CPC(HEP & NP), 2009, **33**(Suppl. I): 134—136 **Chinese Physics C** Vol. 33, Suppl. I , Mar., 2009

Solution to the eigenstates of pairing Hamiltonian in finite nuclei

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Abstract We apply the algebraic Bethe technique to the nuclear pairing problem with certain limits. We obtain the exact energies and eigenstates, and find the symmetry between the states corresponding to less and more than half full shell. We also proved that the problem of solving BAE can be transformed into the problem of finding the roots of a hypergeometric polynomial, which is much simpler.

Key words shell model, pairing, BAE, hypergeometric polynomial, Hamiltonian

PACS 21.60.Fw, 03.65.Fd, 71.10.Li

1 Introduction

Pairing is known to play an important role in the quantum many-body problems. In particular, pairing force is a key ingredient of the residual interaction between nucleons in a nucleus. An algebraic approach to pairing was given by Richardson some time ago[1]. Richardson's formalism was rather complex and was not widely used in nuclear physics. For a limited class of seniority-conserving pairing interactions, the quasispin formalism of Kerman[2] was used to treat a large number of cases. In this article, we wish to explore an approach related to Richardson's formalism which can be reduced to the quasispin formalism in the appropriate limit. In the nuclear shell model, the nucleus is pictured as a system of fermions moving in a central field with well defined single particle energy levels ϵ_j, arising from spin-orbit interactions. We consider nucleons at time-reversed states$|\,jm\rangle$ and $(-)^{j-m}\mid J-m\rangle$, interacting with a pairing force and when the pairing strength is separable $c_{jj'}=c_j^*c_{j'}$, the Hamiltonian takes the form described by the Hamiltonian

$$\hat{H}=\sum_{jm}\epsilon_j a_{jm}^+a_{jm}-|G|\sum_{jj'}c_j^*c_{j'}\hat{S}_j^+\hat{S}_{j'}^- \ , \tag{1}$$

here, a_{jm}^+ and a_{jm} are the creation and annihilation operators for nucleons in level j, $\hat{S}_j^+$ and $\hat{S}_{j'}^-$ are the quasispin operators, they create or destroy a single pair of nucleons in the time reversed states on level j. If we also define

$$\hat{S}_j^0=1/2\sum_{m>0}(a_{jm}^+a_{jm}+a_{j-m}^+a_{j-m}-1), \tag{2}$$

then, we have a set of orthogonal $SU(2)$ algebras

$$[\hat{S}_j^+,\hat{S}_{j'}^-]=2\delta_{jj'}\hat{S}_j^0, \qquad [\hat{S}_j^0,\hat{S}_{j'}^{\pm}]=\pm\delta_{jj'}\hat{S}_j^{\pm}. \tag{3}$$

Now, we define

$$\hat{S}^+(0)=\sum_j c_j^*\hat{S}_j^+ \qquad \text{and} \qquad \hat{S}^-(0)=\sum_j c_j\hat{S}_j^-, \tag{4}$$

furthermore, if we assume that the energy levels are degenerate, the first term in Eq. (1) is a constant for a given number of pairs because the Hamiltonian is number conserving. Ignoring this term, we obtain

$$\hat{H}=-|G|\hat{S}^+(0)\hat{S}^-(0). \tag{5}$$

Physically, the operator $\hat{S}^+(0)$ creates a single fermion pair and c_j^* can be viewed as the probability amplitude that this pair is created at level j. This interpretation implies, however, that these constants are normalized:

$$\sum_j|c_j|^2=1\ . \tag{6}$$

Note that a state with no pairs, here denoted by $|0\rangle$, is the lowest weight state for all the $SU(2)$ algebras corresponding to different levels. In other words, it

Received 3 September 2008

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obeys

$$\hat{S}_j^- \,|\,0\rangle = 0 \ \text{ and } \ \hat{S}_j^0 \,|\,0\rangle = -\Omega_j/2\,|\,0\rangle \ \text{ for every } j, \quad (7)$$

therefore, this state is annihilated by the Hamiltonian in Eq. (5)

$$\hat{H}\,|\,0\rangle = 0 \ . \quad (8)$$

The state which represents the full shell, denoted by $|\,\bar{0}\rangle$ in this paper, is the highest weight state of all $SU(2)$ algebras corresponding to different levels. It obeys

$$\hat{S}_j^+ \,|\,\bar{0}\rangle = 0 \ \text{ and } \ \hat{S}_j^0 \,|\,\bar{0}\rangle = \Omega_j/2\,|\,\bar{0}\rangle \ \text{ for every } j, \quad (9)$$

