On analytic formulas of Feynman propagators in position space*

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Abstract We correct an inaccurate result of previous work on the Feynman propagator in position space of a free Dirac field in $(3+1)$-dimensional spacetime; we derive the generalized analytic formulas of both the scalar Feynman propagator and the spinor Feynman propagator in position space in arbitrary $(D+1)$-dimensional spacetime; and we further find a recurrence relation among the spinor Feynman propagator in $(D+1)$-dimensional spacetime and the scalar Feynman propagators in $(D+1)$-, $(D-1)$- and $(D+3)$-dimensional spacetimes.

Key words Feynman propagator, Klein-Gordon field, Dirac field, arbitrary dimensional spacetime

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1 Introduction

Although one cannot adopt the extreme view that the set of all Feynman rules represents the full theory of quantized fields, the approach of the Feynman graphs and rules plays an important role in perturbative quantum field theories. For a generic quantum theory involving interacting fields, the set of its Feynman rules includes some vertices and propagators. While for a free field theory there is only one graph, that is, the Feynman propagator, in the set of its Feynman rules. Thus the Feynman propagator describes most, if not all, of the physical contents of the free field theory. It is true that the real world is not governed by any free field theory. However, this kind of theory is the basis of a perturbatively interacting field theory, and the issues of a free theory are usually in the simplest situation and thus their study will be attempted at the first step of investigations. In statistical field theory, the Feynman propagator is usually called the correlation function. We need to know its formula in position space to figure out the critical exponent η and the correlation length ξ, which is related to another exponent ν. In the path integral language of quantum field theory, the Feynman propagator in position space can be physically understood as the energy due to the presence of the two external sources located at two different points in space and acting on each other (See Ref. [1] Chapt.I.4).

It has been mentioned in Refs. [1–8] that in (3+1)-dimensional spacetime, after integrating over momentum, the Feynman propagator of a free Klein-Gordon field can be expressed in terms of Bessel or modified Bessel functions, which depends on whether the separation of two spacetime points is timelike or spacelike. By changing variables to hyperbolic functions and using the integral representation of the Hankel function of the second kind, the authors of Ref. [2] derived the full analytic formulas of the Feynman propagors of free Klein-Gordon and Dirac fields in (3+1)-dimensional spacetime. The expressions of the Feynman propagators of a free Klein-Gordon field in (1+1)- and that in (2+1)-dimensional spacetime can be found in Refs. [9] and [10], respectively. However, in Ref. [2] the expression for the Feynman propagator of a Dirac spinor field is inaccurate, since there is at least a redundant term in their results. In this paper we will show that the term actually

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vanishes and we will give the correct expression. Furthermore, we will generalize the results of previous work and derive the full analytic formulas of the Feynman propagators in position space of, respectively, the Klein-Gordon scalar and the Dirac spinor in arbitrary $(D+1)$-dimensional spacetime. Eventually we will find an interesting recurrence relation between the spinor Feynman propagator in $(D+1)$-dimensional spacetime and the scalar Feynman propagators in $(D+1)$- and alternate-successive dimensional spacetime.

This paper is organized as follows. In Section 2, we will briefly review the derivation of the analytic formulas of the Feynman propagators of a free Klein-Gordon field in (1+1)- and (2+1)-dimensional spacetime, and we will compute, once and for all, the scalar Feynman propagator in $(D+1)$-dimensional spacetime. After completion of this work, we became aware of the analytic formula of the scalar Feynman propagator in position space in arbitrary dimensional spacetime which is also given in Ref. [11]. We will demonstrate that our result exactly agrees with that of Ref. [11] in this section. In Section 3 we will make use of the obtained formula of the scalar Feynman propagator to compute the expression of the spinor Feynman propagator in $(D+1)$-dimensional spacetime and obtain a recurrence relation. We will also compare our result for $D=3$ with that of Ref. [2], and we will show that one additional term in Ref. [2] actually has no contribution and that the method they used to prove the term nonzero was inappropriate. The last section is devoted to conclusions.

2 Feynman propagator of Klein-Gordon theory

Following the notation of Refs. [1, 4], we write the Feynman propagator of a free Klein-Gordon field $\phi(x)\equiv\phi(t,\vec{x})$ in $(D+1)$-dimensional spacetime as the time-ordered two-point correlation function,

$$\begin{aligned} D_{\mathrm{F}}(x) &\equiv \langle 0|T\phi(x)\phi(0)|0\rangle \\ &= \theta(x^0)D(x)+\theta(-x^0)D(-x)\,, \end{aligned} \tag{1}$$

with the unordered two-point correlation function,

$$D(x)\equiv\langle 0|\phi(x)\phi(0)|0\rangle=\int\frac{\mathrm{d}^D\vec{p}}{(2\pi)^D}\frac{1}{2E_{\vec{p}}}\mathrm{e}^{-\mathrm{i}(E_{\vec{p}}t-\vec{p}\cdot\vec{x})}. \tag{2}$$

