , \\ |n\rangle_q &= \frac{(a_q^+)^n}{\sqrt{[n]_q!}}|0\rangle_q , \end{aligned} \tag{2}$$

where the function $[n]_q$ is defined as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} , \tag{3}$$

where the $q$ factorial $[n]_q!$ is given by the relation

$$[n]_q! = [n]_q[n-1]_q\ \cdots\ [1]_q . \tag{4}$$

From the above it follows that

$$\begin{aligned} a_q|n\rangle_q &= \sqrt{[n]_q}|n-1\rangle_q , \\ a_q^+|n\rangle_q &= \sqrt{[n+1]_q}|n+1\rangle_q . \end{aligned} \tag{5}$$

The operator $N_q$ is such that

$$N_q|n\rangle_q = n|n\rangle_q , \tag{6}$$

and

$$\begin{aligned} a_q a_q^+ &= [N_q+1]_q, \\ a_q^+ a_q &= [N_q]_q. \end{aligned} \tag{7}$$

Using the standard procedure, one can decompose $|z\rangle_q$ in the basis $|n\rangle_q$

$$|z\rangle_q = \left[\exp_q(|\alpha|^2)\right]^{-1/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{[n]!}}|n\rangle_q = \sum_{n=0}^{\infty}P_q(n)|n\rangle_q , \tag{8}$$

where $\alpha$ is the complex number ($\alpha=|\alpha|\mathrm{e}^{\mathrm{i}\phi}$), the $q$ analogue of the exponential function $\exp_q(x)$ is defined as

$$\exp_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]!} . \tag{9}$$

We can easily find the probability distribution of the $q$ bosons:

$$P_q^2(n) = \left[\exp_q(|\alpha|^2)\right]^{-1}\frac{|\alpha|^{2n}}{[n]!} . \tag{10}$$

## 3 Nonlinear theory of interaction

We shall consider the Hamitonian for the Ξ-type three-level atom (Fig. 1) coupled to the $q$ analogue of the single-mode quantized cavity field in the following form (in what follows we adopt $\hbar=1$) in the rotating-wave approximations,

$$\begin{aligned} H &= H_0 + H_{\mathrm{I}} , \\ H_0 &= \sum_{j=1}^{3}\omega_j|j\rangle\langle j| + \omega_0 a_q^+ a_q , \\ H_{\mathrm{I}} &= g_1[R_{21}a_q + R_{12}a_q^+] + g_2[R_{32}a_q + R_{23}a_q^+]. \end{aligned} \tag{11}$$

$|3\rangle$, $|2\rangle$, $|1\rangle$ denote the top, middle, and bottom levels respectively, and atomic transition between $|1\rangle \longleftrightarrow |2\rangle$ and $|2\rangle \longleftrightarrow |3\rangle$ couple with the single-mode quantized cavity field whose frequency is $\omega_0$ (dipole is allowed), but the transition $|1\rangle \longleftrightarrow |3\rangle$ is dipole forbidden. $a_q^+(a_q)$ is the one photon creation (annihilation) operator, parameter $g_i(i=1,2)$ is a coupling constant of light with atom. $R_{ij}=|i\rangle\langle j|(i,j=1,2,3)$ is the atomic operators, $\omega_j(j=1,2,3)$ is the eigen frequency of $|j\rangle$. $a_q^+$, $a_q$ are the creation and annihilation operators of the cavity mode, respectively.

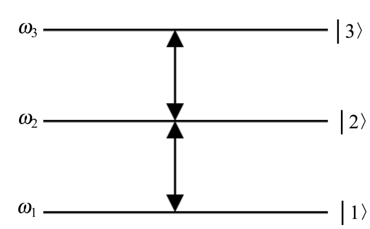


Fig. 1. Energy structure of the Ξ-type three level atom in the system.

