

Two loop electroweak corrections from heavy fermions to $b \rightarrow s + \gamma$ *

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Abstract Applying an effective Lagrangian method and an on-shell scheme, we analyze the electroweak corrections to the rare decay $b \rightarrow s + \gamma$ from some special two loop diagrams in which a closed heavy fermion loop is attached to the virtual charged gauge bosons or Higgs. At the decoupling limit where the virtual fermions in the inner loop are much heavier than the electroweak scale, we verify the final results satisfying the decoupling theorem explicitly when the interactions among Higgs and heavy fermions do not contain the nondecoupling couplings. Adopting the universal assumptions on the relevant couplings and mass spectrum of new physics, we find that the relative corrections from those two loop diagrams to the SM theoretical prediction on the branching ratio of $B \rightarrow X_s \gamma$ can reach 5% as the energy scale of new physics $\Lambda_{NP} = 200$ GeV.

Key words two-loop, inclusive decay

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1 Introduction

The measurements of the branching ratios at CLEO, ALEPH and BELLE [1] give the combined result

$$BR(B \rightarrow X_s \gamma) = (3.52 \pm 0.23 \pm 0.09) \times 10^{-4}, \quad (1)$$

which agrees with the next-to-next-to-leading order (NNLO) standard model (SM) prediction [2],

$$BR(B \rightarrow X_s \gamma) = (3.15 \pm 0.23) \times 10^{-4}. \quad (2)$$

Good agreement between the experiment and the theoretical prediction of the SM implies that the new physics scale should lie well above the electroweak (EW) scale Λ_{EW} . The systematic analysis of new physics corrections to $B \rightarrow X_s \gamma$ up to two-loop order can help us understand where the new physics scale sets in, and the distribution of new physical particle masses around this scale. In principle, the two-loop corrections can be large when some additional parameters are involved at this perturbation order besides the parameters appearing in one loop results. In other words, including the two-loop contributions, one can obtain a more exact constraint on the new physics parameter space from the present experimental results.

Though the QCD corrections to the rare B decays are discussed extensively in the literature, the pure two-loop EW corrections to the branching ratio of $b \rightarrow s \gamma$ are less advanced because of the well known difficulty in calculation. Strumia has evaluated the two-loop EW corrections to $b \rightarrow s \gamma$ from the top quark using heavy mass expansion in gaugeless limit of the SM [3]. At the limit of large $\tan\beta$ in supersymmetry, Ref. [4] analyzes the two loop corrections to the branching ratio of $B \rightarrow X_s \gamma$ from the virtual charged Higgs and gluino-squark sector.

Employing the effective Lagrangian method and on-shell scheme, we present the corrections to the branch ratio of $B \rightarrow X_s \gamma$ from some special diagrams in which a closed heavy fermion loop is attached to the virtual charged gauge bosons or Higgs here. The effective Lagrangian method can yield one loop EW corrections to the effective Lagrangian of $b \rightarrow s \gamma$ exactly in the SM and beyond, and it has been adopted to calculate the two loop supersymmetric corrections for the branching ratio of $b \rightarrow s \gamma$ [5], neutron EDM [6] and lepton MDMs and EDMs [7, 8]. In concrete calculation, we assume that all external quarks and photon are off-shell, then expand the amplitude of corresponding triangle diagrams according to the ex-

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ternal momenta of quarks and photons. Using loop momentum translating invariance, we formulate the sum of amplitude from those triangle diagrams corresponding to the same self energy in the form which explicitly satisfies the Ward identity required by the QED gauge symmetry, then get all dimension 6 operators together with their coefficients. After the equations of motion are applied to external quarks, higher dimensional operators, such as dimension 8 operators, also contribute to the branching ratio of $B \rightarrow X_s \gamma$, in principle. However, the contributions of dimension 8 operators contain the additional suppression factor m_b^2/Λ_{EW}^2 compared with that of dimension 6 operators, where m_b is the mass of the bottom quark. Setting $\Lambda_{EW} \sim 100$ GeV, one obtains easily that this suppression factor is about 10^{-3} for the $b \rightarrow s \gamma$. Under the current experimental precision, it implies that the contributions of all higher dimension operators ($D \geq 8$) can be neglected safely.

We adopt the naive dimensional regularization with the anticommuting γ_5 scheme, where there is no distinction between the first 4 dimensions and the remaining $D - 4$ dimensions. Since the bare effective Lagrangian contains the ultraviolet divergence, which is induced by divergent subdiagrams, we give the renormalized results in the on-mass-shell scheme [9]. Additionally, we adopt the nonlinear R_ξ gauge with $\xi = 1$ for simplification [10]. This special gauge-fixing term guarantees explicit electromagnetic gauge invariance throughout the calculation, not just at the end because the choice of gauge-fixing term eliminates the $\gamma W^\pm G^\mp$ vertex in the Lagrangian.

This paper is ordered as follows. In Section 2, we introduce the effective Lagrangian method and our notations. We will demonstrate how to obtain the identities among two loop integrals from the loop momentum translating invariance through an example, then obtain the corrections from the relevant diagrams to the effective Lagrangian of $b \rightarrow s \gamma$. Section 3 is devoted to the numerical discussion under universal assumptions on the parameters of new physics. In Section 4, we give our conclusion. Some tedious formulae are collected in the appendices.

2 The Wilson coefficients from the two-loop diagrams

In a conventional form, the effective Hamilton is written as

$$H_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_i C_i(\mu) \mathcal{O}_i, \quad (3)$$

where V is the CKM matrix and $G_F = 1.16639 \times 10^{-5}$ GeV $^{-2}$ is the 4-fermion coupling. The definitions of those dimension six operators are [11]

$$\begin{aligned} \mathcal{O}_1 &= \frac{1}{(4\pi)^2} \bar{s}(i\mathcal{D})^3 \omega_- b, \\ \mathcal{O}_2 &= \frac{eQ_d}{(4\pi)^2} \left[\overline{(i\mathcal{D}_\mu s)} \gamma^\mu F \cdot \sigma \omega_- b + \bar{s} F \cdot \sigma \gamma^\mu \omega_- (i\mathcal{D}_\mu b) \right], \\ \mathcal{O}_3 &= \frac{eQ_d}{(4\pi)^2} \bar{s} (\partial^\mu F_{\mu\nu}) \gamma^\nu \omega_- b, \\ \mathcal{O}_4 &= \frac{1}{(4\pi)^2} \bar{s}(i\mathcal{D})^2 (m_b \omega_+ + m_s \omega_-) b, \\ \mathcal{O}_5 &= \frac{eQ_d}{(4\pi)^2} \bar{s} \sigma^{\mu\nu} (m_b \omega_+ + m_s \omega_-) b F_{\mu\nu}, \\ \mathcal{O}_6 &= \frac{g_s}{(4\pi)^2} \left[\overline{(i\mathcal{D}_\mu s)} \gamma^\mu G \cdot \sigma \omega_- b + \bar{s} G \cdot \sigma \gamma^\mu \omega_- (i\mathcal{D}_\mu b) \right], \\ \mathcal{O}_7 &= \frac{g_s}{(4\pi)^2} \bar{s} (\partial^\mu G_{\mu\nu}) \gamma^\nu \omega_- b, \\ \mathcal{O}_8 &= \frac{g_s}{(4\pi)^2} \bar{s} T^a \sigma^{\mu\nu} (m_b \omega_+ + m_s \omega_-) b G_{\mu\nu}^a, \\ \mathcal{O}_9 &= -\frac{eQ_d}{(4\pi)^2} \left[\overline{(i\mathcal{D}_\mu s)} \gamma^\mu F \cdot \sigma \omega_- b \right. \\ &\quad \left. - \bar{s} F \cdot \sigma \gamma^\mu \omega_- (i\mathcal{D}_\mu b) \right], \\ \mathcal{O}_{10} &= \frac{1}{(4\pi)^2} \bar{s}(i\mathcal{D})^2 (m_b \omega_+ - m_s \omega_-) b, \\ \mathcal{O}_{11} &= \frac{eQ_d}{(4\pi)^2} \bar{s} \sigma^{\mu\nu} (m_b \omega_+ - m_s \omega_-) b F_{\mu\nu}, \\ \mathcal{O}_{12} &= -\frac{g_s}{(4\pi)^2} \left[\overline{(i\mathcal{D}_\mu s)} \gamma^\mu G \cdot \sigma \omega_- b \right. \\ &\quad \left. - \bar{s} G \cdot \sigma \gamma^\mu \omega_- (i\mathcal{D}_\mu b) \right], \\ \mathcal{O}_{13} &= \frac{g_s}{(4\pi)^2} \bar{s} T^a \sigma^{\mu\nu} (m_b \omega_+ - m_s \omega_-) b G_{\mu\nu}^a, \\ \mathcal{O}_{14} &= (\bar{s}_\alpha \gamma^\mu \omega_- c_\alpha) (\bar{c}_\beta \gamma^\mu \omega_- b_\beta), \end{aligned} \quad (4)$$

where $F_{\mu\nu}$ and $G_{\mu\nu} = G_{\mu\nu}^a T^a$ are the field strengths of the photon and gluon, respectively, and T^a ($a = 1, \dots, 8$) are $SU(3)_c$ generators. In addition, e and g_s represent the EW and strong couplings, respectively.

After expanding the amplitude of corresponding triangle diagrams, we extract the Wilson coefficients of operators in Eq. (4), which are formulated in the linear combinations of one and two loop vacuum integrals in momentum space, then obtain the corrections to the branching ratio of $B \rightarrow X_s \gamma$.

2.1 The corrections from the diagrams where a closed heavy fermion loop is inserted into the self energy of W^\pm gauge boson

In order to get the amplitude of the diagrams in Fig. 1(a), one can write the renormalizable interaction among the charged EW gauge boson W^\pm and the heavy fermions $F_{\alpha,\beta}$ in a more universal form as

$$\mathcal{L}_{\text{WFF}} = \frac{e}{s_w} W^{-,\mu} \bar{F}_\alpha \gamma_\mu (\zeta_{\alpha\beta}^L \omega_- + \zeta_{\alpha\beta}^R \omega_+) F_\beta + \text{h.c.}, \quad (5)$$

where the concrete expressions of $\zeta_{\alpha\beta}^{L,R}$ depend on the models employed in our calculation. The conservation of electric charge requires $Q_\beta - Q_\alpha = 1$, where $Q_{\alpha,\beta}$ denote the electric charge of the heavy fermions

$F_{\alpha,\beta}$, respectively.