Combining the above two equations gives

$$D_{\mathrm{F}}(t,\vec{x})=D(|t|,\vec{x})=\int\frac{\mathrm{d}^D\vec{p}}{(2\pi)^D}\frac{1}{2E_{\vec{p}}}\mathrm{e}^{-\mathrm{i}(E_{\vec{p}}|t|-\vec{p}\cdot\vec{x})}. \tag{3}$$

Thus, we can obtain the analytic formula of the Feynman propagator in position space by integrating over the D-dimensional momentum in the expression of $D(|t|,\vec{x})$. This integral in D-dimensional Euclidean space can be evaluated by changing the variables from Cartesian coordinates to spherical coordinates. Since the angular integral parts look a little different in $(1+1)$-, $(2+1)$- and general $(D+1)$-dimensional (for $D\geqslant 3$) spacetimes, in order to be more careful in our derivation, let us consider these situations case by case. Eventually we will show that the general result of $(D+1)$-dimensional spacetime holds for $D=1,2$ as well.

2.1 (1+1)-dimensional spacetime

When the spatial dimension $D=1$, Eq. (3) becomes

$$D_{\mathrm{F}}(t,r)=\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\frac{1}{2E}\mathrm{e}^{-\mathrm{i}(E|t|-pr)}, \tag{4}$$

with $E=\sqrt{p^2+m^2}$. Using the substitution $E=m\cosh\eta$, $p=m\sinh\eta$ (with $-\infty<\eta<\infty$) in the above integral, we have

$$D_{\mathrm{F}}(t,r)=\frac{1}{4\pi}\int_{-\infty}^{+\infty}\mathrm{d}\eta\mathrm{e}^{-\mathrm{i}m(|t|\cosh\eta-r\sinh\eta)}. \tag{5}$$

Due to the Lorentz invariance, the Feynman propagator can depend only on the interval $x^2\equiv t^2-r^2$. If the interval is timelike, $x^2>0$, we can make a Lorentz transformation such that x is purely in the time-direction, $x^0=\theta(t)\sqrt{t^2-r^2-\mathrm{i}\epsilon}$, $r=0$. Note that $\sqrt{s}$ is not a single-valued-function of s and here and henceforth the cut line in the complex plane of s is chosen to be the negative real axis. The negative infinitesimal imaginary part, $-\mathrm{i}\epsilon$, is because of the Feynman description of the Wick rotation, i.e., $x^2\to x^2-\mathrm{i}\epsilon$ in position space and correspondingly $k^2\to k^2+\mathrm{i}\epsilon$ in momentum space. Thus,

$$\begin{aligned} D_{\mathrm{F}}(t,r) &= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\mathrm{d}\eta\mathrm{e}^{-\mathrm{i}m\sqrt{t^2-r^2-\mathrm{i}\epsilon}\cosh\eta} \\ &= -\frac{\mathrm{i}}{4}H_0^{(2)}(m\sqrt{t^2-r^2-\mathrm{i}\epsilon}), \end{aligned} \tag{6}$$

where $H_0^{(2)}(x)$ is the Hankel function of the second kind, and where we have used the identity in # 3.337 of Ref. [12],

$$\int_{-\infty}^{+\infty}\mathrm{d}\eta\mathrm{e}^{-\mathrm{i}\beta\cosh\eta}=-\mathrm{i}\pi H_0^{(2)}(\beta),\ (-\pi<\arg\beta<0). \tag{7}$$

Likewise, if the interval is spacelike, $x^2<0$, we have

$$\begin{aligned} D_{\mathrm{F}}(t,r) &= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\mathrm{d}\eta\mathrm{e}^{\mathrm{i}m\sqrt{r^2-t^2+\mathrm{i}\epsilon}\sinh\eta} \\ &= \frac{1}{2\pi}K_0(m\sqrt{r^2-t^2+\mathrm{i}\epsilon}), \end{aligned} \tag{8}$$

where $K_0(x)$ is the modified Bessel function, and where we have used the identity in # 3.714 of Ref. [12],

$$\int_0^\infty \mathrm{d}\eta \cos(\beta \sinh\eta) = K_0(\beta), \quad \left(-\frac{\pi}{2} < \arg\beta < \frac{\pi}{2}\right). \quad (9)$$

