Critical behavior of the XY spin chain with three-site interaction studied in terms of the Loschmidt echo*

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Abstract: The effect of the three-site interaction (α) on the critical behaviors of the XY spin chain is studied in terms of the Loschmidt echo (LE). The critical lines can be shifted by α, and the anisotropy parameter of the XY chain has no effect on the critical lines. The scaling behaviors of the LE reveal that the dynamical behaviors of the LE are reliable for characterizing quantum phase transition (QPT).

Key words: Loschmidt echo, quantum phase transition, three-site interaction

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1 Introduction

Quantum phase transition (QPT), which is different from traditional thermodynamics phase transition, takes place at zero temperature and is induced by the variation in an internal parameter to a critical value [1, 2]. Traditionally, order parameter and symmetry breaking with the Landau-Ginzburg paradigm are used to describe QPT [3, 4]. Recently, many new perspectives have been put forward to characterize QPT [5–12]. Wu et al. addressed the fact that the non-analyticities in bipartite entanglement can characterize QPT [5]. Zanardi and Paunkovic characterized QPT in terms of the overlap function between two ground states obtained from two different external parameter values [6]. Additionally, fidelity or fidelity susceptibility (FS) [7–9], pair-wise quantum discord [10], the Berry phase [11, 12], the matrix product state [13] and the photon bunching effect [14] have also shown a close relationship with QPT.

Quan et al. found that the Loschmidt echo (LE) of a coupled system, which consists of a two-level system and the surrounding system modeled as an Ising spin chain in a transverse field, presented a deep valley at the critical point and was invariant at the critical point under the scaling transformation [15]. Subsequently, the role of the LE in detecting QPT has also been revealed in many other systems, including the Dicke model [16], the Bose-Einstein condensate model [17, 18], XY and XXZ spin models [19, 20], and so on. In particular, the LE was directly used to detect the critical points in the Ising model [21]. All these studies have shown that the LE is powerful in studying QPT.

Quantum spin chains play a fundamental role in studying many-body systems. For instance, entanglement in general and in spin chains has attracted much attention [22–24]. QPT in spin chains is also studied widely [7, 9, 10]. The three-site interaction is always present in a real spin system [25–27]. For example, the three-site interaction is helpful in describing the magnetic properties of solid ^{3}He [26]. In particular, the general three-site interaction [28] and four-site interaction [29] have been successfully demonstrated in experiments recently. Here, the XY spin chain with three-site interaction is selected and the effect of the three-site interaction on the critical behaviors is studied in terms of the LE.

2 The general formalism

We choose an environmental spin chain described by an XY spin chain with the $XZX+YZY$ type of three-site interaction in a transverse magnetic field.

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A central spin is transversely coupled to the environmental spin chain. The total Hamiltonian H is

$$H=\mu S^z+\nu S^x+J\sum_j^N\left(\frac{1+\gamma}{2}S_j^xS_{j+1}^x+\frac{1-\gamma}{2}S_j^yS_{j+1}^y+\frac{\lambda}{2}S_j^z\right)+J^*\sum_j^N(S_j^xS_{j+2}^x+S_j^yS_{j+2}^y)S_{j+1}^z+\frac{J\delta}{N}\sum_j^N S^zS_j^z=H_{\rm S}+H_{\rm E}^{\lambda}+H_{\rm I},\qquad(1)$$