Applying Eq. (5), we can write firstly the amplitude of those two loop diagrams corresponding to the self-energy diagram Fig. 1(a). After integrating over loop momenta, one gets the following terms in the effective Lagrangian,

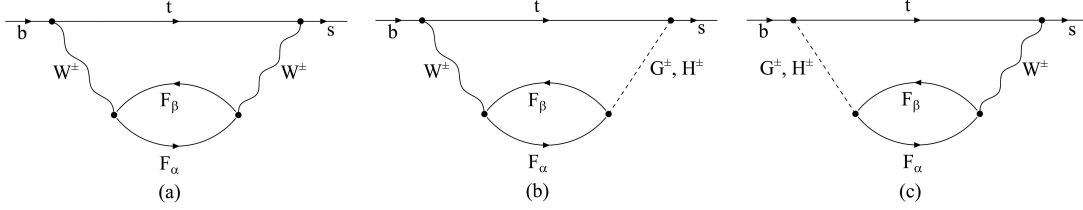


Fig. 1. The relating two-loop diagrams in which a closed heavy fermion loop is attached to virtual W^\pm bosons or G^\pm (H^\pm), where a real photon or gluon is attached in all possible ways.

$$\begin{aligned} \mathcal{L}_{\text{W}}^{\text{eff}} = & \frac{\sqrt{2}G_{\text{F}}\alpha_e x_w}{\pi s_w^2 Q_{\text{d}}} V_{\text{ts}}^* V_{\text{tb}} (4\pi x_{\text{R}})^{2\varepsilon} \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)^2} \left\{ \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) \times \left[\frac{1}{24\varepsilon} \left\{ -\psi_1 \right. \right. \right. \\ & + (x_{F_\alpha} + x_{F_\beta}) \psi_2 \left. \right\} (x_w, x_t) - \frac{1}{24} \varrho_{2,1}(x_{F_\alpha}, x_{F_\beta}) \psi_2(x_w, x_t) - \frac{x_{F_\alpha} + x_{F_\beta}}{144} \psi_3(x_w, x_t) \\ & + \phi_1(x_{F_\alpha}, x_{F_\beta}) \psi_1(x_w, x_t) + \psi_4(x_w, x_t) + F_1(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) + Q_{\text{u}} \left(\frac{1}{24\varepsilon} \left\{ \psi_5 \right. \right. \\ & + (x_{F_\alpha} + x_{F_\beta}) \psi_6 \left. \right\} (x_w, x_t) - \frac{1}{24} \varrho_{2,1}(x_{F_\alpha}, x_{F_\beta}) \psi_6(x_w, x_t) + \frac{x_{F_\alpha} + x_{F_\beta}}{144} \psi_7(x_w, x_t) \\ & + \frac{1}{8} \phi_2(x_{F_\alpha}, x_{F_\beta}) \psi_5(x_w, x_t) + \psi_8(x_w, x_t) + F_2(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \left. \right] \mathcal{O}_2 \\ & + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L - \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) F_3(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \mathcal{O}_2 + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) (x_{F_\alpha} x_{F_\beta})^{1/2} \\ & \times \left[-\frac{1}{12\varepsilon} \psi_2(x_w, x_t) + \frac{1}{12} \varrho_{1,1}(x_{F_\alpha}, x_{F_\beta}) \psi_2(x_w, x_t) + \phi_3(x_{F_\alpha}, x_{F_\beta}) \psi_1(x_w, x_t) + \psi_9(x_w, x_t) \right. \\ & + F_4(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) + Q_{\text{u}} \left(-\frac{1}{12\varepsilon} \psi_6(x_w, x_t) + \frac{1}{12} \varrho_{1,1}(x_{F_\alpha}, x_{F_\beta}) \psi_6(x_w, x_t) + \phi_4(x_{F_\alpha}, x_{F_\beta}) \psi_5(x_w, x_t) \right. \\ & + \psi_{10}(x_w, x_t) + F_5(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \left. \right] \mathcal{O}_2 + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^{R*} - \zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R \right) (x_{F_\alpha} x_{F_\beta})^{1/2} F_6(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \mathcal{O}_9 \\ & + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) \left[\frac{1}{24\varepsilon} \left\{ \psi_5 + (x_{F_\alpha} + x_{F_\beta}) \psi_6 \right\} (x_w, x_t) - \frac{1}{24} \varrho_{2,1}(x_{F_\alpha}, x_{F_\beta}) \psi_6(x_w, x_t) \right. \\ & + \frac{x_{F_\alpha} + x_{F_\beta}}{144} \psi_7(x_w, x_t) + \frac{1}{8} \phi_2(x_{F_\alpha}, x_{F_\beta}) \psi_5(x_w, x_t) + \psi_8(x_w, x_t) + F_2(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \\ & + T_\alpha^c F_7(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \left. \right] \mathcal{O}_6 + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) (x_{F_\alpha} x_{F_\beta})^{1/2} \left[-\frac{1}{12\varepsilon} \psi_6(x_w, x_t) \right. \\ & + \frac{1}{12} \varrho_{1,1}(x_{F_\alpha}, x_{F_\beta}) \psi_6(x_w, x_t) + \phi_4(x_{F_\alpha}, x_{F_\beta}) \psi_5(x_w, x_t) + \psi_{10}(x_w, x_t) + F_5(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \\ & + T_\alpha^c F_8(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \left. \right] \mathcal{O}_6 + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L - \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) T_\alpha^c F_9(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \mathcal{O}_6 \\ & + \left. \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R - \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) (x_{F_\alpha} x_{F_\beta})^{1/2} T_\alpha^c F_{10}(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \mathcal{O}_{12} \right\} + \dots, \quad (6) \end{aligned}$$

where $\alpha_e = e^2/4\pi$ and $Q_{\text{d}} = -1/3, Q_{\text{u}} = 2/3$ represent the charge of down- and up-type quarks, respectively. $T_\alpha^c = 1$ when the heavy virtual fermions take part in the strong interaction, otherwise $T_\alpha^c = 0$. With an energy

scale Λ , we define $x_i = m_i^2/\Lambda^2$, $x_R = \Lambda_{\text{RE}}^2/\Lambda^2$ with Λ_{RE} denoting the renormalization scale that can take any value in the range from the EW scale Λ_{EW} to the new physics scale Λ_{NP} naturally. Certainly, the final result does not depend on the choice of Λ because of the invariance of energy scale. Here, we adopt the abbreviations $c_w = \cos\theta_w$ and $s_w = \sin\theta_w$, with θ_w denoting the Weinberg angle. The functions ψ_i , ϕ_i are defined as

$$\begin{aligned}
\psi_1(x, y) &= \frac{\partial^4 \varrho_{4,1}}{\partial x^4}(x, y) - 3 \frac{\partial^3 \varrho_{3,1}}{\partial x^3}(x, y), \\
\psi_2(x, y) &= \frac{\partial^4 \varrho_{3,1}}{\partial x^4}(x, y) + 3 \frac{\partial^3 \varrho_{2,1}}{\partial x^3}(x, y), \\
\psi_3(x, y) &= \left\{ 4 \frac{\partial^4 \varrho_{3,1}}{\partial x^4} - 18 \frac{\partial^3 \varrho_{2,1}}{\partial x^3} + 3 \frac{\partial^4 \varrho_{3,2}}{\partial x^4} + 9 \frac{\partial^3 \varrho_{2,2}}{\partial x^3} \right\}(x, y), \\
\psi_4(x, y) &= \left\{ \frac{1}{48} \frac{\partial^4 \varrho_{4,1}}{\partial x^4} - \frac{23}{144} \frac{\partial^3 \varrho_{3,1}}{\partial x^3} + \frac{1}{4} \frac{\partial^2 \varrho_{2,1}}{\partial x^2} + \frac{1}{48} \frac{\partial^4 \varrho_{4,2}}{\partial x^4} - \frac{1}{16} \frac{\partial^3 \varrho_{3,2}}{\partial x^3} \right\}(x, y), \\
\psi_5(x, y) &= \frac{\partial^4 \varrho_{4,1}}{\partial x^4}(x, y) - 6 \frac{\partial^3 \varrho_{3,1}}{\partial x^3}(x, y) + 6 \frac{\partial^2 \varrho_{2,1}}{\partial x^2}(x, y), \\
\psi_6(x, y) &= 6 \frac{\partial^2 \varrho_{1,1}}{\partial x^2}(x, y) - \frac{\partial^4 \varrho_{3,1}}{\partial x^4}(x, y), \\
\psi_7(x, y) &= \left\{ 4 \frac{\partial^4 \varrho_{3,1}}{\partial x^4} - 36 \frac{\partial^3 \varrho_{2,1}}{\partial x^3} + 18 \frac{\partial^2 \varrho_{1,1}}{\partial x^2} + 3 \frac{\partial^4 \varrho_{3,2}}{\partial x^4} - 18 \frac{\partial^2 \varrho_{1,2}}{\partial x^2} \right\}(x, y), \\
\psi_8(x, y) &= \left\{ -\frac{1}{48} \frac{\partial^4 \varrho_{4,1}}{\partial x^4} + \frac{19}{72} \frac{\partial^3 \varrho_{3,1}}{\partial x^3} - \frac{2}{3} \frac{\partial^2 \varrho_{2,1}}{\partial x^2} + \frac{1}{3} \frac{\partial \varrho_{1,1}}{\partial x} - \frac{1}{48} \frac{\partial^4 \varrho_{4,2}}{\partial x^4} + \frac{1}{8} \frac{\partial^3 \varrho_{3,2}}{\partial x^3} - \frac{1}{8} \frac{\partial^2 \varrho_{2,2}}{\partial x^2} \right\}(x, y), \\
\psi_9(x, y) &= \left\{ \frac{1}{72} \frac{\partial^4 \varrho_{3,1}}{\partial x^4} - \frac{3}{8} \frac{\partial^3 \varrho_{2,1}}{\partial x^3} + \frac{1}{24} \frac{\partial^4 \varrho_{3,2}}{\partial x^4} + \frac{1}{8} \frac{\partial^3 \varrho_{2,2}}{\partial x^3} \right\}(x, y), \\
\psi_{10}(x, y) &= \left\{ -\frac{1}{72} \frac{\partial^4 \varrho_{3,1}}{\partial x^4} + \frac{1}{2} \frac{\partial^3 \varrho_{2,1}}{\partial x^3} - \frac{1}{2} \frac{\partial^2 \varrho_{1,1}}{\partial x^2} - \frac{1}{24} \frac{\partial^4 \varrho_{3,2}}{\partial x^4} + \frac{1}{4} \frac{\partial^2 \varrho_{1,2}}{\partial x^2} \right\}(x, y), \\
\phi_1(x, y) &= \left\{ \frac{1}{8} \frac{\partial \varrho_{2,1}}{\partial x} - \frac{1}{24} \frac{\partial^2 \varrho_{3,1}}{\partial x^2} - \frac{3x_w}{32} \frac{\partial^2 \varrho_{2,1}}{\partial x^2} + \frac{x_w}{16} \frac{\partial^3 \varrho_{3,1}}{\partial x^3} - \frac{x_w}{128} \frac{\partial^4 \varrho_{4,1}}{\partial x^4} \right\}(x, y), \\
\phi_2(x, y) &= \left\{ -\frac{\partial \varrho_{2,1}}{\partial x} + \frac{1}{3} \frac{\partial^2 \varrho_{3,1}}{\partial x^2} + \frac{3x_w}{4} \frac{\partial^2 \varrho_{2,1}}{\partial x^2} - \frac{x_w}{2} \frac{\partial^3 \varrho_{3,1}}{\partial x^3} + \frac{x_w}{16} \frac{\partial^4 \varrho_{4,1}}{\partial x^4} \right\}(x, y), \\
\phi_3(x, y) &= \left\{ \frac{1}{16} \frac{\partial^2 \varrho_{2,1}}{\partial x^2} - \frac{1}{8} \frac{\partial \varrho_{1,1}}{\partial x} + \frac{x_w}{16} \frac{\partial^2 \varrho_{1,1}}{\partial x^2} - \frac{x_w}{16} \frac{\partial^3 \varrho_{2,1}}{\partial x^3} + \frac{x_w}{96} \frac{\partial^4 \varrho_{3,1}}{\partial x^4} \right\}(x, y), \\
\phi_4(x, y) &= \left\{ -\frac{1}{16} \frac{\partial^2 \varrho_{2,1}}{\partial x^2} + \frac{1}{8} \frac{\partial \varrho_{1,1}}{\partial x} - \frac{x_w}{16} \frac{\partial^2 \varrho_{1,1}}{\partial x^2} + \frac{x_w}{16} \frac{\partial^3 \varrho_{2,1}}{\partial x^3} - \frac{x_w}{96} \frac{\partial^4 \varrho_{3,1}}{\partial x^4} \right\}(x, y). \tag{7}
\end{aligned}$$

Note that the result in Eq. (6) does not depend on the concrete choice of energy scale Λ , and the concrete expressions of $F_i(x, y, z, u)$, $\varrho_{i,j}(x, y)$ ($i, j = 1, 2, \dots$) can be found in the appendix.