the state $|\,\bar{0}\rangle$ is also an eigenstate of the Hamiltonian

$$\hat{H}\,|\,\bar{0}\rangle = E_1\,|\,\bar{0}\rangle \ , \quad (10)$$

its energy is given by

$$E_1 = -|G|\sum_j \Omega_j |c_j|^2 \ . \quad (11)$$

2 Eigenstates and eigenvalues of the nuclear pairing Hamiltonian

We begin by introducing the operators

$$\hat{S}^+(x) = \sum_j \frac{c_j^*}{1-|c_j|^2 x}\hat{S}^+(0), \quad (12)$$

$$\hat{S}^-(x) = \sum_j \frac{c_j}{1-|c_j|^2 x}\hat{S}^-(0), \quad (13)$$

$$\hat{K}^0(x) = \sum_j \frac{1}{1/|c_j|^2 - x}\hat{S}_j^0, \quad (14)$$

and we obtain

$$[\hat{S}^+(x), \hat{S}^-(0)] = [\hat{S}^+(0), \hat{S}^-(x)] = 2\hat{K}^0(x), \quad (15)$$

$$[\hat{K}^0(x), \hat{S}^\pm(y)] = \pm\frac{\hat{S}^\pm(x) - \hat{S}^\pm(y)}{x-y}. \quad (16)$$

The states $|0\rangle$ and $|\,\bar{0}\rangle$, are eigenstates of $\hat{K}^0(x)$:

$$\hat{K}^0(x)\,|\,0\rangle = -\left(\sum_j \frac{\Omega_j/2}{1/|c_j|^2 - x}\right)|\,0\rangle, \quad (17)$$

$$\hat{K}^0(x)\,|\,\bar{0}\rangle = \left(\sum_j \frac{\Omega_j/2}{1/|c_j|^2 - x}\right)|\,\bar{0}\rangle. \quad (18)$$

Let us first form a Bethe ansatz state as follows:

$$\hat{S}^+(x^{(1)})\,|\,0\rangle, \quad (19)$$

we denoted our variable by $(x^{(1)})$ in order to emphasize that this state has one pair of nucleons. Using the form of the Hamiltonian given in Eq. (5), we can show that the Hamiltonian acting on this state gives[3]

$$\hat{H}\hat{S}^+(x^{(1)})\,|\,0\rangle = -2|G|\left(\sum_j \frac{\Omega_j/2}{1/|c_j|^2 - x^{(1)}}\right)\hat{S}^+(0)\,|\,0\rangle \quad (20)$$

alternatively, we can choose $(x^{(1)})$ as a solution of

$$-\sum_j \frac{\Omega_j/2}{1/|c_j|^2 - x^{(1)}} = 0 \ , \quad (21)$$

then

$$\hat{H}\hat{S}^+(x^{(1)})\,|\,0\rangle = 0 \ , \quad (22)$$

here Eq. (21) is our one-pair Bethe ansatz equation because it determines the value of $(x^{(1)})$.

Now, we can calculate eigenstates with more than one pair of nucleons. Using the form of the Hamiltonian given in Eq. (5), it is possible to show that the state

$$\hat{S}^+(0)\hat{S}^+(z_1^{(N)})\ldots\hat{S}^+(z_{N-1}^{(N)})\,|\,0\rangle \quad (23)$$

is an eigenstate of the Hamiltonian if the parameters $(z_k^{(N)})$ obey the following Bethe ansatz equations

$$-\sum_j \frac{\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1}\frac{1}{z_m^{(N)} - z_k^{(N)}} \ , \quad (24)$$

here, $m = 1, 2, \cdots, N$, and we assuming $2 \leqslant N \leqslant N_{\max}/2$. then we can obtain the energy

$$E_{\mathrm{N}} = -|G|\left(\sum_j \Omega_j|c_j|^2 - \sum_{k=1}^{N-1}\frac{2}{z_k^{(N)}}\right). \quad (25)$$

If the shell is more than half full, we choose to work with hole pairs instead of particle pairs. Therefore the state (23) represents a shell which is at most half full. Here, we would like to emphasize that the BAE's (24) are a set of $N-1$ coupled equations in $N-1$ variables. The parameters $z_m^{(N)}$ are all different from one another and they are also different from zero. Using the same parameters that appear in Eq. (23), we now form the state

$$\hat{S}^-(z_1^{(N)})\hat{S}^-(z_2^{(N)})\ldots\hat{S}^-(z_{N-1}^{(N)})\,|\,\bar{0}\rangle. \quad (26)$$

We can show that the states (23) and (26) have the same energy in Eq. (25) which is given in terms of the variables . In principle, Eq. (24) may have more than one solution in which case each solution gives us two eigenstates. One should substitute each solution in the states (23) and (26) in order to find corresponding eigenstates and then in Eq. (25) in order to find their energy. We summarize our results in Table 1.