It is worthwhile noting that the $-\mathrm{i}\epsilon$ description assures the proper phase angles of $\sqrt{x^2-\mathrm{i}\epsilon}$ in Eq. (6) and $\sqrt{-x^2+\mathrm{i}\epsilon}$ in Eq. (8), respectively, so that the mathematical identities (7) and (9) can be applied to figure out these two expressions. In the limit of $x^2\to 0$, both Eqs. (6) and (8) are divergent and are approaching

$$\lim_{x^2\to 0} -\frac{\mathrm{i}}{4}H_0^{(2)}(m\sqrt{x^2-\mathrm{i}\epsilon}) \sim \frac{1}{4\pi}\ln\frac{1}{x^2-\mathrm{i}\epsilon}, \quad (10)$$

$$\lim_{x^2\to 0} \frac{1}{2\pi}K_0(m\sqrt{-x^2+\mathrm{i}\epsilon}) \sim \frac{1}{4\pi}\ln\frac{1}{x^2-\mathrm{i}\epsilon}, \quad (11)$$

which are of the same form and do not depend on the mass m, and which can be recognized as the scalar Feynman propagator on the lightcone. The above expression is indeed the Feynman propagator of a massless scalar field,

$$D_{\mathrm{F}}(x) = \frac{1}{4\pi}\ln\frac{1}{x^2-\mathrm{i}\epsilon}, \quad \text{for} \quad m=0. \quad (12)$$

In summary, the scalar Feynman propagator in position space may be written in a compact way as

$$\begin{aligned} D_{\mathrm{F}}(x) &= \theta(x^2)\left(-\frac{\mathrm{i}}{4}H_0^{(2)}(m\sqrt{x^2-\mathrm{i}\epsilon})\right) \\ &+ \theta(-x^2)\frac{1}{2\pi}K_0(m\sqrt{-x^2+\mathrm{i}\epsilon}), \end{aligned} \quad (13)$$

where the theta function $\theta(x)$ is defined as

$$\theta(x) = \int_{-\infty}^{x}\delta(y)\mathrm{d}y = \begin{cases} 1\,, & (x>0) \\ 0\,, & (x<0) \end{cases} \quad (14)$$

and the value of $\theta(x=0)$ depends on whether the argument x is approaching 0 from the positive or negative real axis, that is, $\theta(0^+)=1$ and $\theta(0^-)=0$.

2.2 (2+1)-dimensional spacetime

In $(2+1)$-dimensional spacetime, the Feynman propagator of a free scalar field is

$$D_{\mathrm{F}}(t,r) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty \mathrm{d}p\frac{p}{2E}\mathrm{e}^{-\mathrm{i}(E|t|-pr\cos\theta)}, \quad (15)$$

with $E=\sqrt{p^2+m^2}$. Using a similar calculation procedure, we can obtain the analytic formula of the scalar Feynman propagator in (2+1)-dimensional spacetime as follows,

$$\begin{aligned} D_{\mathrm{F}}(x) &= \theta(x^2)\frac{-\mathrm{i}}{4\pi\sqrt{x^2-\mathrm{i}\epsilon}}\mathrm{e}^{-\mathrm{i}m\sqrt{x^2-\mathrm{i}\epsilon}} \\ &+ \theta(-x^2)\frac{1}{4\pi\sqrt{-x^2+\mathrm{i}\epsilon}}\mathrm{e}^{-m\sqrt{-x^2+\mathrm{i}\epsilon}}. \end{aligned} \quad (16)$$

Eqs. (13) and (16) agree well with the results of previous work [9, 10].

2.3 $(D+1)$-dimensional spacetime (for $D\geqslant 3$)

Now, let us proceed to compute the scalar Feynman propagator in $(D+1)$-dimensional spacetime (for $D\geqslant 3$). Since the method we use in the following differs from that used in Ref. [2], let us wait to see whether the results from these two approaches are consistent or not. Changing the variables from Cartesian coordinates to spherical coordinates, Eq. (3) becomes

$$\begin{aligned} D_{\mathrm{F}}(t,r) &= \frac{1}{(2\pi)^D}\frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\dfrac{D-1}{2}\right)}\int_0^\pi \sin^{D-2}\theta\mathrm{d}\theta \\ &\times\int_0^\infty \mathrm{d}p\frac{p^{D-1}}{2E}\mathrm{e}^{-\mathrm{i}(E|t|-pr\cos\theta)}, \end{aligned} \quad (17)$$

with $E=\sqrt{p^2+m^2}$. To evaluate the above integral, we need to figure out the angular integral $\int_0^\pi \mathrm{d}\theta\sin^{D-2}\theta\mathrm{e}^{\mathrm{i}pr\cos\theta}$. From the formula # 3.387 of Ref. [12],

$$\begin{aligned} &\int_{-1}^{1}\mathrm{d}x(1-x^2)^{\nu-1}\mathrm{e}^{\mathrm{i}\mu x} \\ &= \sqrt{\pi}\left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)J_{\nu-\frac{1}{2}}(\mu), \ (\mathrm{Re}\ \nu>0), \end{aligned} \quad (18)$$

we can easily find that

$$\int_0^\pi \mathrm{d}\theta\sin^k\theta\mathrm{e}^{\mathrm{i}pr\cos\theta} = \sqrt{\pi}\left(\frac{2}{pr}\right)^{\frac{k}{2}}\Gamma\left(\frac{k+1}{2}\right)J_{\frac{k}{2}}(pr). \quad (19)$$