The Schrödinger equation in the interaction representation is

$$\mathrm{i}\frac{\partial|\psi(t)\rangle}{\partial t} = H_{\mathrm{I}}|\psi(t)\rangle . \tag{12}$$

The atom is prepared initially in the excited level $|3\rangle$ and the field in the $q$ analogue of Glauber's coherent state $|z\rangle_q$, so the initial state of the system is

$$|\psi(0)\rangle_q = |\psi(0)\rangle_a \otimes |\psi(0)\rangle_f = \sum_{n=0}^{\infty}P_q(n)|3,n\rangle_q . \tag{13}$$

At a time $t>0$, the state vector becomes

$$|\psi(t)\rangle_q = \sum_{n=0}^{\infty} P_q(n)[A_n(t)|3,n\rangle_q + B_{n+1}(t)|2,n+1\rangle_q + C_{n+2}(t)|1,n+2\rangle_q]. \quad (14)$$

Using the Schrödinger equation in the interaction picture and letting $g=g_1=g_2$, a set of coupled equations can be obtained

$$\begin{cases} \mathrm{i}\dot{A}_n(t) = g\sqrt{[n+1]}B_{n+1}(t), \\ \mathrm{i}\dot{B}_{n+1}(t) = g\sqrt{[n+2]}C_{n+2}(t) + g\sqrt{[n+1]}A_n(t), \\ \mathrm{i}\dot{C}_{n+2}(t) = g\sqrt{[n+2]}B_{n+1}(t). \end{cases} \quad (15)$$

The equations can be solved

$$\begin{cases} A_n(t) = \dfrac{[n+2]}{\mu} + \dfrac{[n+1]}{\mu}\cos(\sqrt{\mu}gt), \\ B_{n+1}(t) = -\mathrm{i}\dfrac{\sqrt{[n+1]}}{\sqrt{\mu}}\sin(\sqrt{\mu}gt), \\ C_{n+2}(t) = \dfrac{\sqrt{[n+1][n+2]}}{\mu}[\cos(\sqrt{\mu}gt)-1], \end{cases} \quad (16)$$

where

$$\mu = [n+1]+[n+2].$$

We can find the exact solution of the Schrödinger equation for the state vector

$$|\psi(t)\rangle_q = \sum_{n=0}^{\infty} P_q(n)\left\{\left[\frac{[n+2]}{\mu} + \frac{[n+1]}{\mu}\cos(\sqrt{\mu}gt)\right] \times |3,n\rangle_q - \mathrm{i}\frac{[n+1]}{\sqrt{\mu}}\sin(\sqrt{\mu}gt)|2,n+1\rangle_q + \frac{[n+1][n+2]}{\mu}(\cos(\sqrt{\mu}gt)-1)|1,n+2\rangle_q\right\}. \quad (17)$$

## 4 Photon higher-order squeezing effects

Two quadrature components of the light field amplitude $X_1^q$ and $X_2^q$ are defined as follows:

$$\begin{aligned} X_1^q(N) &= \frac{1}{2}[(a_q^+)^N + (a_q)^N], \\ X_2^q(N) &= \frac{\mathrm{i}}{2}[(a_q^+)^N - (a_q)^N]. \end{aligned} \quad (18)$$

Satisfying

$$\left[X_1^q(N),\ X_2^q(N)\right] = \frac{\mathrm{i}}{2}\left[(a_q)^N,\ (a_q^+)^N\right].$$

The corresponding uncertainty relation is

$$\langle[\Delta X_1^q(N)]^2\rangle\langle[\Delta X_2^q(N)]^2\rangle \geqslant \frac{1}{4}\left|\langle[X_1^q(N),\ X_2^q(N)]\rangle\right|^2,$$

where variances $\langle[\Delta X_{1,2}^q(N)]^2\rangle$ are defined by

$$\langle[\Delta X_{1,2}^q(N)]^2\rangle = \langle[X_{1,2}^q(N)]^2\rangle - \left(\langle X_{1,2}^q(N)\rangle\right)^2.$$

Let us define

$$S_{1,2} = {}_q\langle[\Delta X_{1,2}^q(N)]^2\rangle_q - \frac{1}{2}|{}_q\langle[X_1^q(N),\ X_2^q(N)]\rangle_q|, \quad (19)$$