The charged gauge boson self energy composed of a closed heavy fermion loop induces the ultraviolet divergence in the Wilson coefficients of the effective Lagrangian, and the unrenormalized W^\pm self energy is generally expressed as

$$\Sigma_{\mu\nu}^W(p, \Lambda_{\text{RE}}) = \Lambda^2 A_0^W g_{\mu\nu} + \left(A_1^W + \frac{p^2}{\Lambda^2} A_2^W + \dots \right) (p^2 g_{\mu\nu} - p_\mu p_\nu) + \left(B_1^W + \frac{p^2}{\Lambda^2} B_2^W + \dots \right) p_\mu p_\nu, \tag{8}$$

where the form factors $A_{0,1,2}^W$ and $B_{1,2}^W$ only depend on the virtual field masses and renormalization scale. Here, we omit those terms which are strongly suppressed at the limit of heavy virtual fermion masses. The corresponding counter terms are given as

$$\Sigma_{\mu\nu}^{\text{WC}}(p, \Lambda_{\text{RE}}) = - \left[\delta m_w^2(\Lambda_{\text{RE}}) + m_w^2 \delta Z_w(\Lambda_{\text{RE}}) \right] g_{\mu\nu} - \delta Z_w(\Lambda_{\text{RE}}) \left[p^2 g_{\mu\nu} - p_\mu p_\nu \right]. \tag{9}$$

The renormalized self energy is given by

$$\hat{\Sigma}_{\mu\nu}^W(p, A_{\text{RE}}) = \Sigma_{\mu\nu}^W(p, A_{\text{RE}}) + \Sigma_{\mu\nu}^{\text{WC}}(p, A_{\text{RE}}). \quad (10)$$

For on-shell external gauge boson W^\pm , we have [9]

$$\hat{\Sigma}_{\mu\nu}^W(p, m_w)\epsilon^\nu(p) \Big|_{p^2=m_w^2} = 0, \quad \lim_{p^2 \rightarrow m_w^2} \frac{1}{p^2 - m_w^2} \hat{\Sigma}_{\mu\nu}^W(p, m_w)\epsilon^\nu(p) = \epsilon_\mu(p), \quad (11)$$

where $\epsilon(p)$ is the polarization vector of the W^\pm gauge boson. Inserting Eq. (8) and Eq. (9) into Eq. (11), we derive the counter terms for the W^\pm self energy in the on-shell scheme as

$$\delta Z_w^{\text{os}} = A_1^w + \frac{m_w^2}{A^2} A_2^w = A_1^w + x_z A_2^w, \quad \delta m_w^{2,\text{os}} = A_0^w A^2 - m_w^2 \delta Z_w^{\text{os}}. \quad (12)$$

To cancel the ultraviolet divergence and those dangerous terms violating the decoupling theorem completely, we should derive the counter term for the vertex $\gamma W^+ W^-$ here since the corresponding coupling is not zero at tree level. In the nonlinear R_ξ gauge with $\xi = 1$, the counter term for the vertex $\gamma W^+ W^-$ is

$$i\delta C_{\gamma W^+ W^-} = i\epsilon \cdot \delta Z_w(A_{\text{RE}}) [g_{\mu\nu}(k_1 - k_2)_\rho + g_{\nu\rho}(k_2 - k_3)_\mu + g_{\rho\mu}(k_3 - k_1)_\nu], \quad (13)$$

where k_i ($i = 1, 2, 3$) denote the incoming momenta of W^\pm and photon, and μ, ν, ρ denote the corresponding Lorentz indices, respectively.

We can verify that the sum of amplitude from counter diagrams satisfies the Ward identity required by the QED gauge invariance obviously. Accordingly, the effective Lagrangian from the counter term diagrams is written as

$$\begin{aligned} \delta \mathcal{L}_W^{\text{C}} = & -\frac{\sqrt{2}G_F \alpha_e x_w}{\pi S_W^2 Q_d} V_{ts}^* V_{tb} (4\pi x_R)^{2\epsilon} \frac{\Gamma^2(1+\epsilon)}{(1-\epsilon)^2} \left\{ \left(\zeta_{\alpha\beta}^{\text{L}*} \zeta_{\alpha\beta}^{\text{L}} + \zeta_{\alpha\beta}^{\text{R}*} \zeta_{\alpha\beta}^{\text{R}} \right) \times \left[\frac{1}{24\epsilon} \left\{ -\psi_1 \right. \right. \right. \\ & + (x_{F_\alpha} + x_{F_\beta}) \psi_2 \left. \right\} (x_w, x_t) - \frac{1}{24} \varrho_{2,1}(x_{F_\alpha}, x_{F_\beta}) \psi_2(x_w, x_t) - \frac{x_{F_\alpha} + x_{F_\beta}}{144} \psi_3(x_w, x_t) \\ & + \phi_1(x_{F_\alpha}, x_{F_\beta}) \psi_1(x_w, x_t) + \psi_4(x_w, x_t) + Q_u \left(\frac{1}{24\epsilon} \left\{ \psi_5 + (x_{F_\alpha} + x_{F_\beta}) \psi_6 \right\} (x_w, x_t) \right. \\ & - \frac{1}{24} \varrho_{2,1}(x_{F_\alpha}, x_{F_\beta}) \psi_6(x_w, x_t) + \frac{x_{F_\alpha} + x_{F_\beta}}{144} \psi_7(x_w, x_t) + \frac{1}{8} \phi_2(x_{F_\alpha}, x_{F_\beta}) \psi_5(x_w, x_t) \\ & \left. \left. + \psi_8(x_w, x_t) \right) \right] \mathcal{O}_2 + \left(\zeta_{\alpha\beta}^{\text{L}*} \zeta_{\alpha\beta}^{\text{R}} + \zeta_{\alpha\beta}^{\text{R}*} \zeta_{\alpha\beta}^{\text{L}} \right) (x_{F_\alpha} x_{F_\beta})^{1/2} \left[-\frac{1}{12\epsilon} \psi_2(x_w, x_t) \right. \\ & + \frac{1}{12} \varrho_{1,1}(x_{F_\alpha}, x_{F_\beta}) \psi_2(x_w, x_t) + \phi_3(x_{F_\alpha}, x_{F_\beta}) \psi_1(x_w, x_t) + \psi_9(x_w, x_t) + Q_u \left(-\frac{1}{12\epsilon} \psi_6(x_w, x_t) \right. \\ & \left. \left. + \frac{1}{12} \varrho_{1,1}(x_{F_\alpha}, x_{F_\beta}) \psi_6(x_w, x_t) + \phi_4(x_{F_\alpha}, x_{F_\beta}) \psi_5(x_w, x_t) + \psi_{10}(x_w, x_t) \right) \right] \mathcal{O}_2 \\ & + \left(\zeta_{\alpha\beta}^{\text{L}*} \zeta_{\alpha\beta}^{\text{L}} + \zeta_{\alpha\beta}^{\text{R}*} \zeta_{\alpha\beta}^{\text{R}} \right) \left[\frac{1}{24\epsilon} \left\{ \psi_5 + (x_{F_\alpha} + x_{F_\beta}) \psi_6 \right\} (x_w, x_t) + \frac{x_{F_\alpha} + x_{F_\beta}}{144} \psi_7(x_w, x_t) \right. \\ & \left. - \frac{1}{24} \varrho_{2,1}(x_{F_\alpha}, x_{F_\beta}) \psi_6(x_w, x_t) + \frac{1}{8} \phi_2(x_{F_\alpha}, x_{F_\beta}) \psi_5(x_w, x_t) + \psi_8(x_w, x_t) \right] \mathcal{O}_6 \\ & + \left(\zeta_{\alpha\beta}^{\text{L}*} \zeta_{\alpha\beta}^{\text{R}} + \zeta_{\alpha\beta}^{\text{R}*} \zeta_{\alpha\beta}^{\text{L}} \right) (x_{F_\alpha} x_{F_\beta})^{1/2} \left[-\frac{1}{12\epsilon} \psi_6(x_w, x_t) + \frac{1}{12} \varrho_{1,1}(x_{F_\alpha}, x_{F_\beta}) \psi_6(x_w, x_t) \right. \\ & \left. \left. + \phi_4(x_{F_\alpha}, x_{F_\beta}) \psi_5(x_w, x_t) + \psi_{10}(x_w, x_t) \right] \mathcal{O}_6 \right\} + \dots \end{aligned} \quad (14)$$

Adding the counter terms to bare Lagrangian Eq. (6), we cancel the ultraviolet divergence there. Under

our approximation, the resulting effective Lagrangian is written as

$$\begin{aligned}
\widehat{\mathcal{L}}_W^{\text{eff}} = & \frac{\sqrt{2}G_F\alpha_e x_w}{\pi s_w^2 Q_d} V_{ts}^* V_{tb} \left\{ \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) \left[F_1(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) + Q_u F_2(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \right] \mathcal{O}_2 \right. \\
& + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L - \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) F_3(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \mathcal{O}_2 \\
& + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) (x_{F_\alpha} x_{F_\beta})^{1/2} \left[F_4(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) + Q_u F_5(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \right] \mathcal{O}_2 \\
& + \left(\zeta_{\alpha\beta}^L \zeta_{\alpha\beta}^{R*} - \zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R \right) (x_{F_\alpha} x_{F_\beta})^{1/2} F_6(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \mathcal{O}_9 \\
& + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) \left[F_2(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) + T_\alpha^c F_7(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \right] \mathcal{O}_6 \\
& + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) (x_{F_\alpha} x_{F_\beta})^{1/2} \left[F_5(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) + T_\alpha^c F_8(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \right] \mathcal{O}_6 \\
& + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L - \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) T_\alpha^c F_9(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \mathcal{O}_6 \\
& \left. + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R - \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) (x_{F_\alpha} x_{F_\beta})^{1/2} T_\alpha^c F_{10}(x_w, x_t, x_{F_\alpha}, x_{F_\beta}) \mathcal{O}_{12} \right\} + \dots, \tag{15}
\end{aligned}$$

which only depends on the masses of virtual fields. It should be clarified that the corrections to the coefficients of $\mathcal{O}_{9,12}$ do not depend on the concrete renormalization scheme adopted here since the relevant terms from the bare Lagrangian do not contain the ultraviolet divergence. In the limit $z \ll x, y$, the function $\Phi(x, y, z)$ can be approximated in powers of z as