Then, substituting Eq. (19) with $k=D-2$ into Eq. (17), we obtain

$$D_{\mathrm{F}}(t,r) = \frac{1}{2(2\pi)^{\frac{D}{2}}r^{\frac{D}{2}-1}}\int_0^\infty \mathrm{d}p\frac{p^{\frac{D}{2}}}{E}J_{\frac{D}{2}-1}(pr)\mathrm{e}^{-\mathrm{i}E|t|}, \quad (20)$$

which, by changing the variable of integration to $x=E/m$, leads to

$$\begin{aligned} D_{\mathrm{F}}(t,r) &= \frac{m^{\frac{D}{2}}}{2(2\pi)^{\frac{D}{2}}r^{\frac{D}{2}-1}}\int_1^\infty \mathrm{d}x(x^2-1)^{\frac{1}{2}(\frac{D}{2}-1)} \\ &\times J_{\frac{D}{2}-1}(mr\sqrt{x^2-1})\mathrm{e}^{-\mathrm{i}m|t|x}. \end{aligned} \quad (21)$$

To compute the above integral, we can make the analytical continuation of the 2nd formula of # 6.645 of Ref. [12],

$$\int_1^\infty \mathrm{d}x(x^2-1)^{\frac{1}{2}\nu}\mathrm{e}^{-\alpha x}J_\nu(\beta\sqrt{x^2-1}) = \sqrt{\frac{2}{\pi}}\beta^\nu(\alpha^2+\beta^2+\mathrm{i}\epsilon)^{-\frac{1}{2}\nu-\frac{1}{4}}K_{\nu+\frac{1}{2}}\left(\sqrt{\alpha^2+\beta^2+\mathrm{i}\epsilon}\right) \tag{22}$$

and obtain the following identity:

$$\int_1^\infty \mathrm{d}x(x^2-1)^{\frac{1}{2}\nu}\mathrm{e}^{-\mathrm{i}ax}J_\nu(b\sqrt{x^2-1}) = \begin{cases}\sqrt{\frac{2}{\pi}}b^\nu(b^2-a^2+\mathrm{i}\epsilon)^{-\frac{1}{2}\nu-\frac{1}{4}}K_{\nu+\frac{1}{2}}(\sqrt{b^2-a^2+\mathrm{i}\epsilon}), & (b>a>0)\\ \sqrt{\frac{\pi}{2}}b^\nu\frac{(-\mathrm{i})^{2(\nu+1)}}{(\sqrt{a^2-b^2-\mathrm{i}\epsilon})^{\nu+\frac{1}{2}}}H^{(2)}_{\nu+\frac{1}{2}}\left(\sqrt{a^2-b^2-\mathrm{i}\epsilon}\right), & (a>b>0)\end{cases}. \tag{23}$$

Substituting Eq. (23) (with $\nu=\frac{D}{2}-1,\ a=mt,\ b=mr$) into Eq. (21), we obtain

$$D_{\mathrm{F}}(t,r) = \frac{(-\mathrm{i})^D m^{\frac{D-1}{2}}}{2^{\frac{D+3}{2}}\pi^{\frac{D-1}{2}}(t^2-r^2-\mathrm{i}\epsilon)^{\frac{D-1}{4}}}H^{(2)}_{\frac{D-1}{2}}(m\sqrt{t^2-r^2-\mathrm{i}\epsilon}),\ (\text{for } t^2-r^2>0), \tag{24}$$

$$D_{\mathrm{F}}(t,r) = \frac{m^{\frac{D-1}{2}}}{(2\pi)^{\frac{D+1}{2}}(r^2-t^2+\mathrm{i}\epsilon)^{\frac{D-1}{4}}}K_{\frac{D-1}{2}}(m\sqrt{r^2-t^2+\mathrm{i}\epsilon}),\ (\text{for } r^2-t^2>0). \tag{25}$$

In the limit of $x^2\to 0$, the above two formulas are approaching a common asymptotic expression, that is, the scalar Feynman propagator on the lightcone,

$$D_{\mathrm{F}}(x)\sim\frac{\Gamma\left(\frac{D-1}{2}\right)}{4\pi^{\frac{D+1}{2}}}\left(-\frac{1}{x^2-\mathrm{i}\epsilon}\right)^{\frac{D-1}{2}},\ (\text{for } x^2\to 0), \tag{26}$$

which is indeed the exact formula of the Feynman propagator of a massless scalar field. In summary, the full analytic expression of the scalar Feynman propagator in $(D+1)$-dimensional spacetime is given by

$$\begin{aligned}D_{\mathrm{F}}(x) &= \theta(x^2)\frac{(-\mathrm{i})^D m^{\frac{D-1}{2}}}{2^{\frac{D+3}{2}}\pi^{\frac{D-1}{2}}(x^2-\mathrm{i}\epsilon)^{\frac{D-1}{4}}}H^{(2)}_{\frac{D-1}{2}}(m\sqrt{x^2-i\epsilon})\\ &\quad+\theta(-x^2)\frac{m^{\frac{D-1}{2}}}{(2\pi)^{\frac{D+1}{2}}(-x^2+\mathrm{i}\epsilon)^{\frac{D-1}{4}}}\\ &\quad\times K_{\frac{D-1}{2}}(m\sqrt{-x^2+\mathrm{i}\epsilon}).\end{aligned} \tag{27}$$