where, $H_{\rm S}$, $H_{\rm E}^{\lambda}$ and $H_{\rm I}$ are the Hamiltonians of the central spin, the environmental spin chain and their interaction, respectively; μ and ν are dimensionless constants; N is the size of the environmental spin chain; J is the isotropic XY exchange interaction; J^* is the three-site interaction strength; and δ is the strength of the coupling between the central spin and the environmental spin chain. As usual, spin operators S_j^x, S_j^y and S_j^z describe the environmental spin and S^z describes the central spin; λ is the intensity of the magnetic field; and γ is the anisotropy in the in-plane interaction of the environmental spin chain. It is assumed that the central spin is initially in the superposition state of the ground and excited states. In this case, the evolution of the environmental spin chain is driven by the two different effective Hamiltonians, $H_{\rm E}^{\lambda\pm g}$, which are given by the replacement of λ in $H_{\rm E}^{\lambda}$ with $\lambda\pm g$ and $g=\delta\cos\theta/N$ with $\tan\theta=\nu/\mu$. At the same time, the environmental spin chain is assumed to be prepared initially in the ground state of $H_{\rm E}^{\lambda-g}$. Then, the LE obeys the following expression:

$$LE(t)=\prod_{k=1}^{M}LE_k(t)=\prod_{k=1}^{M}[1-\sin^2(2\varPhi_k)\sin^2(\varLambda_k^{\lambda+g}t)].\qquad(2)$$

Here, k=1, 2, $\cdots$, M $(M=N/2)$; t is evolution time; $\varPhi_k=(\theta_k^{\lambda+g}-\theta_k^{\lambda-g})/2$ and $\theta_k^{\lambda\pm g}=\arctan\left[-\gamma\sin(x_k)/(\lambda\pm g+\cos(x_k)-\alpha\cos(2x_k))\right]$, in which $\alpha=J^*/J$ is the strength of the $XZX+YZY$ type of three-site interaction in units of J and $x_k=2\pi k/N$; and $\varLambda_k^{\lambda\pm g}=[(\lambda\pm g+\cos(x_k)-\alpha\cos(2x_k))^2+(\gamma\sin(x_k))^2]^{0.5}$ are energy spectra of the effective Hamiltonian $H_{\rm E}^{\lambda\pm g}$. The deduction of the LE expression has been described in our previous work in detail [30]. In the following, we will discuss the features of the LE analytically based on Eq. (2).

Since each factor $LE_k(t)$ in Eq. (2) is less than unity, one can expect the $LE(t)$ to decrease to zero in the large N limit under some reasonable conditions. For this purpose, we make a heuristic analysis of the features of the $LE(t)$ according to the method reported in Ref. [15]. By introducing a cutoff frequency K_c, we define the partial product for the $LE(t)$ as follows

$$LE_c\equiv\prod_{k=1}^{K_c}LE_k\geqslant LE(t).\qquad(3)$$

The corresponding partial sum can be readily obtained as $S(t)$=ln$LE_c=-\sum_{k=1}^{Kc}|\ln LE_k|$. For small k and large N, we have $\varLambda_k^{\lambda\pm g}\approx|\lambda\pm g+1-\alpha|$ and $\sin(2\varPhi_k)\approx(-4\pi\gamma gk)/N\ (\lambda-g+1-\alpha)(\lambda+g+1-\alpha)$. As a result, if N is large enough and k is small, the approximation of $S(t)$ can be obtained as

$$S(t)\approx-\frac{4E(K_c)\gamma^2g^2\sin^2(|\lambda\pm g+1-\alpha|t)}{(\lambda-g+1-\alpha)^2(\lambda+g+1-\alpha)^2},\qquad(4)$$

where, $E(K_c)=4\pi^2K_c(K_c+1)(2K_c+1)/(6N^2)$. In the weak coupling regime $g\ll1$, when $\lambda\rightarrow-1+\alpha$, we have

$$LE_c(t)\approx\exp(-\eta t^2),\qquad(5)$$

where, $\eta=4E(K_c)\gamma^2g^4/(\lambda-g+1-\alpha)^2(\lambda+g+1-\alpha)^2$.

One can see from Eq. (5) that when N is large enough and λ is adjusted to the critical value of $-1+\alpha$, the $LE_c(t)$ will decay exceptionally in the second power of time. Note that $LE(t)$ is less than $LE_c(t)$. It can be expected that the $LE(t)$ has similar behaviors.