$$\begin{aligned}
\Phi(x, y, z) = & \varphi_0(x, y) + z\varphi_1(x, y) + \frac{z^2}{2!}\varphi_2(x, y) + \frac{z^3}{3!}\varphi_3(x, y) + 2z(\ln z - 1)\pi_1(x, y) \\
& + 2z^2 \left(\frac{\ln z}{2!} - \frac{3}{4} \right) \pi_2(x, y) + 2z^3 \left(\frac{\ln z}{3!} - \frac{11}{36} \right) \pi_3(x, y) + \dots \tag{16}
\end{aligned}$$

with

$$\begin{aligned}
\pi_1(x, y) = & 1 + \varrho_{1,1}(x, y), \quad \pi_2(x, y) = -\frac{x+y}{(x-y)^2} - \frac{2xy}{(x-y)^3} \ln \frac{y}{x}, \\
\pi_3(x, y) = & -\frac{1}{(x-y)^2} - \frac{12xy}{(x-y)^4} - \frac{6xy(x+y)}{(x-y)^5} \ln \frac{y}{x}, \tag{17}
\end{aligned}$$

and the concrete expressions of function $\varphi_i(x, y)$ ($i=0, 1, 2, 3$) can be found in the appendix. Using the asymptotic expressions in Eq. (16), we derive the leading contributions contained in Eq. (15) under the assumption $m_F = m_{F_\alpha} = m_{F_\beta} \gg m_w$,

$$\begin{aligned}
\widehat{\mathcal{L}}_W^{\text{eff}} \approx & \frac{\sqrt{2}G_F\alpha_e x_w}{\pi s_w^2 Q_d} V_{ts}^* V_{tb} \left\{ \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) \times \left[\left\{ -\frac{1-3Q_\beta}{8} \frac{\partial^2 \varrho_{2,1}}{\partial x_w^2} - \frac{2-3Q_\beta}{8} \frac{\partial \varrho_{1,1}}{\partial x_w} \right. \right. \\
& \left. \left. - \frac{1}{144} \frac{\partial^4 \varrho_{4,1}}{\partial x_w^4} - \frac{1}{48} \frac{\partial^3 \varrho_{3,1}}{\partial x_w^3} \right\} (x_w, x_t) + Q_u \left\{ \frac{1}{144} \frac{\partial^4 \varrho_{4,1}}{\partial x_w^4} - \frac{1}{12} \frac{\partial^3 \varrho_{3,1}}{\partial x_w^3} - \frac{29}{72} \frac{\partial^2 \varrho_{2,1}}{\partial x_w^2} - \frac{11}{12} \frac{\partial \varrho_{1,1}}{\partial x_w} \right\} (x_w, x_t) \right] \mathcal{O}_2 \\
& - \frac{1-Q_\beta}{8} \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L - \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) \frac{\partial \varrho_{1,1}}{\partial x_w} (x_w, x_t) \mathcal{O}_2 + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) \left[\left\{ \frac{1}{144} \frac{\partial^4 \varrho_{4,1}}{\partial x_w^4} - \frac{1}{16} \frac{\partial^3 \varrho_{3,1}}{\partial x_w^3} \right. \right. \\
& \left. \left. + \frac{1}{4} \frac{\partial^2 \varrho_{2,1}}{\partial x_w^2} + \frac{1}{16} \frac{\partial \varrho_{1,1}}{\partial x_w} \right\} (x_w, x_t) + Q_u \left\{ -\frac{1}{144} \frac{\partial^4 \varrho_{4,1}}{\partial x_w^4} + \frac{1}{12} \frac{\partial^3 \varrho_{3,1}}{\partial x_w^3} - \frac{5}{24} \frac{\partial^2 \varrho_{2,1}}{\partial x_w^2} - \frac{1}{12} \frac{\partial \varrho_{1,1}}{\partial x_w} \right\} (x_w, x_t) \right] \mathcal{O}_2 \\
& + \frac{1}{8} \left(\zeta_{\alpha\beta}^L \zeta_{\alpha\beta}^{R*} - \zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R \right) \frac{\partial^2 \varrho_{2,1}}{\partial x_w^2} (x_w, x_t) \mathcal{O}_9 + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^L + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^R \right) \left[\left\{ \frac{1}{144} \frac{\partial^4 \varrho_{4,1}}{\partial x_w^4} - \frac{1}{12} \frac{\partial^3 \varrho_{3,1}}{\partial x_w^3} \right. \right. \\
& \left. \left. - \frac{29}{72} \frac{\partial^2 \varrho_{2,1}}{\partial x_w^2} - \frac{11}{12} \frac{\partial \varrho_{1,1}}{\partial x_w} \right\} (x_w, x_t) + T_\alpha^c \left\{ \frac{3}{8} \frac{\partial^2 \varrho_{2,1}}{\partial x^2} (x, y) + \frac{5}{4} \frac{\partial \varrho_{1,1}}{\partial x} \right\} (x_w, x_t) \right] \mathcal{O}_6
\end{aligned}$$

$$\begin{aligned}
& + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R + \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) \left\{ -\frac{1}{144} \frac{\partial^4 \varrho_{4,1}}{\partial x_w^4} + \frac{1}{12} \frac{\partial^3 \varrho_{3,1}}{\partial x_w^3} - \frac{5}{24} \frac{\partial^2 \varrho_{2,1}}{\partial x_w^2} - \frac{1}{12} \frac{\partial \varrho_{1,1}}{\partial x_w} \right\} (x_w, x_t) \mathcal{O}_6 \\
& + \left(\zeta_{\alpha\beta}^{L*} \zeta_{\alpha\beta}^R - \zeta_{\alpha\beta}^{R*} \zeta_{\alpha\beta}^L \right) T_\alpha^c \left\{ \frac{1}{16} \frac{\partial \varrho_{1,1}}{\partial x_w} + \frac{7}{24} \frac{\partial^2 \varrho_{2,1}}{\partial x_w^2} \right\} (x_w, x_t) \mathcal{O}_{12} \Big\} + \dots, \tag{18}
\end{aligned}$$

where ellipses represent those relatively unimportant corrections. Comparing the result in Eq. (15), the contributions from the corresponding diagrams contain the additional suppressed factor m_b^2/Λ_{EW}^2 when both of the virtual charged gauge bosons in Fig. 1(a) are replaced with the charged Goldstone G^\pm . However, we should consider the corrections from those two loop diagrams in which one of virtual charged gauge bosons is replaced with the charged Goldstone G^\pm since it represents the longitudinal component of the charged gauge boson in the nonlinear R_ξ gauge. As the closed fermion loop is attached to the virtual W^\pm gauge boson and charged Higgs simultaneously, the corresponding triangle diagrams belong to the famous Barr-Zee type diagrams [12]. It is shown [13] that this type of diagram contributes to the important corrections to the effective Lagrangian. For the reason mentioned above, we also generalize the result directly to the diagrams in which a closed heavy loop is attached to the virtual H^\pm and W^\pm fields simultaneously.

2.2 The corrections from the diagrams where a closed heavy fermion loop is attached to the virtual W^\pm , G^\pm (H^\pm) bosons

Similarly, the renormalizable interaction among the EW charged Goldstone/Higgs G^\pm (H^\pm) and the heavy fermions $F_{\alpha,\beta}$ can be expressed in a more uni-

versal form as

$$\begin{aligned}
\mathcal{L}_{S^\pm FF} &= \frac{e}{s_w} \left[G^- \bar{F}_\alpha (\mathcal{G}_{\alpha\beta}^{c,L} \omega_- + \mathcal{G}_{\alpha\beta}^{c,R} \omega_+) F_\beta \right. \\
&\quad \left. + H^- \bar{F}_\alpha (\mathcal{H}_{\alpha\beta}^{c,L} \omega_- + \mathcal{H}_{\alpha\beta}^{c,R} \omega_+) F_\beta \right] + \text{h.c.}, \tag{19}
\end{aligned}$$

where the concrete expressions of $\mathcal{G}_{\alpha\beta}^{c,L,R}$, $\mathcal{H}_{\alpha\beta}^{c,L,R}$ depend on the models employed in our calculation, the conservation of electric charge requires $Q_\beta - Q_\alpha = 1$. Generally, the couplings among the charged Goldstone/Higgs and quarks are written as

$$\begin{aligned}
\mathcal{L}_{S^\pm \bar{d}u} &= \frac{eV_{ud}^*}{\sqrt{2}s_w} \left\{ G^- \bar{d} \left[\frac{m_u}{m_w} \omega_+ + \frac{m_d}{m_w} \omega_- \right] u \right. \\
&\quad \left. + H^- \bar{d} \left[\frac{m_u}{m_w} \omega_+ - \mathcal{B}_c \frac{m_d}{m_w} \omega_- \right] u \right\} + \text{h.c.}, \tag{20}
\end{aligned}$$

where the parameter \mathcal{B}_c also depends on the concrete models adopted in our analysis. In full theory, the couplings in Eq. (19) induce the corrections to the effective Lagrangian for $b \rightarrow s\gamma$ through the diagrams presented in Fig. 1(b), (c).

Since there is no mixing between the charged gauge boson and the charged Higgs/Goldstone at tree level, the corresponding corrections from the diagrams presented in Fig. 1(b), (c) to the bare effective Lagrangian do not include the ultraviolet divergence, and can be formulated as

$$\begin{aligned}
\widehat{\mathcal{L}}_{WH}^{\text{eff}} &= \frac{\sqrt{2}G_F\alpha_e\mathcal{B}_c}{\pi s_w^2 Q_d} V_{ts}^* V_{tb} \left\{ (x_{F_\beta} x_w)^{1/2} P_1(x_w, x_{H^\pm}, x_t, x_{F_\alpha}, x_{F_\beta}) \times \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_5 \right. \right. \\
&\quad - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_{11} \Big] + (x_w x_{F_\alpha})^{1/2} P_2(x_w, x_{H^\pm}, x_t, x_{F_\alpha}, x_{F_\beta}) \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_5 \right. \\
&\quad - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_{11} \Big] + (x_w x_{F_\beta})^{1/2} P_3(x_w, x_{H^\pm}, x_t, x_{F_\alpha}, x_{F_\beta}) \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_5 \right. \\
&\quad - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_{11} \Big] + (x_w x_{F_\alpha})^{1/2} P_4(x_w, x_{H^\pm}, x_t, x_{F_\alpha}, x_{F_\beta}) \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_5 \right. \\
&\quad - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_{11} \Big] + (x_{F_\beta} x_w)^{1/2} P_5(x_w, x_{H^\pm}, x_t, x_{F_\alpha}, x_{F_\beta}) \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_8 \right. \\
&\quad - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_{13} \Big] + (x_{F_\beta} x_w)^{1/2} P_5(x_w, x_{H^\pm}, x_t, x_{F_\alpha}, x_{F_\beta}) \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_8 \right. \\
&\quad + i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_{13} \Big] + (x_{F_\beta} x_w)^{1/2} P_6(x_w, x_{H^\pm}, x_t, x_{F_\alpha}, x_{F_\beta}) \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_8 \right. \\
&\quad - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_{13} \Big] - (x_{F_\beta} x_w)^{1/2} P_6(x_w, x_{H^\pm}, x_t, x_{F_\alpha}, x_{F_\beta}) \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_8 \right. \\
&\quad \left. + i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_{13} \right] \Big\}, \quad \widehat{\mathcal{L}}_{WG}^{\text{eff}} = \widehat{\mathcal{L}}_{WH}^{\text{eff}} (\mathcal{B}_c \rightarrow 1, \mathcal{G}_{\beta\alpha}^{c,L,R} \rightarrow \mathcal{H}_{\beta\alpha}^{c,L,R}, x_{H^\pm} \rightarrow x_w). \tag{21}
\end{aligned}$$

The expressions of form factors $P_i(x, y, z, u, w)$ ($i=1, \dots, 4$) can be found in appendix.