In particular, taking $D=3$, it follows from the above equation that

$$\begin{aligned}D_{\mathrm{F}}(x) &= \theta(x^2)\frac{\mathrm{i}m}{8\pi\sqrt{x^2-\mathrm{i}\epsilon}}H^{(2)}_1(m\sqrt{x^2-\mathrm{i}\epsilon})\\ &\quad+\theta(-x^2)\frac{m}{4\pi^2\sqrt{-x^2+\mathrm{i}\epsilon}}K_1(m\sqrt{-x^2+\mathrm{i}\epsilon}),\end{aligned} \tag{28}$$

which is consistent with the results in (3 + 1)-dimensional spacetime of Ref. [2]. Moreover, Eq. (27) holds not only for $D\geqslant 3$ but also for $D=1,2$. Noting the facts that

$$H^{(2)}_{\frac{1}{2}}(x)=\mathrm{i}\sqrt{\frac{2}{\pi x}}\mathrm{e}^{-ix},\ K_{\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}, \tag{29}$$

it can easily be verified that if the spatial dimension is taken to be $D=1,2$, Eq. (27) will reduce to Eqs. (13),(16), respectively. Therefore, in the following we will use Eq. (27) to describe the scalar Feynman propagator in $(D+1)$-dimensional spacetime for $D\geqslant 1$. The shapes of the scalar Feynman propagators with spacelike or timelike separations in different dimensional spacetime are shown in Figs. 1 and 2, respectively. The figures show that in any dimensional spacetime, the spacelike propagation amplitude is dominated by the exponential decay, while the timelike propagation amplitude behaves as the damped oscillation; and in both cases the more the dimension of spacetime, the more rapidly the propagation amplitude decreases.

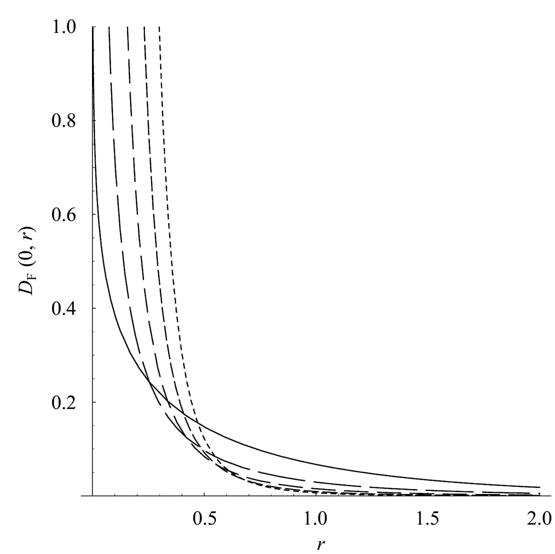

Fig. 1. The scalar Feynman propagator $D_{\mathrm{F}}(0,r)$ with spacelike separation r in $(D+1)$-dimensional spacetime, where we have set the mass parameter $m=1$, and the solid-line corresponds to $D=1$, while the dashed-lines, from long to short, correspond to $D=2,3,4,5$, respectively.