In general, in order to determine the phase transition regions for a finite size system, one needs to do the scaling transformation. Here, we investigate the properties of the LE in the vicinity of QPT under the scaling transformation. From Eq. (2), one can notice that only the LE_k's deviating prominently from unity has a remarkable effect on the shape and amplitude of the LE. And so the coefficients $\sin(2\varPhi_k)'$s in these LE_k's should considerably be nonzero. So we check the values of $\sin(2\varPhi_k)$. For this purpose, we introduce the frequency $k_c^{\lambda\pm g}$ and make $\left|\varepsilon_{k_c}^{\lambda\pm g}\right|=\left|\lambda\pm g+\cos(2\pi k_c^{\lambda\pm g}/N)-\alpha\cos(4\pi k_{k_c}^{\lambda\pm g}/N)\right|$ as small as possible [31]. For the case of small g and in the vicinity of QPT, one can see that $k_c^{\lambda\pm g}\ll M$. We can rewrite the coefficient $\sin(2\varPhi_k)$ as follows

$$\sin(2\varPhi_k)=\frac{-2\gamma g\sin(2\pi k/N)}{\varLambda_k^{\lambda+g}\varLambda_k^{\lambda-g}}.\qquad(6)$$

In order to make Eq. (6) be considerably nonzero, the k in Eq. (6) should be close to $k_c^{\lambda+g}$ and $k_c^{\lambda-g}$, which makes the value of $\gamma\sin(2\pi k/N)$ comparable with $\varepsilon_{k_c}^{\lambda+g}$ and $\varepsilon_{k_c}^{\lambda-g}$. For the small g, the k can approach $k_c^{\lambda+g}$ and $k_c^{\lambda-g}$ at the same time. Because of $k_c^{\lambda\pm g}\ll M$, $\sin(2\pi k/N)$ is a small value when k is approaching $k_c^{\lambda+g}$ and $k_c^{\lambda-g}$. Under these conditions, one can obtain

$$\varLambda_k^{\lambda+g}=\varLambda_k^{\lambda-g}\approx\sqrt{g^2+4\gamma^2\pi^2k^2/N^2}.\qquad(7)$$

Combining Eq. (6) and Eq. (7), one can find that the transformation $\gamma/N \rightarrow \tau\gamma/N$ and $g \rightarrow \tau g$ leads to $\Lambda_k^{\lambda+g} \rightarrow \tau\Lambda_k^{\lambda+g}$ and $\Lambda_k^{\lambda-g} \rightarrow \tau\Lambda_k^{\lambda-g}$, which makes the coefficient $\sin(2\Phi_k)$ remain invariant. Here, τ is the scaling factor. So, the LE is invariant in the vicinity of QPT under the transformation $t \rightarrow t/\tau$.

In order to further manifest the quantum criticality of the XY spin chain with three-site interaction, we resort to the numerical calculation to characterize the features of the LE in the following sections.

3 Results and discussion

As is well known, in the space of parameters λ and γ, the XY spin chain without the three-site interaction (for α=0) exhibits two critical lines $\lambda = \pm 1$ as $\gamma \neq 0$. Given the two critical lines, we discuss the effect of the three-site interaction on the critical behaviors by using the dynamical behavior of the LE in the following.