Using the asymptotic expressions of $\Phi(x, y, z)$ at the limit $x, y \gg z$ in Eq. (16), we simplify the expressions of Eq. (21) in the limit $m_F = m_{F_\alpha} = m_{F_\beta} \gg m_w$ as:

$$\begin{aligned}
\widehat{\mathcal{L}}_{\text{WH}}^{\text{eff}} \approx & \frac{\sqrt{2}G_F\alpha_e\mathcal{B}_c m_w}{\pi s_w^2 Q_d m_F} V_{ts}^* V_{tb} \left\{ \left[\frac{21}{64} - \frac{5}{288} Q_\beta \right. \right. \\
& + \left. \left(\frac{3}{16} + \frac{Q_\beta}{48} \right) \left(\ln m_F^2 - \frac{\varrho_{2,1}(m_w^2, m_t^2) - \varrho_{2,1}(m_{H^\pm}^2, m_t^2)}{m_w^2 - m_{H^\pm}^2} \right) \right] \\
& \times \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_5 - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_{11} \right] \\
& + \left[\frac{19 - 20Q_\beta}{144} + \frac{2 - 4Q_\beta}{48} \left(\ln m_F^2 - \frac{\varrho_{2,1}(m_w^2, m_t^2) - \varrho_{2,1}(m_{H^\pm}^2, m_t^2)}{m_w^2 - m_{H^\pm}^2} \right) \right] \\
& \times \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_5 - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_{11} \right] \\
& - \left[\frac{16}{144} + \frac{2 + 6Q_\beta}{48} \left(\ln m_F^2 - \frac{\varrho_{2,1}(m_w^2, m_t^2) - \varrho_{2,1}(m_{H^\pm}^2, m_t^2)}{m_w^2 - m_{H^\pm}^2} \right) \right] \\
& \times \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_5 - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_{11} \right] \\
& - \left[\frac{2Q_\beta}{144} + \frac{6 - 2Q_\beta}{48} \left(\ln m_F^2 - \frac{\varrho_{2,1}(m_w^2, m_t^2) - \varrho_{2,1}(m_{H^\pm}^2, m_t^2)}{m_w^2 - m_{H^\pm}^2} \right) \right] \\
& \times \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_5 - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_{11} \right] \\
& - \frac{1}{8\sqrt{2}} \left[1 + \ln m_F^2 - \frac{\varrho_{2,1}(m_w^2, m_t^2) - \varrho_{2,1}(m_{H^\pm}^2, m_t^2)}{m_w^2 - m_{H^\pm}^2} \right] \\
& \times \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_8 - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_{13} \right] \\
& - \frac{1}{8\sqrt{2}} \left[1 + \ln m_F^2 - \frac{\varrho_{2,1}(m_w^2, m_t^2) - \varrho_{2,1}(m_{H^\pm}^2, m_t^2)}{m_w^2 - m_{H^\pm}^2} \right] \\
& \times \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_8 + i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_{13} \right] \\
& - \frac{1}{4\sqrt{2}} \left[1 + \ln m_F^2 - \frac{\varrho_{2,1}(m_w^2, m_t^2) - \varrho_{2,1}(m_{H^\pm}^2, m_t^2)}{m_w^2 - m_{H^\pm}^2} \right] \\
& \times \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_8 - i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^L - \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^R \right) \mathcal{O}_{13} \right] \\
& + \frac{1}{4\sqrt{2}} \left[1 + \ln m_F^2 - \frac{\varrho_{2,1}(m_w^2, m_t^2) - \varrho_{2,1}(m_{H^\pm}^2, m_t^2)}{m_w^2 - m_{H^\pm}^2} \right] \\
& \times \left. \left[\Re \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_8 + i\Im \left(\mathcal{H}_{\beta\alpha}^{c,L} \zeta_{\alpha\beta}^R + \mathcal{H}_{\beta\alpha}^{c,R} \zeta_{\alpha\beta}^L \right) \mathcal{O}_{13} \right] \right\}. \tag{22}
\end{aligned}$$

The results indicate that the corrections to the effective Lagrangian from the diagrams presented in Fig. 1(b), (c) are suppressed in the limit $m_F = m_{F_\alpha} = m_{F_\beta} \gg m_w$ unless the couplings $\mathcal{H}_{\beta\alpha}^{c,L,R}$ violate the decoupling theorem.

It is well known that the short distance QCD affects the rare B decay strongly. At the NLO level [14],

the Wilson coefficients at the bottom quark scale are given as

$$\begin{aligned}
\tilde{C}_5(\mu_b) & \approx 0.67(\tilde{C}_5(\mu_w) - 0.42\tilde{C}_8(\mu_w) - 0.88), \\
\tilde{C}_8(\mu_b) & \approx 0.7(\tilde{C}_8(\mu_w) + 0.12), \tag{23}
\end{aligned}$$

where the corresponding Wilson coefficients at EW

scale are written as

$$\begin{aligned}\tilde{C}_5(\mu_w) &= C_2(\mu_w) + C_5(\mu_w) + C_9(\mu_w) + C_{11}(\mu_w), \\ \tilde{C}_8(\mu_w) &= C_6(\mu_w) + C_8(\mu_w) + C_{12}(\mu_w) + C_{13}(\mu_w).\end{aligned}\quad (24)$$

As an application, we investigate the relative corrections to the branching ratio of rare decay $B \rightarrow X_s \gamma$ originating from those sectors.

3 The corrections to branching ratio of $B \rightarrow X_s \gamma$

In order to eliminate the strong dependence on the b -quark mass, the branching ratio is usually normalized by the decay rate of the B meson semileptonic decay,

$$\begin{aligned}\frac{\Gamma(B \rightarrow X_s \gamma)}{\Gamma(B \rightarrow X_c e \bar{\nu})} &= \frac{\Gamma(b \rightarrow s \gamma)}{\Gamma(b \rightarrow c e \bar{\nu})} \\ &= \frac{2\alpha_e}{3\pi\rho(y)\chi(y)} |\tilde{C}_7(\mu_b)|^2,\end{aligned}\quad (25)$$

where $\rho(y) = 1 - 8y + 8y^3 - y^4 - 12y^2 \ln y$ is the phase-space factor with $y = (m_c/m_b)^2$, and

$$\chi(y) = 1 - \frac{2\alpha_s(m_b)}{3\pi} f(y),$$

with $f(m_c^2/m_b^2) \approx 2.4$. From now on we shall assume the value $BR(B \rightarrow X_c e \bar{\nu}) = 10.5\%$ for the semileptonic branching ratio, $\alpha_s(m_z) = 0.118$, $\alpha_e(m_z) = 1/127$. For the mass spectrum of SM, we take $m_t = 174$ GeV, $m_b = 4.2$ GeV, $m_w = 80.42$ GeV and $m_z = 91.19$ GeV. In the CKM matrix, we apply the Wolfenstein parameterization and set $A = 0.85$, $\lambda = 0.22$, $\rho = 0.22$ and $\eta = 0.35$ [15].

Without loss of generality, we adopt the universal assumptions on those couplings and mass spectrum of new physics as

$$\begin{aligned}\zeta_{\alpha\beta}^L &= \zeta_{\alpha\beta}^R = \mathcal{H}_{\beta\alpha}^{c,L} = \mathcal{H}_{\beta\alpha}^{c,R} = \mathcal{G}_{\beta\alpha}^{c,L} = \mathcal{G}_{\beta\alpha}^{c,R} = e^{i\phi_{CP}}, \\ m_{F_\alpha} &= m_{F_\beta} = m_{H^\pm} = \Lambda_{NP}.\end{aligned}\quad (26)$$

To continue our discussion, we assume the electric charge of heavy fermions as $Q_\beta = 2/3, 1, -1/3, 4/3, 5/3$, which corresponds to the electric charge of another heavy fermion in inner loop $Q_\alpha = -1/3, 0, 4/3, 1/3, 2/3$, respectively. In addition, we also assume that those heavy fermions with fractional electric charge all take part in the strong interaction.

Many extensions of the SM include the heavy fermion fields with $Q_\beta = 2/3, Q_\alpha = -1/3$. In the extensions of SM with large [16] or warped [17] extra dimensions, the KK excitations of up- and down-type quarks form a closed fermion loop which can be attached to the zero modes of charged gauge boson and Higgs. In the minimal supersymmetric extension of

SM (MSSM) [18], the closed fermion loop composed by chargino ($Q_\beta = 1$) and neutralino ($Q_\alpha = 0$) can be attached to the charged gauge boson and Higgs. In the $3-3-1$ model [19], the electric charge of exotic quarks are assigned as $Q_\beta = 4/3, 5/3$.

In many EW extensions of the SM, the couplings among the charged Higgs and quarks contain an enhancing factor \mathcal{B}_c . For example, $\mathcal{B}_c = \tan\beta$ in the MSSM is a strong enhancing factor at large $\tan\beta$ limit. In other EW theories, such as the littlest Higgs [20], $3-3-1$ model, the couplings among the charged Higgs and quarks also contain a nontrivial enhancing factor $\mathcal{B}_c \gg 1$. In our numerical discussion, we assume the possible enhancing factor with a trivial value $\mathcal{B}_c = 1$ or a nontrivial value $\mathcal{B}_c = 10$.

Including NLO QCD effects, we plot the relative corrections to one loop SM theoretical prediction on the branching ratio of the inclusive $B \rightarrow X_s \gamma$ decay versus the possible CP violation phase ϕ_{CP} with $\mathcal{B}_c = 1$ in Fig. 2(a). Depending on concrete choices of

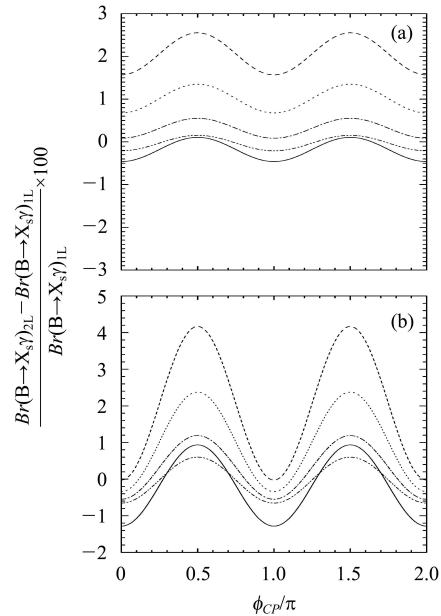


Fig. 2. The relative correction to the branching ratio of the inclusive $B \rightarrow X_s \gamma$ decay versus the possible CP violation phases ϕ_{CP} . Here the solid-line represents the theoretical correction with $Q_\beta = 1, Q_\alpha = 0$, the dashed-line represents the theoretical correction with $Q_\beta = -1/3, Q_\alpha = -4/3$, the dotted-line represents the theoretical correction with $Q_\beta = 2/3, Q_\alpha = -1/3$, the dashed-dotted-line represents the theoretical correction with $Q_\beta = 4/3, Q_\alpha = 1/3$, and the dash-dotted-dotted-line represents the theoretical correction with $Q_\beta = 5/3, Q_\alpha = 2/3$, respectively. The enhancing factor is chosen as the trivial $\mathcal{B}_c = 1$ in (a), or a nontrivial $\mathcal{B}_c = 10$ in (b).