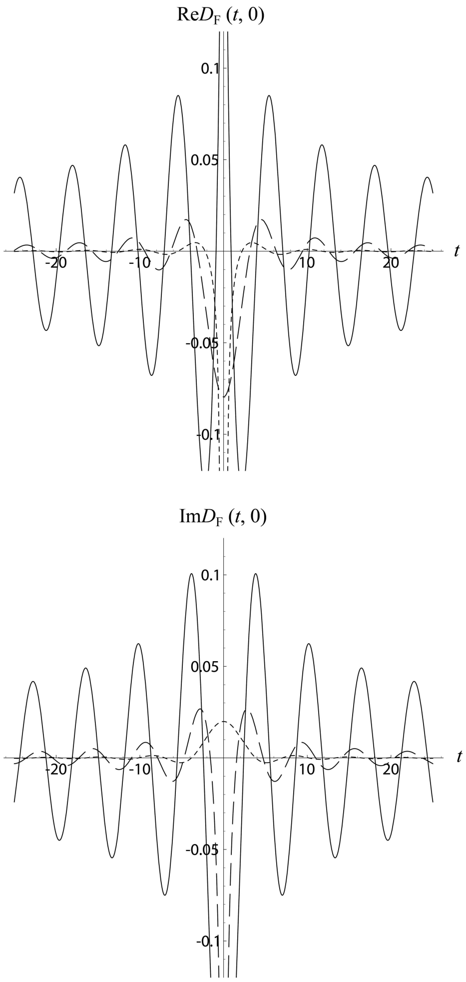

Fig. 2. The real and imaginary parts of the scalar Feynman propagator $D_{\rm F}(t,0)$ with timelike separation t in $(D+1)$-dimensional spacetime, where we have set the mass parameter $m=1$, and the solid-line corresponds to $D=1$, the long dashed-line $D=2$ and the short dashed-line $D=3$.

After completion of this work, we became aware of the analytic formula of the scalar Feynman propagator in position space in arbitrary dimensional spacetime which is also presented in Ref. [11],

$$D_{\rm F}(x)=\frac{(-{\rm i})^D m^{\frac{D-1}{2}}}{2^{\frac{D+3}{2}}\pi^{\frac{D-1}{2}}(x^2-{\rm i}\epsilon)^{\frac{D-1}{4}}}H^{(2)}_{\frac{D-1}{2}}(m\sqrt{x^2-{\rm i}\epsilon}). \tag{30}$$

In the following, let us demonstrate that Eq. (27) is equivalent to Eq. (30). That is, we need to prove the two parts in Eq. (27) are actually equal in the spirit of analytical continuation. For simplicity of the notation, let us take $D=3$ as an example and show that

$$\frac{{\rm i}m}{8\pi\sqrt{x^2-{\rm i}\epsilon}}H_1^{(2)}(m\sqrt{x^2-{\rm i}\epsilon}) =\frac{m}{4\pi^2\sqrt{-x^2+{\rm i}\epsilon}}K_1(m\sqrt{-x^2+{\rm i}\epsilon}). \tag{31}$$

Proof: Let the cut line in the complex plane be the negative real axis, so that the argument range of the variable x of the complex function $f(x)=\sqrt{x}$ must be $(-\pi/2,\ \pi/2)$. First of all, let us show that

$$\sqrt{-x^2+{\rm i}\epsilon}={\rm i}\sqrt{x^2-{\rm i}\epsilon} \tag{32}$$

holds in the spirit of analytical continuation:

1) If $x^2>0$, we have $\sqrt{x^2-{\rm i}\epsilon}=|x^2|^{\frac{1}{2}}$. On the other hand, $\sqrt{-x^2+{\rm i}\epsilon}=\sqrt{|x^2|{\rm e}^{{\rm i}\pi}}=|x^2|^{\frac{1}{2}}{\rm e}^{{\rm i}\frac{\pi}{2}}={\rm i}|x^2|^{\frac{1}{2}}$. Thus Eq.(32) holds.

2) If $x^2<0$, we have $\sqrt{-x^2+{\rm i}\epsilon}=|x^2|^{\frac{1}{2}}$. On the other hand, $\sqrt{x^2-{\rm i}\epsilon}=\sqrt{|x^2|{\rm e}^{-{\rm i}\pi}}=|x^2|^{\frac{1}{2}}{\rm e}^{-{\rm i}\frac{\pi}{2}}=-{\rm i}|x^2|^{\frac{1}{2}}$. Thus Eq.(32) holds as well.

Now, to prove Eq. (31), it is sufficient to prove

$$\frac{{\rm i}m}{8\pi z}H_1^{(2)}(mz)=\frac{m}{4\pi^2{\rm i}z}K_1({\rm i}mz), \tag{33}$$

where the argument of z is ${\rm Arg}z\in(-\pi/2,\ 0)$, and which is equivalent to

$$K_1({\rm i}z)=-\frac{\pi}{2}H_1^{(2)}(z), \tag{34}$$

which can be checked from the definition of the function $K_\nu(z)$. Therefore, we complete our proof of Eq. (31). Since any function $f(x)$ can be written as $f(x)=\theta(x)f(x)+\theta(-x)f(x)$, we finally obtain

$$\begin{aligned} D_{\rm F}(x) &= \theta(x^2)\frac{{\rm i}m}{8\pi\sqrt{x^2-{\rm i}\epsilon}}H_1^{(2)}(m\sqrt{x^2-{\rm i}\epsilon}) \\ &\quad +\theta(-x^2)\frac{m}{4\pi^2\sqrt{-x^2+{\rm i}\epsilon}}K_1(m\sqrt{-x^2+{\rm i}\epsilon}) \\ &= \frac{{\rm i}m}{8\pi\sqrt{x^2-{\rm i}\epsilon}}H_1^{(2)}(m\sqrt{x^2-{\rm i}\epsilon}) \\ &= \frac{m}{4\pi^2\sqrt{-x^2+{\rm i}\epsilon}}K_1(m\sqrt{-x^2+{\rm i}\epsilon}), \end{aligned} \tag{35}$$

which demonstrates the equivalence of our result and that of Ref. [11].

