Comment on phase conventions in helicity formalism*

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Abstract: Using the sequential decay process $e^+e^- \to J/\psi \to \Lambda\bar{\Lambda}$, $\Lambda \to p\pi^-$, $\bar{\Lambda} \to \bar{p}\pi^+$ as an example, the procedure for deducing the full angular distribution is illustrated by adopting both the Jacob-Wick and Jackson conventions in the helicity formalism. To make sure that the final physical result is free of phase conventions, we point out that the coefficients that relate the angular momentum states in different coordinate systems of reference frames have to be taken into account properly in the procedure. The fact that those coefficients are constants suggests that the Jackson convention is favorable in dealing with the processes with sequential decays.

Key words: Jacob-Wick convention, Jackson convention, helicity formalism

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1 Introduction

The helicity formalism, proposed by M. Jacob and G. C. Wick in 1959 [1], is nowadays widely used in particle physics to deal with collisions and decays involving particles with spin. In the formalism, the basic state of a particle is labeled by its momentum $\boldsymbol{p}$ and its helicity quantum number λ. Without an explicit wave equation, the relative phases of the basic states have to be defined by a special convention. The commonly used conventions are the Jacob-Wick convention proposed in Ref. [1] and the Jackson convention later proposed in Ref. [2].

Physical observables, such as differential cross-sections and differential decay widths, are expected to be independent of the phase conventions. However, different results did appear in the literature from time to time when different conventions were applied to calculate the angular distribution of the same process. For example, the sequential decay process of $e^+e^- \to J/\psi \to \Lambda\bar{\Lambda}$, $\Lambda \to p\pi^-$, $\bar{\Lambda} \to \bar{p}\pi^+$ was discussed by applying the Jacob-Wick convention in Ref. [3]. The angular distribution so obtained was found to be different from the recent result in Ref. [4] where the Jackson convention was applied. It is therefore of interest to understand the reason why such differences arise.

The angular distribution of $e^+e^- \to J/\psi \to \Lambda\bar{\Lambda}$, $\Lambda \to p\pi^-$, $\bar{\Lambda} \to \bar{p}\pi^+$ will be re-examined by applying both the Jacob-Wick and the Jackson conventions in this paper. Using this as an example, some subtle points in dealing with sequential decays will be explored.

2 Phase conventions

Some key points of phase conventions in the helicity formalism relevant to the following discussions are listed in this section. Detailed information can be found in Refs. [1, 2].

For a particle with non-zero mass and spin $\mathbf{s}$, the state with momentum $\boldsymbol{p}=(p,\theta,\phi)$ and helicity quantum number λ can be defined from the angular momentum eigenstate $|ss_z\rangle$ as

$$|p\theta\phi\,\lambda\rangle = R(\phi,\theta,\chi)L_z(p)|ss_z=\lambda\rangle, \tag{1}$$

where θ,ϕ are respectively the polar and azimuth angle of $\boldsymbol{p}$, and $|\boldsymbol{p}|=p$; R is the standard rotation operator

$$R(\phi,\theta,\chi) = e^{-i\phi J_z}e^{-i\theta J_y}e^{-i\chi J_z} \tag{2}$$

with $J_{x,y,z}$ the angular momentum operators of the particle; $L_z(p)$ is the Lorentz boost operator, which

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boosts a particle at rest to momentum $\boldsymbol{p}$ along the z-axis. Since J_z commutes with $L_z(p)$, the $\mathrm{e}^{-\mathrm{i}\chi J_z}$ term in Eq. (1) only contributes an arbitrary phase. Generally, χ may be chosen as a function of θ, ϕ and the functional form could be fixed by a specific phase convention. Actually,

$$\chi(\theta,\phi)=\chi_{\mathrm{JW}}(\theta,\phi)\equiv -\phi \tag{3}$$

is taken for the Jacob-Wick convention, and

$$\chi(\theta,\phi)=\chi_{\mathrm{JK}}(\theta,\phi)\equiv 0 \tag{4}$$

for the Jackson convention.

For a two-particle system, with spin $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, respectively, in a state of zero total linear momentum, $\boldsymbol{p_1}=-\boldsymbol{p_2}=\boldsymbol{p}$ (direction θ, ϕ), the two-particle state is defined by

$$|p\theta\phi\,\lambda_1\lambda_2\rangle = R(\phi,\theta,\chi)|p00\,\lambda_1\lambda_2\rangle. \tag{5}$$

Here the rotation operator R takes the same form as given in Eq. (2) but $\boldsymbol{J}=\boldsymbol{J^{(1)}}+\boldsymbol{J^{(2)}}$ is the total angular momentum of the two particles, and

$$|p00\,\lambda_1\lambda_2\rangle = |p00\,\lambda_1\rangle \otimes r^{(2)}\,|p00\,\lambda_2\rangle\,. \tag{6}$$

The definition of the operator $r^{(2)}$ subjects to different phase conventions,

$$r^{(2)}=r^{(2)}_{\mathrm{JW}}\equiv(-1)^{s_2-\lambda_2}\mathrm{e}^{-\mathrm{i}\pi J^{(2)}_y} \tag{7}$$

for the Jacob-Wick convention, and

$$r^{(2)}=r^{(2)}_{\mathrm{JK}}\equiv \mathrm{e}^{-\mathrm{i}\pi J^{(2)}_z}\mathrm{e}^{-\mathrm{i}\pi J^{(2)}_y} \tag{8}$$

for the Jackson convention.