Q_β and the CP violation phase ϕ_{CP} , the relative corrections to the branching ratio from those two loop diagrams can reach 2.5%. Compared with the corrections from QCD, the modifications from those two loop EW diagrams are unimportant certainly. Nevertheless, those effects can be observed possibly in the experiment along with improvement of the theoretical analysis and the increase in experimental precision. Taking the enhancing factor $\mathcal{B}_c = 10$, we plot the relative corrections to one loop SM theoretical prediction on the branching ratio of the inclusive $B \rightarrow X_s \gamma$ decay versus the possible CP violation phase ϕ_{CP} in Fig. 2(b). Because the contributions from two loop Barr-Zee diagrams are enhanced drastically, the relative corrections to one loop SM theoretical prediction on the branching ratio of $B \rightarrow X_s \gamma$ can reach 4.5%. Although the two-loop EW corrections cannot compete with that from QCD, we cannot neglect the corrections with this magnitude.

Assuming $\mathcal{B}_c = 10$ and $\phi_{CP} = \pi/2$, we plot the relative corrections to one loop SM theoretical prediction on the branching ratio of $B \rightarrow X_s \gamma$ varying with the energy scale of new physics Λ_{NP} in Fig. 3. Since the intervention between the top quark and the particles in new physics, the relative corrections reach the maximum ($\sim 4.5\%$) around $\Lambda_{NP} = 200$ GeV. With increasing Λ_{NP} , the relative corrections become

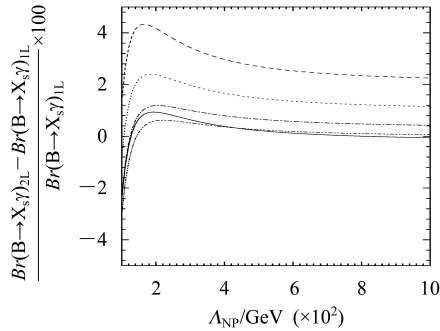


Fig. 3. The relative correction to the branching ratio of the inclusive $B \rightarrow X_s \gamma$ decay versus the energy scale of new physics Λ_{NP} . Here, the solid line represents the theoretical correction with $Q_\beta = 1$, $Q_\alpha = 0$, the dashed line represents the theoretical correction with $Q_\beta = -1/3$, $Q_\alpha = -4/3$, the dotted line represents the theoretical correction with $Q_\beta = 2/3$, $Q_\alpha = -1/3$, the dashed-dotted line represents the theoretical correction with $Q_\beta = 4/3$, $Q_\alpha = 1/3$, and the dash-dotted line represents the theoretical correction with $Q_\beta = 5/3$, $Q_\alpha = 2/3$, respectively.

smaller and smaller. At $\Lambda_{NP} = 1$ TeV, the relative corrections are about 2%.

In the SM, the CP asymmetry of the $B \rightarrow X_s \gamma$ process is calculated to be rather small: $A_{CP} \sim 0.5\%$ [21]. Certainly, the new CP violation phases may induce the observable effects on the CP asymmetry of $B \rightarrow X_s \gamma$. However, the numerical results indicate that the corrections from those two loop diagrams to the CP asymmetry of $B \rightarrow X_s \gamma$ are rather small. The relative correction to one loop SM theoretical prediction on the branching ratio of $B \rightarrow X_s \gamma$ is already above 8% when $\mathcal{B}_c = 30$, $\Lambda_{NP} = 200$ GeV and $\phi_{CP} = \pi/2$. The corresponding correction from those two loop diagrams to the CP asymmetry is still smaller than 1% under our universal assumptions on the parameter space.

As mentioned above, the universal assumptions on the couplings and mass spectrum of new physics are adopted in our numerical analysis. In concrete EW extensions of the SM, this choice is a very simple assumption on parameter space. However, the numerical results given above reflect the typical magnitude of corrections from those two loop diagrams to the branching ratio of $B \rightarrow X_s \gamma$ unless there is contingent cancelation among different sectors of those two loop diagrams in concrete extensions of the SM.

4 Conclusions

Applying an effective Lagrangian method and an on-shell scheme, we analyze the EW corrections to the rare decay $b \rightarrow s + \gamma$ from some special two loop diagrams in which a closed heavy fermion loop is attached to the virtual charged gauge bosons or Higgs. The analysis shows that the final results satisfy the decoupling theorem explicitly when the interactions among Higgs and heavy fermions do not contain the nondecoupling couplings. Adopting the universal assumptions on the relevant couplings and masses of new physics, we present the relative corrections from those two loop diagrams to one loop SM theoretical prediction on the branching ratio of $B \rightarrow X_s \gamma$ varying with the possible CP violation phases and energy scale of new physics. The numerical results indicate that the relative corrections from those two loop diagrams can reach 5% if there is not contingent cancelation among different sectors of corresponding contributions.

Appendix A

Form factors in the two-loop Wilson coefficients

The definition of $\Psi(x, y, z)$ is written as

1) $\lambda^2 > 0$, $\sqrt{y} + \sqrt{z} < \sqrt{x}$:

$$\Psi(x, y, z) = 2 \ln \left(\frac{x+y-z-\lambda}{2x} \right) \ln \left(\frac{x-y+z-\lambda}{2x} \right) - \ln \frac{y}{x} \ln \frac{z}{x} - 2L_{i_2} \left(\frac{x+y-z-\lambda}{2x} \right) - 2L_{i_2} \left(\frac{x-y+z-\lambda}{2x} \right) + \frac{\pi^2}{3}, \quad (\text{A1})$$

where $L_{i_2}(x)$ is the spence function;

2) $\lambda^2 > 0$, $\sqrt{x} + \sqrt{z} < \sqrt{y}$:

$$\Psi(x, y, z) = \text{Eq. (A1)}(x \leftrightarrow y); \quad (\text{A2})$$

3) $\lambda^2 > 0$, $\sqrt{x} + \sqrt{y} < \sqrt{z}$:

$$\Psi(x, y, z) = \text{Eq. (A1)}(x \leftrightarrow z); \quad (\text{A3})$$

4) $\lambda^2 < 0$:

$$\Psi(x, y, z) = 2 \left\{ Cl_2 \left(2 \arccos \left(\frac{-x+y+z}{2\sqrt{yz}} \right) \right) + Cl_2 \left(2 \arccos \left(\frac{x-y+z}{2\sqrt{xz}} \right) \right) + Cl_2 \left(2 \arccos \left(\frac{x+y-z}{2\sqrt{xy}} \right) \right) \right\}, \quad (\text{A4})$$

where $Cl_2(x)$ denotes the Clausen function.

The expressions of $\varphi_0(x, y)$, $\varphi_1(x, y)$, $\varphi_2(x, y)$ and $\varphi_3(x, y)$ are given as

$$\varphi_0(x, y) = \begin{cases} (x+y) \ln x \ln y + (x-y) \Theta(x, y), & x > y; \\ 2x \ln^2 x, & x = y; \\ (x+y) \ln x \ln y + (y-x) \Theta(y, x), & x < y. \end{cases} \quad (\text{A5})$$

$$\varphi_1(x, y) = \begin{cases} -\ln x \ln y - \frac{x+y}{x-y} \Theta(x, y), & x > y; \\ 4 - 2 \ln x - \ln^2 x, & x = y; \\ -\ln x \ln y - \frac{x+y}{y-x} \Theta(y, x), & x < y, \end{cases} \quad (\text{A6})$$

$$\varphi_2(x, y) = \begin{cases} \frac{(2x^2 + 6xy) \ln x - (6xy + 2y^2) \ln y}{(x-y)^3} - \frac{4xy}{(x-y)^3} \Theta(x, y), & x > y; \\ -\frac{5}{9x} + \frac{2}{3x} \ln x, & x = y; \\ \frac{(2x^2 + 6xy) \ln x - (6xy + 2y^2) \ln y}{(x-y)^3} - \frac{4xy}{(y-x)^3} \Theta(y, x), & x < y, \end{cases} \quad (\text{A7})$$

$$\varphi_3(x, y) = \begin{cases} -\frac{12xy(x+y)}{(x-y)^5} \Theta(x, y) - \frac{2(x^2 + 6xy + y^2)}{(x-y)^4} \\ + \frac{2(x^3 + 20x^2y + 11xy^2) \ln x - 2(y^3 + 20xy^2 + 11x^2y) \ln y}{(x-y)^5}, & x > y; \\ -\frac{53}{150x^2} + \frac{1}{5x^2} \ln x, & x = y; \\ -\frac{12xy(x+y)}{(y-x)^5} \Theta(y, x) - \frac{2(x^2 + 6xy + y^2)}{(x-y)^4} \\ + \frac{2(x^3 + 20x^2y + 11xy^2) \ln x - 2(y^3 + 20xy^2 + 11x^2y) \ln y}{(x-y)^5}, & x < y, \end{cases} \quad (\text{A8})$$

with

$$\Theta(x, y) = \ln x \ln \frac{y}{x} - 2 \ln(x-y) \ln \frac{y}{x} - 2L_{i_2} \left(\frac{y}{x} \right) + \frac{\pi^2}{3}. \quad (\text{A9})$$