The changes in LE as functions of α and λ at a given evolution time t=20 are shown in Fig. 1(a), and the corresponding contour map is also plotted. It is found that the dynamical behavior of the LE depends on α and λ values. There exist two anomaly lines, $|\lambda - \alpha|$=1. As the values of α and λ are away from the anomaly lines, the LE is characterized by local oscillatory behavior. While the values of α and λ are near the anomaly lines, an obvious decay appears and the LE presents a deep valley. It is well known that for the pure XY spin chain, corresponding to α=0 in our case, an obvious decay following a deep valley appears as λ approaches the critical lines $\lambda = \pm 1$ [19]. By considering the three-site interaction, i.e., $\alpha \neq 0$, the positions of the deep valleys shift from $|\lambda|$=1 to $|\lambda - \alpha|$=1. Fig. 1(b) shows the dynamical behavior of the LE as a function of λ for the given typical three-site interactions. Here, the typical values of α are given as 0, 0.5 and 1.0, respectively. The corresponding anomaly points are $\lambda = -1$ and 1, $\lambda = -0.5$ and 1.5 and λ=0 and 2.0, respectively. The anomaly behaviors characterized by the deep valley are always closely associated with the QPT of the environmental spin chain [15]. After introducing the three-site interaction, the anomaly region of the XY spin chain is redistributed. Similar anomaly regions were also reported in studying the QPT of an XY spin chain with three-site interaction using the FS approach [9].

The effects of the anisotropy parameter γ on the LE are shown in Fig. 2. The LE as functions of α and γ for the typical value of λ=0.5 is shown in Fig. 2(a). It is easy to see that for the special case of γ=0, the LE always keeps unity, which can be seen from Eq. (2). While for the case of $\gamma \neq 0$, the LE presents the deep valleys along the lines $\alpha = -0.5$ and 1.5. The variation in γ cannot change the positions of the deep valleys. For the typical value of α=0.5, the variation in the LE as functions of λ and γ is shown in Fig. 2(b). Similarly, the positions of the deep valleys along $\lambda = -0.5$ and λ=1.5 do not shift with changing γ values. The results demonstrate that the anisotropy parameter γ has no effect on the anomaly lines $|\lambda - \alpha|$=1 characterized by the deep valleys.

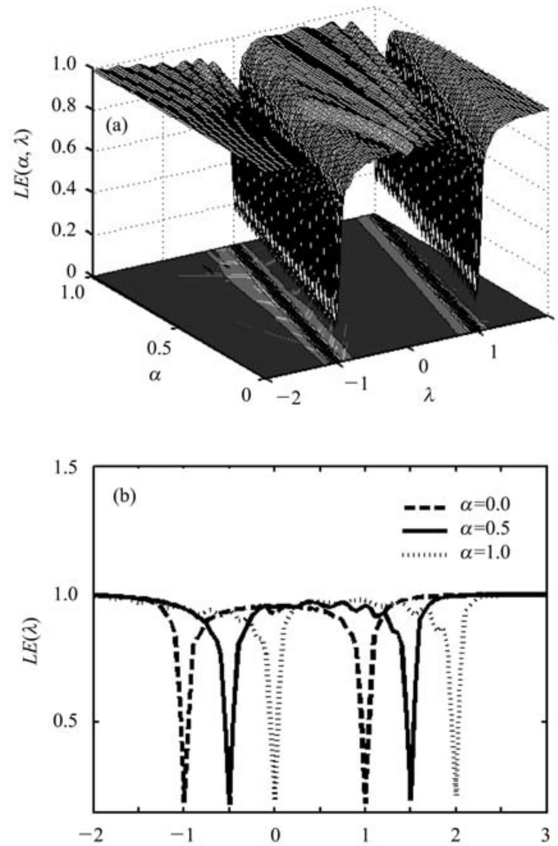

Fig. 1. (a) LE as functions of α and λ, and the corresponding contour map, is plotted. (b) LE as a function of λ in the case of α=0.0 (dashed line), 0.5 (solid line) and 1.0 (dotted line), respectively. Here, N=1000, γ=1.0 (Ising model), g=0.01 (weak coupling),t=20.