$S_{1,2}$ is the squeezing volume. If $S_{1,2}<0$, the light field is squeezed. It means that we can find a quantum state whose quantum noise can be greatly decreased. Considering $N=2$, we have

$$S_1 = \frac{1}{4}(\langle(a_q^+)^4\rangle + \langle(a_q)^4\rangle + \langle(a_q^+)^2(a_q)^2\rangle + \langle(a_q)^2(a_q^+)^2\rangle) - \frac{1}{4}(\langle(a_q^+)^2\rangle + \langle(a_q)^2\rangle)^2 - \frac{1}{4}|\langle(a_q)^2(a_q^+)^2\rangle - \langle(a_q^+)^2(a_q)^2\rangle|, \quad (20)$$

$$S_2 = \frac{1}{4}(-\langle(a_q^+)^4\rangle - \langle(a_q)^4\rangle + \langle(a_q^+)^2(a_q)^2\rangle + \langle(a_q)^2(a_q^+)^2\rangle) + \frac{1}{4}(\langle(a_q^+)^2\rangle + \langle(a_q)^2\rangle)^2 - \frac{1}{4}|\langle(a_q)^2(a_q^+)^2\rangle - \langle(a_q^+)^2(a_q)^2\rangle|. \quad (21)$$

We have analysed the photon higher-order squeezing effects of the $q$ analogue of a single-mode field interaction with the three-level atom initially in the excited level $|3\rangle$ by numerical calculation, $[n]$ will not be changed when $q\to 1/q$, so we only discussed $0<q<1$ (Fig. 2).

It is found that: larger fluctuation occurs in the evolutions of squeezing volume for $q=0.1$, and the region of the squeezing volume greater than zero is almost equal to the region of the squeezing volume less than zero. When the squeezing volume is greater than zero, there are no squeezing effects in the field. For $q=0.3$ the amplitude starts decreasing and the field has squeezing effects all the time. The amplitude still decreases for $q$=0.5, with the increase value of $q$, we found the similar results as $q$=0.8.

Whatever the value of $q$, the field has squeezing effects, and with different values of $q$, the squeezing volume exhibits different periodical change trend. And the more $q$ deviates from 1, the greater fluctuation of the squeezing parameter there is. Here the system reflects strong quantum coherence and nonlinear action, it is the inner reflection of quantum character and nonlinear action of $q$.

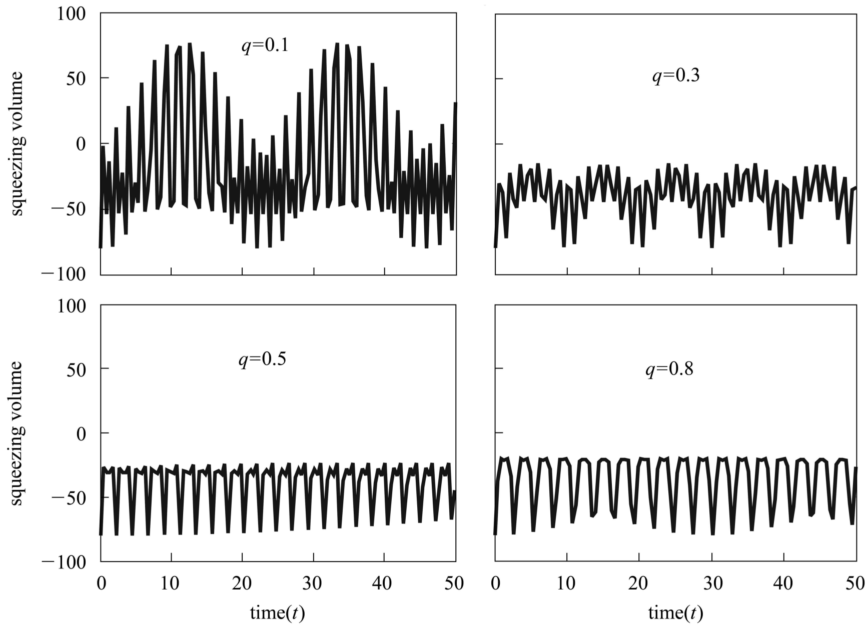


Fig. 2. Time evolutions of squeezing volume with different values of $q$.

## 5 Conclusions

In the present paper, a nonlinear theoretical model of interaction between the $q$ analogue of a single-mode field and a Ξ-type three-level atom has been discussed. Photon higher-order squeezing effects are calculated by using the full quantum theory, and numerical results indicate that variation in the value of $q$ has more abilities to control the squeezing effect of the field. The variation in the value of $q$ has a greater influence on the quantum coherence and quantum properties. If the influence of parameter $q$ on the quantum properties could be controlled in the laboratory we could have a lot of abilities to modulate the field nonclassical properties. When $q$ approaches 1, the theory reduces to the common linear theory.

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