Based on the two-particle state defined in Eq. (5), the state with total angular momentum J and z-component $J_z=M$ can be constructed by

$$|p\,JM\,\lambda_1\lambda_2\rangle = N_J\int \mathrm{d}\Omega\mathscr{D}^{J*}_{M\lambda_1-\lambda_2}(\phi,\theta,\chi)|p\theta\phi\,\lambda_1\lambda_2\rangle, \tag{9}$$

where $\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$. The definition of the standard rotation matrix element $\mathscr{D}^J_{mm'}(\phi,\theta,\chi) = \mathrm{e}^{-\mathrm{i}m\phi}d^J_{mm'}(\theta)\mathrm{e}^{-\mathrm{i}m'\chi}$ can be found, for example, in Ref. [5]. One can easily verify that

$$\langle p\theta\phi\,\lambda_1\lambda_2|p\,JM\,\lambda_1'\lambda_2'\rangle = N_J\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\mathscr{D}^{J*}_{M\lambda_1-\lambda_2}(\phi,\theta,\chi). \tag{10}$$

The normalization factor N_J is irrelevant and will simply be dropped in the following discussions.

3 Angular distribution of the sequential decay process

Assuming Λ, $\bar{\Lambda}$ are approximately on mass shell, the process of $\mathrm{e}^+\mathrm{e}^-\to \mathrm{J}/\psi\to\Lambda\bar{\Lambda}$, $\Lambda\to\mathrm{p}\pi^-$, $\bar{\Lambda}\to\bar{\mathrm{p}}\pi^+$ can be broken down into three sub-processes. The coordinate system O-xyz of the J/ψ rest frame $\mathscr{S}_{\mathrm{L}}$ can be fixed by choosing $\hat{z}$ along the e^+ beam direction and $\hat{y}$ along an arbitrary direction that transverses to $\hat{z}$. For $\mathrm{J}/\psi\to\Lambda\bar{\Lambda}$, as seen in $\mathscr{S}_{\mathrm{L}}$, the decay amplitude is given by

$$T_{M;\lambda_1\lambda_2}(\Theta,\Phi) = {}_{\mathrm{L}}\langle p\Theta\Phi\,\lambda_1\lambda_2|S_{\mathrm{J}/\psi}|JM\rangle_{\mathrm{L}}, \tag{11}$$

where $|JM\rangle_{\mathrm{L}}$ is the angular momentum eigenstate of the J/ψ (at rest), with $J=1$ and M the component along $\hat{z}$. The helicity state denoted by a subscript L is defined in the reference frame $\mathscr{S}_{\mathrm{L}}$; λ_1,λ_2 are respectively the helicities of $\Lambda,\bar{\Lambda}$; $S_{\mathrm{J}/\psi}$ is the transition matrix which governs J/ψ decays. In the $\mathscr{S}_{\mathrm{L}}$ frame, the linear momenta of $\Lambda,\bar{\Lambda}$ are back-to-back, $\boldsymbol{p}_\Lambda=-\boldsymbol{p}_{\bar{\Lambda}}=\boldsymbol{p}$, and Θ, Φ are the polar and azimuth angles of $\boldsymbol{p}$. Using Eq. (10), one can get

$$T_{M;\lambda_1\lambda_2}(\Theta,\Phi) = \alpha_{\lambda_1\lambda_2}\mathscr{D}^{1*}_{M\lambda_1-\lambda_2}(\Phi,\Theta,\chi(\Theta,\Phi)). \tag{12}$$

Here, the transition element

$$\alpha_{\lambda_1\lambda_2} = {}_{\mathrm{L}}\langle p\,JM\,\lambda_1\lambda_2|S_{\mathrm{J}/\psi}|JM\rangle_{\mathrm{L}}$$

is independent of M due to the rotation invariance.