Table 1. Summary of the energy eigenvalues and the eigenstates of the pairing Hamiltonian. Here, $N_{\max}$ denotes the maximum number of pairs which can occupy the shell and $2 \leqslant N \leqslant N_{\max}/2$.

pairs	state	energy/$(-\mid G\mid)$
1	$\hat{S}^{+}(x^{(1)})\mid 0\rangle$	0
1	$\hat{S}^{+}(0)\mid 0\rangle$	$\sum_j \Omega_j \mid c_j\mid^2$
N	$\hat{S}^{+}(x_1^{(N)})\hat{S}^{+}(x_2^{(N)})\cdots\hat{S}^{+}(x_N^{(N)})\mid 0\rangle$	0
N	$\hat{S}^{+}(0)\hat{S}^{+}(z_1^{(N)})\cdots\hat{S}^{+}(z_{N-1}^{(N)})\mid 0\rangle$	$\sum_j \Omega_j\lvert c_j\rvert^2-\sum_{k=1}^{N-1}\frac{2}{z_k^{(N)}}$
$N_{\max}+1-N$	$\hat{S}^{-}(z_1^{(N)})\hat{S}^{-}(z_2^{(N)})\cdots\hat{S}^{-}(z_{N-1}^{(N)})\mid\bar{0}\rangle$	$\sum_j \Omega_j\lvert c_j\rvert^2-\sum_{k=1}^{N-1}\frac{2}{z_k^{(N)}}$
$N_{\max}$	$\mid\bar{0}\rangle$	$\sum_j \Omega_j\lvert c_j\rvert^2$

3 Exact solution for BAE

Here, we consider the Bethe ansatz equations which are to be solved in order to find the states annihilated by the pairing Hamiltonian for two levels and for arbitrary values of the occupation probabilities $|c_{j_1}|^2$ and $|c_{j_2}|^2$. We give analytical solutions of these equations in the form of the roots of some hypergeometric polynomials. The method we use here is adopted from Ref. [3]. When there are only two levels, Eq. (21) can be written as

$$\frac{-\Omega_{j_1}/2}{1/|c_{j_1}|^2-x_i^{(N)}}+\frac{-\Omega_{j_2}/2}{1/|c_{j_2}|^2-x_i^{(N)}}=\sum_{k=1(k\neq i)}^{N}\frac{1}{x_i^{(N)}-x_k^{(N)}}\ , \tag{27}$$

for $i=1,2,\cdots,N$. These equations are to be satisfied for every $x_i^{(N)}$, so that we have a system of N coupled nonlinear equations. Let us begin by introducing the variables $\eta_i^{(N)}$, which are related to $x_i^{(N)}$ with the linear transformation

$$x_i^{(N)}=\frac{1}{|c_{j_2}|^2}+\eta_i^{(N)}\left(\frac{1}{|c_{j_1}|^2}-\frac{1}{|c_{j_2}|^2}\right), \tag{28}$$

we assumed that $|c_{j_1}|^2\neq|c_{j_2}|^2$. When we write the BAE (27) in terms of the new variables introduced in Eq. (28), we find

$$\sum_{k=1(k\neq i)}^{N}\frac{1}{\eta_i^{(N)}-\eta_k^{(N)}}-\frac{\Omega_{j_2}/2}{\eta_i^{(N)}}+\frac{\Omega_{j_1}/2}{1-\eta_i^{(N)}}=0\ , \tag{29}$$

for $i=1,2,\cdots,N$. This way, the dependence of the BAE on the occupation probabilities $|c_{j_1}|^2$ and $|c_{j_2}|^2$ disappears.

In Ref. [4], Stieltjes had shown that the $N^{\rm th}$ order polynomial

$$p_N(z)=\prod_{i=1}^{N}(z-\eta_i^{(N)})\ , \tag{30}$$

whose roots obey Eq. (29), satisfies the hypergeometric differential equation

$$z(1-z)p_N''(z)+[-\Omega_{j_2}+(\Omega_{j_2}+\Omega_{j_1})z]p_{\rm N}'(z)+ N(N-\Omega_{j_2}-\Omega_{j_1}-1)p_N(z)=0\ . \tag{31}$$

We assume the polynomial solution of Eq. (31) is the following hypergeometric function Ref. [5]:

$$p_N(z)\ =\ F(-N,N-\Omega_{j_2}-\Omega_{j_1}-1;z)= \sum_{k=0}^{N}\frac{(-N)_k(N-\Omega_{j_2}-\Omega_{j_1}-1)_k}{(-\Omega_{j_2})_k(k!)}z^k\ . \tag{32}$$

This procedure reduces the problem of solving the BAE's (27) for N pairs to a problem of finding the roots of an order $N^{\rm th}$ hypergeometric polynomial. Once the roots of the polynomial (31) are found, then Eq. (28) can be used to find the variables $x_i^{(N)}$, then we can find the corresponding eigenstates and energy.

4 Summary and concluding remarks

In this paper, we showed that the Bethe ansatz technique can be applied to the nuclear pairing Hamiltonian with separable pairing strengths and degenerate single particle energy levels in a purely algebraic fashion. This algebraic method reveals symmetry between the energy eigenstates corresponding to at most half full shell and those corresponding to more than half full shell. Using this symmetry, we are now able to complete the pairing spectra obtained by Pan et al Ref. [6] for the first nuclear sd shell.We also provide a technique to solve the equations of Bethe ansatz, which is using hypergeometric polynomial.

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