3 Feynman propagator of Dirac theory

In this section, let us calculate the analytic formula of the Feynman propagator in position space of a free Dirac spinor field in $(D+1)$-dimensional spacetime. Since we have obtained the exact expression of the scalar Feynman propagator in any dimensional spacetime, Eq. (27), it is straightforward to get the expression of the spinor Feynman propagator by means of the relation $S_{\rm F}(x)=({\rm i}\partial\!\!\!/+m)D_{\rm F}(x)$. The result we obtain is

$$
\begin{aligned}
S_{\mathrm{F}}(x) =\; & \theta(x^2)\frac{(-\mathrm{i})^{D-1}m^{\frac{D+1}{2}}\not{x}}{2^{\frac{D+5}{2}}\pi^{\frac{D-1}{2}}(x^2-\mathrm{i}\epsilon)^{\frac{D+1}{4}}}\left[H^{(2)}_{\frac{D-3}{2}}(m\sqrt{x^2-\mathrm{i}\epsilon})-H^{(2)}_{\frac{D+1}{2}}(m\sqrt{x^2-\mathrm{i}\epsilon})\right] \\
& +\theta(-x^2)\frac{\mathrm{i}m^{\frac{D+1}{2}}\not{x}}{2^{\frac{D+3}{2}}\pi^{\frac{D+1}{2}}(-x^2+\mathrm{i}\epsilon)^{\frac{D+1}{4}}}\left[K_{\frac{D-3}{2}}(m\sqrt{-x^2+\mathrm{i}\epsilon})+K_{\frac{D+1}{2}}(m\sqrt{-x^2+\mathrm{i}\epsilon})\right] \\
& -\frac{(D-1)}{2}\frac{\mathrm{i}\not{x}}{x^2-\mathrm{i}\epsilon}D_{\mathrm{F}}(x)+mD_{\mathrm{F}}(x) \\
=\; & \frac{(-\mathrm{i})^{D-1}m^{\frac{D+1}{2}}\not{x}}{2^{\frac{D+5}{2}}\pi^{\frac{D-1}{2}}(x^2-\mathrm{i}\epsilon)^{\frac{D+1}{4}}}\left[H^{(2)}_{\frac{D-3}{2}}(m\sqrt{x^2-\mathrm{i}\epsilon})-H^{(2)}_{\frac{D+1}{2}}(m\sqrt{x^2-\mathrm{i}\epsilon})\right] \\
& -\frac{(D-1)}{2}\frac{\mathrm{i}\not{x}}{x^2-\mathrm{i}\epsilon}D_{\mathrm{F}}(x)+mD_{\mathrm{F}}(x), \qquad (36)
\end{aligned}
$$

where $\not{x}\equiv\gamma^\mu x_\mu$, and where we have used the following recurrence relations of the Hankel function $H^{(2)}_\nu(x)$ and the modified Bessel function $K_\nu(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x}H^{(2)}_\nu(x) = \frac{1}{2}\left[H^{(2)}_{\nu-1}(x)-H^{(2)}_{\nu+1}(x)\right], \qquad (37)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}K_\nu(x) = -\frac{1}{2}\left[K_{\nu-1}(x)+K_{\nu+1}(x)\right]. \qquad (38)$$

In particular, when the spatial dimension $D=3$, Eq. (36) becomes

$$
\begin{aligned}
S_{\mathrm{F}}(x) =\; & -\theta(x^2)\frac{m^2\not{x}}{16\pi(x^2-\mathrm{i}\epsilon)}\Big[H^{(2)}_0(m\sqrt{x^2-\mathrm{i}\epsilon}) \\
& -H^{(2)}_2(m\sqrt{x^2-\mathrm{i}\epsilon})\Big] \\
& +\theta(-x^2)\frac{\mathrm{i}m^2\not{x}}{8\pi^2(-x^2+\mathrm{i}\epsilon)}\Big[K_0(m\sqrt{-x^2+\mathrm{i}\epsilon}) \\
& +K_2(m\sqrt{-x^2+\mathrm{i}\epsilon})\Big]-\frac{\mathrm{i}\not{x}}{x^2-\mathrm{i}\epsilon}D_{\mathrm{F}}(x) \\
& +mD_{\mathrm{F}}(x), \qquad (39)
\end{aligned}
$$

which is a little different from the results of Ref. [2], besides the less important factor of i owing to the convention that $D_{\mathrm{F}}(x)$ here equals $\mathrm{i}\varDelta_{\mathrm{F}}(x)$ there. The essential difference between our results and those of Ref. [2] lies in the fact that there is an additional term multiplied by $\delta(x^2)$ in that book, which is proportional to Eq. (34) on page 80 of Ref. [2]. However, we find that this term is redundant, since its proportional factor can be shown to vanish as follows,