For $\Lambda\to\mathrm{p}\pi^-$ in the rest frame of the Λ, $\mathscr{S}_\Lambda$, the decay amplitude may be written as

$$\begin{aligned} t_{\lambda_1\lambda_{\mathrm{p}}}(\theta_1,\phi_1) &= {}_{\Lambda}\left\langle q\theta_1\phi_1\,\lambda_{\mathrm{p}}|S_\Lambda|\frac{1}{2}\lambda_1\right\rangle_{\Lambda} \\ &= \beta_{\lambda_{\mathrm{p}}}\mathscr{D}^{1/2*}_{\lambda_1\lambda_{\mathrm{p}}}(\phi_1,\theta_1,\chi(\theta_1,\phi_1)). \end{aligned} \tag{13}$$

The helicity of p is denoted by λ_{p}, and the zero helicity of π^- has been dropped from the expression for simplicity. The coordinate system O'-$x'y'z'$ of $\mathscr{S}_\Lambda$ is chosen as $\hat{z'}$ is along the direction of $\boldsymbol{p}_\Lambda$; $\hat{y'}$ is along the direction of $\hat{z}\times\hat{z'}$. $|1/2\lambda_1\rangle_\Lambda$ is the angular momentum eigenstate of the Λ with z-component along $\hat{z'}$, and the helicity states with subscript Λ are known being defined in $\mathscr{S}_\Lambda$. The linear momentum of the proton as seen in $\mathscr{S}_\Lambda$ is expressed by $\boldsymbol{q}$ with the polar and azimuth angles of θ_1,ϕ_1. The transition matrix S_Λ represents the dynamics of Λ decays, and $\beta_{\lambda_{\mathrm{p}}}$ is the transition element of $\Lambda\to\mathrm{p}\pi^-$.

For $\mathscr{S}_{\bar{\Lambda}}$, the rest frame of the $\bar{\Lambda}$, the coordinate system O''-$x''y''z''$ is fixed by choosing $\hat{z''}$ along $\boldsymbol{p}_{\bar{\Lambda}}$ and $\hat{y''}$ along $\hat{z}\times\hat{z''}$. Similarly, the decay amplitude for $\bar{\Lambda}\to\bar{\mathrm{p}}\pi^+$ can be expressed as

$$\begin{aligned} \bar{t}_{\lambda_2\lambda_{\bar{\mathrm{p}}}}(\theta_2,\phi_2) &= {}_{\bar{\Lambda}}\left\langle q\theta_2\phi_2\,\lambda_{\bar{\mathrm{p}}}|S_{\bar{\Lambda}}|\frac{1}{2}\lambda_2\right\rangle_{\bar{\Lambda}} \\ &= \bar{\beta}_{\lambda_{\bar{\mathrm{p}}}}\mathscr{D}^{1/2*}_{\lambda_2\lambda_{\bar{\mathrm{p}}}}(\phi_2,\theta_2,\chi(\theta_2,\phi_2)). \end{aligned} \tag{14}$$

Again, the helicity states are defined in $\mathscr{S}_{\bar{\Lambda}}$. All of the other variables, such as $\lambda_{\bar{\mathrm{p}}}$, θ_2, ϕ_2 and $S_{\bar{\Lambda}}$, $\bar{\beta}_{\lambda_{\bar{\mathrm{p}}}}$ in the expression, have similar meanings as given in Eq. (13).

The decay amplitude for $e^+e^- \to J/\psi \to \Lambda\bar{\Lambda}$, $\Lambda \to p\pi^-$, $\bar{\Lambda} \to \bar{p}\pi^+$ can be constructed from the amplitudes of the sub-processes

$$A_{M\lambda_p\lambda_{\bar{p}}} = \sum_{\lambda_1\lambda_2} T_{M;\lambda_1\lambda_2}(\Theta,\Phi) t_{\lambda_1\lambda_p}(\theta_1,\phi_1) \bar{t}_{\lambda_2\lambda_{\bar{p}}}(\theta_2,\phi_2) \times \left({}_{\Lambda}\left\langle \frac{1}{2}\lambda_1 \right| \otimes_{\bar{\Lambda}} \left\langle \frac{1}{2}\lambda_2 \right|\right) |p\Theta\Phi\,\lambda_1\lambda_2\rangle_{\mathrm{L}}. \quad (15)$$

The last line in Eq. (15), which connects the states defined in different coordinate systems of reference frames, can be decomposed into two coefficients defined as

$$\rho_{\lambda_1} = {}_{\Lambda}\left\langle \frac{1}{2}\lambda_1 \right| R(\Phi,\Theta,\chi) L_z(p) \left| \frac{1}{2}\lambda_1 \right\rangle_{\mathrm{L}}, \quad (16)$$

and

$$\bar{\rho}_{\lambda_2} = {}_{\bar{\Lambda}}\left\langle \frac{1}{2}\lambda_2 \right| R(\Phi,\Theta,\chi) r^{(2)} L_z(p) \left| \frac{1}{2}\lambda_2 \right\rangle_{\mathrm{L}}. \quad (17)$$

Here we use the definition of the two-particle state in Eq. (5).