The functions adopted in the text are written as

$$\begin{aligned}
\rho_{i,j}(x,y) &= \frac{x^i \ln^j x - y^i \ln^j y}{x-y}, \quad \Omega_i(x,y;u,v) = \frac{x^i \Phi(x,u,v) - y^i \Phi(y,u,v)}{x-y}, \\
F_1(x,y,z,u) &= \frac{1}{24} \rho_{2,1}(z,u) \left[\frac{\partial^4 \rho_{3,1}}{\partial x^4} + 3 \frac{\partial^3 \rho_{2,1}}{\partial x^3} \right] (y,x) - \left\{ \frac{1}{8} \frac{\partial \rho_{2,1}}{\partial z} - \frac{1}{24} \frac{\partial^2 \rho_{3,1}}{\partial z^2} - \frac{3x}{32} \frac{\partial^2 \rho_{2,1}}{\partial z^2} + \frac{x}{16} \frac{\partial^3 \rho_{3,1}}{\partial z^3} \right. \\
&\quad \left. - \frac{x}{128} \frac{\partial^4 \rho_{4,1}}{\partial z^4} \right\} (z,u) \left\{ \frac{\partial^4 \rho_{4,1}}{\partial x^4} - 3 \frac{\partial^3 \rho_{3,1}}{\partial x^3} \right\} (x,y) - \frac{1}{18} \frac{\partial^4 \rho_{4,1}}{\partial x^4} (x,y) + \frac{1}{24} \frac{\partial^3 \rho_{3,1}}{\partial x^3} (x,y) - \frac{1}{4} \frac{\partial^2 \rho_{2,1}}{\partial x^2} (x,y) \\
&\quad - \frac{1}{48} \left\{ [(z+u) + 2(z \ln z + u \ln u)] \frac{\partial^4 \rho_{3,1}}{\partial x^4} (x,y) - 2(z-u)^2 (1 + \rho_{1,1}(z,u)) \frac{\partial^4 \rho_{2,1}}{\partial x^4} (x,y) \right. \\
&\quad \left. - [9(z+u) + 6z \ln z - 6u \ln u] \frac{\partial^3 \rho_{2,1}}{\partial x^3} (x,y) - (4 + 18Q_\beta + 6(2 - Q_\beta) \ln u) \frac{\partial^2 \rho_{2,1}}{\partial x^2} (x,y) \right. \\
&\quad \left. - [2 + 6(1 - Q_\beta) \ln u] \frac{\partial \rho_{1,1}}{\partial x} (x,y) - 6(z-u)^2 (1 + \rho_{1,1}(z,u)) \frac{\partial^3 \rho_{1,1}}{\partial x^3} (x,y) \right. \\
&\quad \left. + 6[(-4 + 2Q_\beta)(z + z \ln z) + (-2 - 2Q_\beta)u + (1 - 2Q_\beta)u \ln u] \frac{\partial^2 \rho_{1,1}}{\partial x^2} (x,y) \right. \\
&\quad \left. + \frac{\partial^4}{\partial x^4} [(z-u)^2 \Omega_1 - \Omega_3] (x,y; z,u) - 6 \frac{\partial^4}{\partial x^3 \partial u} [u(z-u) \Omega_1 + u \Omega_2] (x,y; z,u) \right. \\
&\quad \left. + 6 \frac{\partial^4}{\partial x^2 \partial u^2} [u(z+u) \Omega_1 - u \Omega_2] (x,y; z,u) - 2 \frac{\partial^4}{\partial x \partial u^3} [u^2(z-u) \Omega_0 + u^2 \Omega_1] (x,y; z,u) \right. \\
&\quad \left. + 3 \frac{\partial^3}{\partial x^3} [(z-u)^2 \Omega_0 + 4(z-u) \Omega_1 + 3 \Omega_2] (x,y; z,u) + 6 \frac{\partial^3}{\partial x \partial u^2} \left[\left(\frac{5}{2} - Q_\beta \right) u(z-u) \Omega_0 + \frac{3}{2} u \Omega_1 \right] (x,y; z,u) \right. \\
&\quad \left. - 3 \frac{\partial^3}{\partial x^2 \partial u} [3u(z-u) \Omega_0 + ((6 - Q_\beta)z + (11 - 3Q_\beta)u) \Omega_1 - (6 - Q_\beta) \Omega_2] (x,y; z,u) \right. \\
&\quad \left. - 3 \frac{\partial^2}{\partial x \partial u} [(7 - 5Q_\beta)(z-u) \Omega_0 + (1 + Q_\beta) \Omega_1] (x,y; z,u) \right. \\
&\quad \left. + 6 \frac{\partial^2}{\partial x^2} \left[\left(\frac{7}{2} - Q_\beta \right) (z-u) \Omega_0 + \left(\frac{9}{2} - 2Q_\beta \right) \Omega_1 \right] (x,y; z,u) \right\}, \\
F_2(x,y,z,u) &= \frac{1}{24} \rho_{2,1}(z,u) \left[- \frac{\partial^4 \rho_{3,1}}{\partial x^4} + 6 \frac{\partial^2 \rho_{1,1}}{\partial x^2} \right] (x,y) + \left\{ \frac{\partial \rho_{2,1}}{\partial z} - \frac{1}{3} \frac{\partial^2 \rho_{3,1}}{\partial z^2} - \frac{3x}{4} \frac{\partial^2 \rho_{2,1}}{\partial z^2} + \frac{x}{2} \frac{\partial^3 \rho_{3,1}}{\partial z^3} \right. \\
&\quad \left. - \frac{x}{16} \frac{\partial^4 \rho_{4,1}}{\partial z^4} \right\} (z,u) \left\{ \frac{1}{8} \frac{\partial^4 \rho_{4,1}}{\partial x^4} - \frac{3}{4} \frac{\partial^3 \rho_{3,1}}{\partial x^3} + \frac{3}{4} \frac{\partial^2 \rho_{2,1}}{\partial x^2} \right\} (x,y) + (z+u) \left\{ \frac{1}{48} \frac{\partial^4 \rho_{3,1}}{\partial x^4} - \frac{1}{6} \frac{\partial^3 \rho_{2,1}}{\partial x^3} \right. \\
&\quad \left. - \frac{1}{72} \frac{\partial^2 \rho_{1,1}}{\partial x^2} + \frac{2}{9} \frac{\partial \rho_{0,1}}{\partial x} \right\} (x,y) + \left\{ \frac{1}{18} \frac{\partial^4 \rho_{4,1}}{\partial x^4} - \frac{3}{8} \frac{\partial^3 \rho_{3,1}}{\partial x^3} + \frac{11}{18} \frac{\partial^2 \rho_{2,1}}{\partial x^2} - \frac{11}{36} \frac{\partial \rho_{1,1}}{\partial x} \right\} (x,y) \\
&\quad + (z \ln z + u \ln u) \left[\frac{1}{24} \frac{\partial^4 \rho_{3,1}}{\partial x^4} - \frac{1}{12} \frac{\partial^3 \rho_{2,1}}{\partial x^3} - \frac{7}{36} \frac{\partial^2 \rho_{1,1}}{\partial x^2} + \frac{1}{9} \frac{\partial \rho_{0,1}}{\partial x} \right] (x,y) \\
&\quad - (z-u)^2 (1 + \rho_{1,1}(z,u)) \left(\frac{1}{24} \frac{\partial^4 \rho_{2,1}}{\partial y^2 \partial x^2} + \frac{1}{9} \frac{\partial^3 \rho_{1,1}}{\partial y^2 \partial x} + \frac{1}{36} \frac{\partial^3 \rho_{1,1}}{\partial y \partial x^2} \right) (x,y) \\
&\quad - \frac{1}{48} \left\{ 2 \frac{\partial^4}{\partial x \partial y^3} [(z+u) \Omega_2(x,y; z,u) - \Omega_3(x,y; z,u)] - \frac{\partial^4}{\partial x^2 \partial y^2} [(z-u)^2 \Omega_1 \right. \\
&\quad \left. - 2(z+u) \Omega_2 + \Omega_3] (x,y; z,u) - \frac{\partial^3}{\partial x \partial y^2} \left[\frac{8}{3} (z-u)^2 \Omega_0 + \frac{20}{3} (z+u) \Omega_1 - \frac{28}{3} \Omega_2 \right] (x,y; z,u) \right. \\
&\quad \left. - \frac{2}{3} \frac{\partial^4}{\partial x^2 \partial y} [(z-u)^2 \Omega_0 - 2(z+u) \Omega_1 + \Omega_2] (x,y; z,u) + 4 \frac{\partial^2}{\partial x \partial y} [(z+u) \Omega_0 - \Omega_1] (x,y; z,u) \right\},
\end{aligned}$$

$$\begin{aligned}
F_3(x, y, z, u) &= -\frac{1}{16} \left\{ 2(2Q_\beta - 1 + Q_\beta \ln u) \frac{\partial^2 \varrho_{2,1}}{\partial x^2}(x, y) + 2(1 - 2Q_\beta + (1 - Q_\beta) \ln u) \frac{\partial \varrho_{1,1}}{\partial x}(x, y) \right. \\
&\quad + 4(z - u + z \ln z - u \ln u) \frac{\partial^2 \varrho_{1,1}}{\partial x^2}(x, y) + \frac{\partial^3}{\partial x \partial u^2} [(1 - 2Q_\beta)u\Omega_1 - u(z - u)\Omega_0](x, y; z, u) \\
&\quad - \frac{\partial^2}{\partial x \partial u} [(3 - 5Q_\beta)\Omega_1 - (3 - Q_\beta)(z - u)\Omega_0](x, y; z, u) - \frac{\partial^3}{\partial x^2 \partial u} [Q_\beta \Omega_2 - (Q_\beta z \\
&\quad + (2 - Q_\beta)u)\Omega_1](x, y; z, u) - 2 \frac{\partial^2}{\partial x^2} [\Omega_1 + (z - u)\Omega_0](x, y; z, u) \left. \right\}, \\
F_4(x, y, z, u) &= -\frac{1}{12} \varrho_{1,1}(z, u) \left\{ \frac{\partial^4 \varrho_{3,1}}{\partial x^4} + 3 \frac{\partial^3 \varrho_{2,1}}{\partial x^3} \right\}(x, y) - \left\{ \frac{1}{16} \frac{\partial^2 \varrho_{2,1}}{\partial z^2} - \frac{1}{8} \frac{\partial \varrho_{1,1}}{\partial z} + \frac{x}{16} \frac{\partial^2 \varrho_{1,1}}{\partial z^2} - \frac{x}{16} \frac{\partial^3 \varrho_{2,1}}{\partial z^3} \right. \\
&\quad + \left. \frac{x}{96} \frac{\partial^4 \varrho_{3,1}}{\partial z^4} \right\}(z, u) \left\{ \frac{\partial^4 \varrho_{4,1}}{\partial x^4} - 3 \frac{\partial^3 \varrho_{3,1}}{\partial x^3} \right\}(x, y) - \left\{ \frac{1}{12} \frac{\partial^4 \varrho_{3,1}}{\partial x^4} + \frac{1}{2} \frac{\partial^3 \varrho_{2,1}}{\partial x^3} + \frac{1}{2} \frac{\partial^2 \varrho_{1,1}}{\partial x^2} \right\}(x, y) \\
&\quad + \frac{1 - 3Q_\beta}{24u} \frac{\partial \varrho_{1,1}}{\partial x}(x, y) + \frac{1 - Q_\beta}{8} \ln z \frac{\partial^2 \varrho_{1,1}}{\partial x^2}(x, y) - \frac{1}{8} \ln u \left[\frac{\partial^3 \varrho_{2,1}}{\partial x^3} + (3 - Q_\beta) \frac{\partial^2 \varrho_{1,1}}{\partial x^2} \right](x, y) \\
&\quad - \frac{1}{48} \left\{ \frac{\partial^4}{\partial x \partial u^3} [u(z - u)\Omega_0 - u\Omega_1](x, y; z, u) - 3(1 - Q_\beta) \frac{\partial^3}{\partial x \partial u^2} [(z - u)\Omega_0 - \Omega_1](x, y; z, u) \right. \\
&\quad + 3(1 - Q_\beta) \frac{\partial^3}{\partial x \partial z \partial u} [(z - u)\Omega_0 - \Omega_1](x, y; z, u) - 2 \frac{\partial^4 \Omega_2}{\partial x^4}(x, y; z, u) \\
&\quad + 3 \frac{\partial^4}{\partial x^3 \partial u} [(z - u)\Omega_1 - \Omega_2](x, y; z, u) - 6 \frac{\partial^4}{\partial x^2 \partial u^2} (u\Omega_1(x, y; z, u)) - 6 \frac{\partial^3 \Omega_1}{\partial x^3}(x, y; z, u) \\
&\quad + 3 \frac{\partial^3}{\partial x^2 \partial u} [(3 - Q_\beta)(z - u)\Omega_0 + (1 - Q_\beta)\Omega_1](x, y; z, u) \\
&\quad \left. + 3(1 - Q_\beta) \frac{\partial^3}{\partial x^2 \partial z} [(z - u)\Omega_0 - \Omega_1](x, y; z, u) \right\}, \\
F_5(x, y, z, u) &= \frac{1}{12} \varrho_{1,1}(z, u) \left\{ \frac{\partial^4 \varrho_{3,1}}{\partial x^4} - 6 \frac{\partial^2 \varrho_{1,1}}{\partial x^2} \right\}(x, y) + \left\{ \frac{1}{16} \frac{\partial^2 \varrho_{2,1}}{\partial z^2} - \frac{1}{8} \frac{\partial \varrho_{1,1}}{\partial z} + \frac{x}{16} \frac{\partial^2 \varrho_{1,1}}{\partial z^2} - \frac{x}{16} \frac{\partial^3 \varrho_{2,1}}{\partial z^3} \right. \\
&\quad + \left. \frac{x}{96} \frac{\partial^4 \varrho_{3,1}}{\partial z^4} \right\}(z, u) \left\{ \frac{\partial^4 \varrho_{4,1}}{\partial x^4} - 6 \frac{\partial^3 \varrho_{3,1}}{\partial x^3} + 6 \frac{\partial^2 \varrho_{2,1}}{\partial x^2} \right\}(x, y) + \left\{ \frac{1}{12} \frac{\partial^4 \varrho_{3,1}}{\partial x^4} - \frac{1}{2} \frac{\partial^2 \varrho_{1,1}}{\partial x^2} \right\}(x, y) \\
&\quad + \frac{1}{24} \frac{\partial^4}{\partial x \partial y^3} \Omega_2(x, y; z, u) - \frac{3}{8} \frac{\partial^3}{\partial x \partial y^2} \Omega_1(x, y; z, u) + \frac{1}{2} \frac{\partial^2}{\partial x \partial y} \Omega_0(x, y; z, u), \\
F_6(x, y, z, u) &= -\frac{1}{16} \left\{ Q_\beta \left[\frac{2}{u} \frac{\partial \varrho_{1,1}}{\partial x}(x, y) - 2 \frac{\partial^3 \Omega_1}{\partial x^2 \partial u}(x, y; z, u) + \frac{\partial^3}{\partial x \partial u^2} ((z - u)\Omega_0 - \Omega_1)(x, y; z, u) \right] \right. \\
&\quad \left. - Q_\alpha \left[2 \frac{\partial^3 \Omega_1}{\partial x^2 \partial z}(x, y; z, u) - \frac{\partial^3}{\partial x \partial z \partial u} ((z - u)\Omega_0 - \Omega_1)(x, y; z, u) \right] \right\}, \\
F_7(x, y, z, u) &= -\frac{1}{8} \left\{ -10 \frac{\partial \varrho_{1,1}}{\partial x}(x, y) + \ln u \left(\frac{\partial \varrho_{1,1}}{\partial x} + \frac{\partial^2 \varrho_{2,1}}{\partial x^2} \right)(x, y) + 2(z - u) \left(1 + \varrho_{1,1}(z, u) \right) \frac{\partial^2 \varrho_{1,1}}{\partial x^2}(x, y) \right. \\
&\quad - \frac{\partial^3}{\partial x \partial u^2} [(zu - u^2)\Omega_0](x, y; z, u) + \frac{1}{2} \frac{\partial^3}{\partial x^2 \partial u} [(z - 3u)\Omega_1 - \Omega_2](x, y; z, u) \\
&\quad \left. - \frac{1}{2} \frac{\partial^2}{\partial x \partial u} [\Omega_1 - 5(z - u)\Omega_0](x, y; z, u) - \frac{\partial^2}{\partial x^2} [(z - u)\Omega_0 + 2\Omega_1](x, y; z, u) \right\}, \\
F_8(x, y, z, u) &= -\frac{1}{16} \left\{ \frac{2}{u} \frac{\partial \varrho_{1,1}}{\partial x}(x, y) + 2(\ln z - \ln u) \frac{\partial^2 \varrho_{1,1}}{\partial x^2}(x, y) - \frac{\partial^3}{\partial x^2 \partial u} [\Omega_1 + (z - u)\Omega_0](x, y; z, u) \right. \\
&\quad - \frac{\partial^3}{\partial x \partial u^2} [\Omega_1 - (z - u)\Omega_0](x, y; z, u) + \frac{\partial^3}{\partial x^2 \partial z} [\Omega_1 - (z - u)\Omega_0](x, y; z, u) \\
&\quad \left. + \frac{\partial^3}{\partial x \partial z \partial u} [\Omega_1 - (z - u)\Omega_0](x, y; z, u) \right\},
\end{aligned}$$