The scaling transformation of the LE is analyzed in the following. The exact numerical results of the LE for an XY spin chain with α=1 under the scaling transformation at the anomaly point λ=2.0 is shown in Fig. 3. The values of the system parameters in Fig. 3(a) are γ=1.0 and g=0.01 with N=50, 100, 200

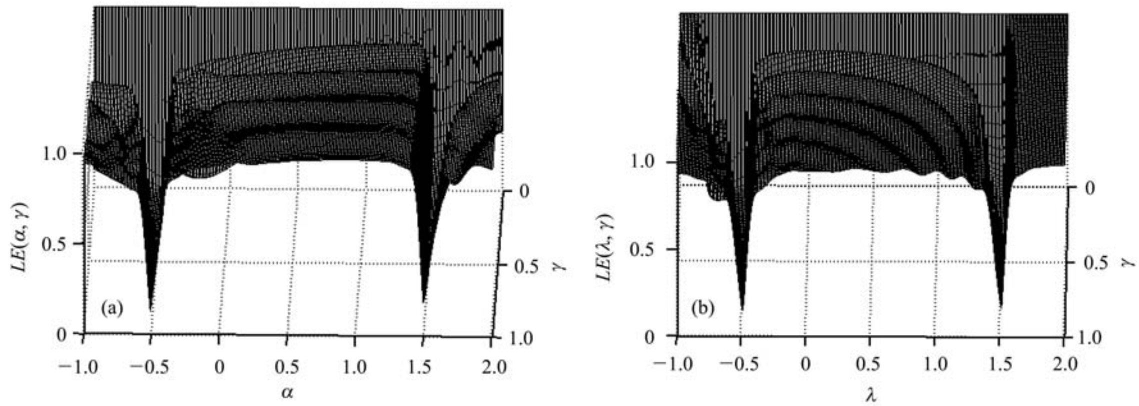

Fig. 2. (a) LE as functions of α and γ for the case of λ=0.5; (b) LE as functions of λ and γ for the case of α=0.5. Here, N=1000, g=0.01, t=20.