Since the two reference frames $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{S}_{\Lambda}$ are actually connected by a Lorentz boost and a rotation

$$\left| \frac{1}{2}\lambda_1 \right\rangle_{\Lambda} = R(\Phi,\Theta,0) L_z(p) \left| \frac{1}{2}\lambda_1 \right\rangle_{\mathrm{L}}, \quad (18)$$

so that

$$\rho_{\lambda_1} = e^{-i\lambda_1\chi(\Theta,\Phi)}. \quad (19)$$

Similarly, by using the connection between $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{S}_{\bar{\Lambda}}$

$$\left| \frac{1}{2}\lambda_2 \right\rangle_{\bar{\Lambda}} = R(\pi+\Phi,\pi-\Theta,0) L_z(p) \left| \frac{1}{2}\lambda_2 \right\rangle_{\mathrm{L}}, \quad (20)$$

one can obtain

$$\begin{aligned} \bar{\rho}_{\lambda_2} &= (-1)^{s_2} e^{i\lambda_2\chi(\Theta,\Phi)} && \text{for the Jacob-Wick convention,} \\ \bar{\rho}_{\lambda_2} &= e^{i\lambda_2\chi(\Theta,\Phi)} && \text{for the Jackson convention.} \end{aligned} \quad (21)$$

Finally, the full angular distribution of the sequential decay process reads

$$\begin{aligned} \frac{d\sigma}{d\Omega d\Omega_1 d\Omega_2} &\propto \sum_{M\lambda_p\lambda_{\bar{p}}} |A_{M\lambda_p\lambda_{\bar{p}}}|^2 \\ &= \sum_{M\lambda_p\lambda_{\bar{p}}} |\beta_{\lambda_p}|^2 |\bar{\beta}_{\lambda_{\bar{p}}}|^2 \sum_{\lambda_1\lambda_2\lambda_1'\lambda_2'} \alpha_{\lambda_1\lambda_2}\alpha^*_{\lambda_1'\lambda_2'} \\ &\quad \times e^{i[(\lambda_1-\lambda_1')\phi_1+(\lambda_2-\lambda_2')\phi_2]} d^1_{M\lambda_1-\lambda_2}(\Theta) d^1_{M\lambda_1'-\lambda_2'}(\Theta) \\ &\quad \times d^{\frac{1}{2}}_{\lambda_1\lambda_p}(\theta_1) d^{\frac{1}{2}}_{\lambda_1'\lambda_p}(\theta_1) d^{\frac{1}{2}}_{\lambda_2\lambda_{\bar{p}}}(\theta_2) d^{\frac{1}{2}}_{\lambda_2'\lambda_{\bar{p}}}(\theta_2). \end{aligned} \quad (22)$$

One can see that the expression given above is completely independent of χ and free from phase conventions. It is also interesting to note that the full angular distribution is independent of Φ as physically it should be. Otherwise, without the two coefficients, ρ_{λ_1} and $\bar{\rho}_{\lambda_2}$, in the expression of the decay amplitude in Eq. (15), the resulting full angular distribution would be dependent on Φ, as in Ref. [3].

4 Discussions and conclusion

The result given in Eq. (22) is the same as in Ref. [4], where the Jackson convention was applied, and different from that in Ref. [3] with the application of the Jacob-Wick convention. The resulting full angular distribution in Ref. [3] is dependent on Φ, which is obviously incorrect. As indicated in section **3**, the dependence on Φ is a consequence of the ignorance of the two coefficients, ρ_{λ_1} and $\bar{\rho}_{\lambda_2}$. If the two coefficients are taken into account, the same result as here can be achieved. Actually, ρ_{λ_1} and $\bar{\rho}_{\lambda_2}$, which play a crucial role to get a convention-free result, are the transition matrix elements of the states between different coordinate systems of reference frames, by which the relative phases of amplitudes according to the sub-processes are defined.

It is interesting to note that $\chi_{\mathrm{JK}} = 0$ in the Jackson convention, the two coefficients, i.e., the transition matrix elements, are simply $\rho_{\lambda_1} = \bar{\rho}_{\lambda_2} = 1$. That explains why correct results, for example in Refs. [4, 6–8], could be obtained even without mentioning the existence of the coefficients. It seems safer (and of course simpler) to adopt the Jackson convention in the helicity formalism.

The Φ dependence of the full angular distribution was also noticed in the study of the spin correlations in τ-pair decays when using the Jacob-Wick convention [9]. The arguments to fix the problem are, however, inadequate.

To conclude, some subtle points on phase conventions in dealing with sequential decays have been discussed in this article. Using $e^+e^- \to J/\psi \to \Lambda\bar{\Lambda}$, $\Lambda \to p\pi^-$, $\bar{\Lambda} \to \bar{p}\pi^+$ as an example, we have shown that the convention-free result could be reached by either the Jacob-Wick or the Jackson convention.

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