$$\begin{aligned}
F_9(x, y, z, u) &= -\frac{1}{16} \left\{ 2(2 + \ln u) \left(\frac{\partial \varrho_{1,1}}{\partial x} - \frac{\partial^2 \varrho_{2,1}}{\partial x^2} \right) (x, y) + 2 \frac{\partial^3}{\partial x \partial u^2} [u \Omega_1] (x, y; z, u) \right. \\
&\quad \left. - \frac{\partial^3}{\partial x^2 \partial u} [(z-u) \Omega_1 - \Omega_2] (x, y; z, u) + \frac{\partial^2}{\partial x \partial u} [(z-u) \Omega_0 - \Omega_1] (x, y; z, u) \right\}, \\
F_{10}(x, y, z, u) &= -\frac{1}{32} \left\{ \frac{2}{u} \frac{\partial \varrho_{1,1}}{\partial x} (x, y) - 2(2 + \ln z) \frac{\partial^2 \varrho_{1,1}}{\partial x^2} (x, y) - 4 \frac{\partial^3 \Omega_1}{\partial x^2 \partial u} (x, y; z, u) \right. \\
&\quad \left. - 2 \frac{\partial^3}{\partial x \partial u^2} [\Omega_1 - (z-u) \Omega_0] (x, y; z, u) - \frac{\partial^3}{\partial x^2 \partial z} [5 \Omega_1 - (z-u) \Omega_0] (x, y; z, u) \right\}, \\
P_1(x, y, z, u, w) &= \frac{1}{16} \left\{ 2((2 - Q_\beta) \ln w + 1 - 2Q_\beta) \frac{\varrho_{1,1}(x, z) - \varrho_{1,1}(y, z)}{x-y} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \left[3 \frac{\varrho_{3,1}(x, z) - \varrho_{3,1}(y, z)}{x-y} \right. \right. \\
&\quad \left. \left. - \frac{\varrho_{3,2}(x, z) - \varrho_{3,2}(y, z)}{x-y} + 2(u-w + u \ln u - w \ln w) \frac{\varrho_{2,1}(x, z) - \varrho_{2,1}(y, z)}{x-y} \right] \right. \\
&\quad \left. + 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left[2(u-w + u \ln u - w \ln w) \frac{\varrho_{1,1}(x, z) - \varrho_{1,1}(y, z)}{x-y} \right. \right. \\
&\quad \left. \left. + 2 \ln w \frac{\varrho_{2,1}(x, z) - \varrho_{2,1}(y, z)}{x-y} + \frac{\varrho_{2,2}(x, z) - \varrho_{2,2}(y, z)}{x-y} \right] \right. \\
&\quad \left. - \frac{\partial^2}{\partial w^2} \left[w(u-w) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x-y} + w \frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} \right] \right. \\
&\quad \left. + (2 - Q_\beta) \frac{\partial}{\partial w} \left[(u-w) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x-y} - \frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} \right] \right. \\
&\quad \left. - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \left[\frac{\Omega_2(x, z; u, w) - \Omega_2(y, z; u, w)}{x-y} + (u-w) \frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} \right] \right. \\
&\quad \left. - 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{\partial}{\partial w} \left[\frac{\Omega_2(x, z; u, w) - \Omega_2(y, z; u, w)}{x-y} + (u+w) \frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} \right] \right. \\
&\quad \left. - 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left[\frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} + (u-w) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x-y} \right] \right\}, \\
P_2(x, y, z, u, w) &= \frac{1}{16} \left\{ 2(\ln w - 3 + 2Q_\beta - (1 - Q_\beta) \ln u) \frac{\varrho_{1,1}(x, z) - \varrho_{1,1}(y, z)}{x-y} \right. \\
&\quad \left. + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \left[-3 \frac{\varrho_{3,1}(x, z) - \varrho_{3,1}(y, z)}{x-y} + \frac{\varrho_{3,2}(x, z) - \varrho_{3,2}(y, z)}{x-y} + 2(u-w + u \ln u \right. \right. \\
&\quad \left. \left. - w \ln w) \frac{\varrho_{2,1}(x, z) - \varrho_{2,1}(y, z)}{x-y} \right] + 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left(2(u-w + u \ln u - w \ln w) \frac{\varrho_{1,1}(x, z) - \varrho_{1,1}(y, z)}{x-y} \right. \right. \\
&\quad \left. \left. - \frac{\varrho_{2,2}(x, z) - \varrho_{2,2}(y, z)}{x-y} \right) - \frac{\partial^2}{\partial w^2} \left[w(u-w) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x-y} \right. \right. \\
&\quad \left. \left. - w \frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} \right] + 3 \frac{\partial}{\partial w} \left[(u-w) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x-y} \right. \right. \\
&\quad \left. \left. - \frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} \right] + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \left[\frac{\Omega_2(x, z; u, w) - \Omega_2(y, z; u, w)}{x-y} \right. \right. \\
&\quad \left. \left. - (u-w) \frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} \right] + 4w \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{\partial}{\partial w} \left[\frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} \right] \right. \\
&\quad \left. - 2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left[\frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} + (u-w) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x-y} \right] \right. \\
&\quad \left. + (1 - Q_\beta) \frac{\partial}{\partial u} \left[\frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x-y} - (u-w) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x-y} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
P_3(x, y, z, u, w) &= \frac{1}{16} \left\{ -2(2 + \ln w) \frac{\varrho_{1,1}(x, z) - \varrho_{1,1}(y, z)}{x - y} + (1 - 2Q_\beta) \frac{\partial}{\partial w} \left[\frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x - y} \right] \right. \\
&\quad \left. + \left(1 - (u - w) \frac{\partial}{\partial w} \right) \left[\frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x - y} \right] \right\}, \\
P_4(x, y, z, u, w) &= \frac{1}{16} \left\{ 2(2Q_\beta + \ln w - (1 - Q_\beta) \ln u) \frac{\varrho_{1,1}(x, z) - \varrho_{1,1}(y, z)}{x - y} - \left(Q_\beta - (u - w) \frac{\partial}{\partial w} \right. \right. \\
&\quad \left. \left. - (1 - Q_\beta)(u - w) \frac{\partial}{\partial u} \right) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x - y} \right. \\
&\quad \left. - \left(\frac{\partial}{\partial w} + (1 - Q_\beta) \frac{\partial}{\partial u} \right) \frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x - y} \right\}, \\
P_5(x, y, z, u, w) &= \frac{1}{8\sqrt{2}} \left\{ -2(2 + \ln w) \frac{\varrho_{1,1}(x, z) - \varrho_{1,1}(y, z)}{x - y} - \frac{\partial}{\partial w} \left[\frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x - y} \right] \right. \\
&\quad \left. + (u - w) \frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x - y} \right\}, \\
P_6(x, y, z, u, w) &= -\frac{1}{8\sqrt{2}} \left\{ 2(\ln u - \ln w) \frac{\varrho_{1,1}(x, z) - \varrho_{1,1}(y, z)}{x - y} \right. \\
&\quad \left. + \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial w} \right) \left[\frac{\Omega_1(x, z; u, w) - \Omega_1(y, z; u, w)}{x - y} \right] \right. \\
&\quad \left. - (u - w) \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial w} \right) \left[\frac{\Omega_0(x, z; u, w) - \Omega_0(y, z; u, w)}{x - y} \right] \right\}. \tag{A10}
\end{aligned}$$

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