$$
\begin{aligned}
& \lim_{x^2\to 0}\Bigg[\frac{1}{\sqrt{x^2-\mathrm{i}\epsilon}}H^{(2)}_1(m\sqrt{x^2-\mathrm{i}\epsilon}) \\
& \quad -\frac{\mathrm{i}}{\sqrt{-x^2+\mathrm{i}\epsilon}}H^{(2)}_1(-\mathrm{i}m\sqrt{-x^2+\mathrm{i}\epsilon})\Bigg] \\
& \sim \frac{1}{\sqrt{x^2-\mathrm{i}\epsilon}}\frac{\mathrm{i}}{\pi}\frac{2}{m\sqrt{x^2-\mathrm{i}\epsilon}} \\
& \quad -\frac{\mathrm{i}}{\sqrt{-x^2+\mathrm{i}\epsilon}}\frac{\mathrm{i}}{\pi}\frac{2}{(-\mathrm{i}m\sqrt{-x^2+\mathrm{i}\epsilon})} \\
& = \frac{2\mathrm{i}}{m\pi(x^2-\mathrm{i}\epsilon)}+\frac{2\mathrm{i}}{m\pi(-x^2+\mathrm{i}\epsilon)}=0. \qquad (40)
\end{aligned}
$$

The reason the authors of Ref. [2] regarded the above term to be nonzero may come from the fact that they had taken both x^2 and $-x^2$ to be the absolute value $|x^2|$ simultaneously in their calculation. However, it is obviously impossible that both x^2 and $-x^2$ are equal to $|x^2|$, even if $x^2\to 0$, because x^2 can only approach zero from either the positive or the negative axis direction, that is, in any case x^2 and $-x^2$ always have opposite signs even if they are infinitesimal.

Moreover, from Eq. (36) together with Eq. (27), we obtain an interesting recurrence relation for the spinor Feynman propagator in $(D+1)$-dimensional spacetime and the scalar Feynman propagators in $(D-1)$-, $(D+1)$- and $(D+3)$-dimensional spacetime, as follows,

$$
\begin{aligned}
S^{(D+1)}_{\mathrm{F}}(x) =\; & \left(-\frac{\mathrm{i}\not{x}}{x^2-\mathrm{i}\epsilon}+m\right)D^{(D+1)}_{\mathrm{F}}(x) \\
& -\frac{\mathrm{i}m^2\not{x}}{4\pi(x^2-\mathrm{i}\epsilon)}D^{(D-1)}_{\mathrm{F}}(x) \\
& +\mathrm{i}\pi\not{x}D^{(D+3)}_{\mathrm{F}}(x), \qquad (41)
\end{aligned}
$$

where the superscripts denote the spacetime dimensions of the respective physical quantities. The above relation essentially stems from the fact that the Feynman propagator in any dimensional spacetime can be expressed in terms of Bessel and modified Bessel functions, which has been proved in this paper. And it shows that the free Dirac theory and the free Klein-Gordon theories in alternate-successive dimensional spacetime might be related to each other.

4 Conclusions

In this paper, we have pointed out and corrected an error of the results of previous work on the analytic expression of the Feynman propagator in position space of a Dirac spinor in $(3+1)$-dimensional spacetime, and we have derived the generalized analytic formulas of both the scalar Feynman propagator and the spinor Feynman propagator in position space in any $(D+1)$-dimensional spacetime. The method we have used in this paper is different from that used in Ref. [2]. And the result we have obtained shows that the analytic formula of the Feynman propagator in position space can be also expressed in terms of Hankel functions of the second kind and Modified Bessel functions in a general $(D+1)$-dimensional spacetime, just like the known case in $(3+1)$-dimensional spacetime. From the obtained results, we have found an interesting recurrence relation among the spinor Feynman propagator in $(D+1)$-dimensional spacetime and the scalar Feynman propagators in $(D+1)$-, $(D-1)$- and $(D+3)$-dimensional spacetime. The result we have obtained for the scalar case agrees with the previous result of Ref. [11]. The equivalence of the two results was shown at the end of Section 2. This agreement supports our work and thus supports our correction of Ref. [2] . The analytic formula of the spinor Feynman propagator in position space in arbitrary spacetime, obtained by us, is hitherto absent in literature so far as we know. According to Ref. [1], the Feynman propagator in position space represents the energy due to the two external sources located at two different points in spacetime. Figs. 1 and 2 show that in any dimensinal spacetime, the energy from two spacelike-separated external sources decays exponentially with respect to their distance, while the energy from two timelike-separated external sources is dampedly oscillating with respect to their distance. In both cases, the more the dimension of spacetime, the more rapidly the energy decreases.

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