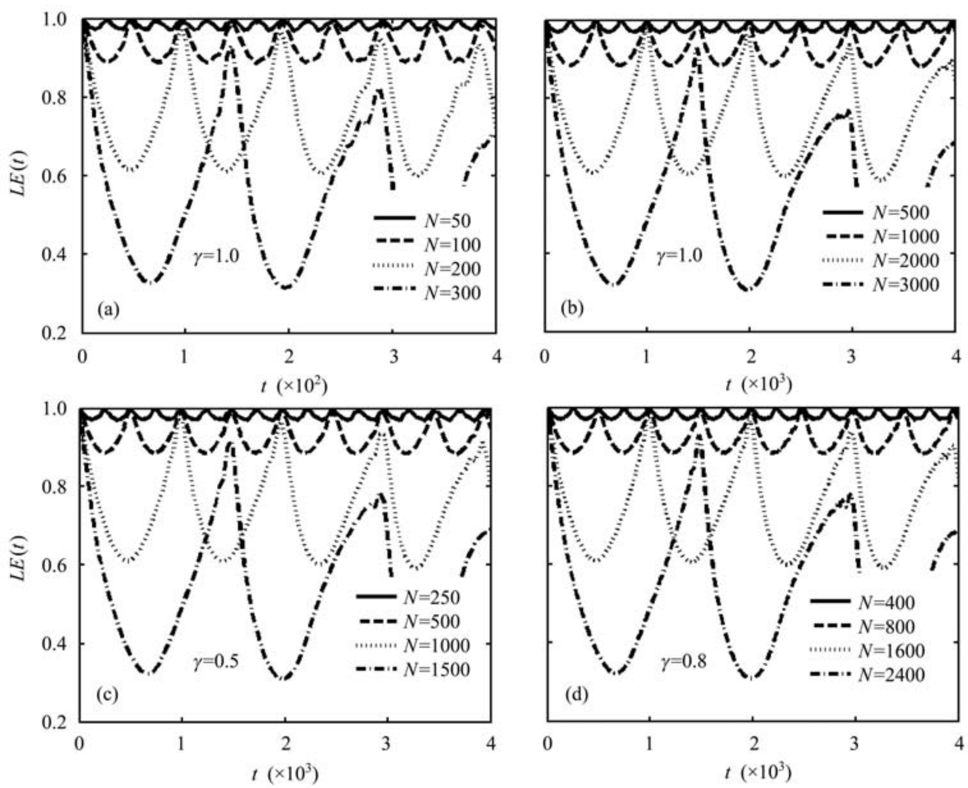

Fig. 3. The scaling behaviors of the LE at the critical point λ=2.0. The values of the system parameters are (a) γ=1.0, g=0.01, N=50, 100, 200, 300; (b) γ=1.0, g=0.001, N=500, 1000, 2000, 3000; (c) γ=0.5, g=0.001, N=250, 500, 1000, 1500; (d) γ=0.8, g=0.001, N=400, 800, 1600, 2400.

and 300, respectively. Fig. 3(b) shows the time evolution of the LE at the anomaly point λ=2.0 after the transformation $t \to t/\tau$, $\gamma/N \to \tau\gamma/N$ and $g \to \tau g$ with τ=0.1 and γ=1.0. Clearly, the shape of the LE during its time evolution is invariant under the scaling transformation at the anomaly point λ=2.0. In order to explore the effect of anisotropy parameter γ, the values of γ are changed to γ=0.5 and γ=0.8 during the scaling transformation. As shown in Fig. 3(c)–(d), a similar phenomenon is observed, i.e., during the time evolution, the shape of the LE is unchanged under scaling transformation at the anomaly point λ=2.0. The numerical results are in accordance with the theoretical analyses. For the anomaly point λ=0, the scaling behaviors are similar to those at λ=2.0 (not shown here). After changing α, similar scaling behaviors are found. Therefore, it is reasonable to expect that the anomaly region characterized by the deep valley of the LE should be the phase transition region of the XY spin chain with three-site interaction and γ cannot change the phase transition region.

By changing the positions of the minimum values of the LE with different spin size N, more information can be obtained. Fig. 4 shows the variation in LE with changing λ for the XY spin chain with α=0.5 and different spin chain size N. There is a dip in each LE-λ curve and the depth of the dip increases with increasing N. With increasing N, the accurate positions of the dips ($\lambda_{\rm m}$) shift to the critical value gradually along the positive direction of the λ axis, i.e., $\lambda_{\rm m}$ approaches $\lambda_1=-0.5$ in Fig. 4(a) and λ_2=1.5 in Fig. 4(b), respectively. The insets in Fig. 4 show the fitting curves $\ln(\lambda_{\rm c}-\lambda_{\rm m})\sim\ln(N)$. Here, $\lambda_{\rm c}$ denotes λ_1 and λ_2, respectively. The results demonstrate that $\lambda_{\rm m}$ approaches $\lambda_1=-0.5$ in the way of $\lambda_{\rm m}=\lambda_1(1-{\rm const}\ N^{-1.213})$, and $\lambda_{\rm m}$ approaches λ_2=1.5 in the way of $\lambda_{\rm m}=\lambda_2(1-{\rm const}\ N^{-1.172})$. In the thermodynamic limit, $N\to\infty$, $\lambda_{\rm m}$ converges towards the critical values $\lambda_1=-0.5$ and λ_2=1.5. On the other hand, from the fitting curves we can see that the convergence rate is different. For the pure XY spin chain, i.e., α=0, we find that the rate approaching the critical value $\lambda_{\rm c}$=1 is almost the same as that approaching $\lambda_{\rm c}=-1$. It can be expected that the symmetry of the dynamical behavior for the XY spin chain may be broken after introducing the three-site interaction. As the anisotropic parameter changes from γ=0.5 to γ=1.0, the LE with α=0.5 always converges towards $\lambda_1=-0.5$ and λ_2=1.5 with increasing spin chain size, as shown in Fig. 4(c). The XY spin chain with different α values shows similar characteristics. Thus, the QPT of the XY spin chain with three-site interaction is reflected faithfully by the dynamical behaviors of the LE.

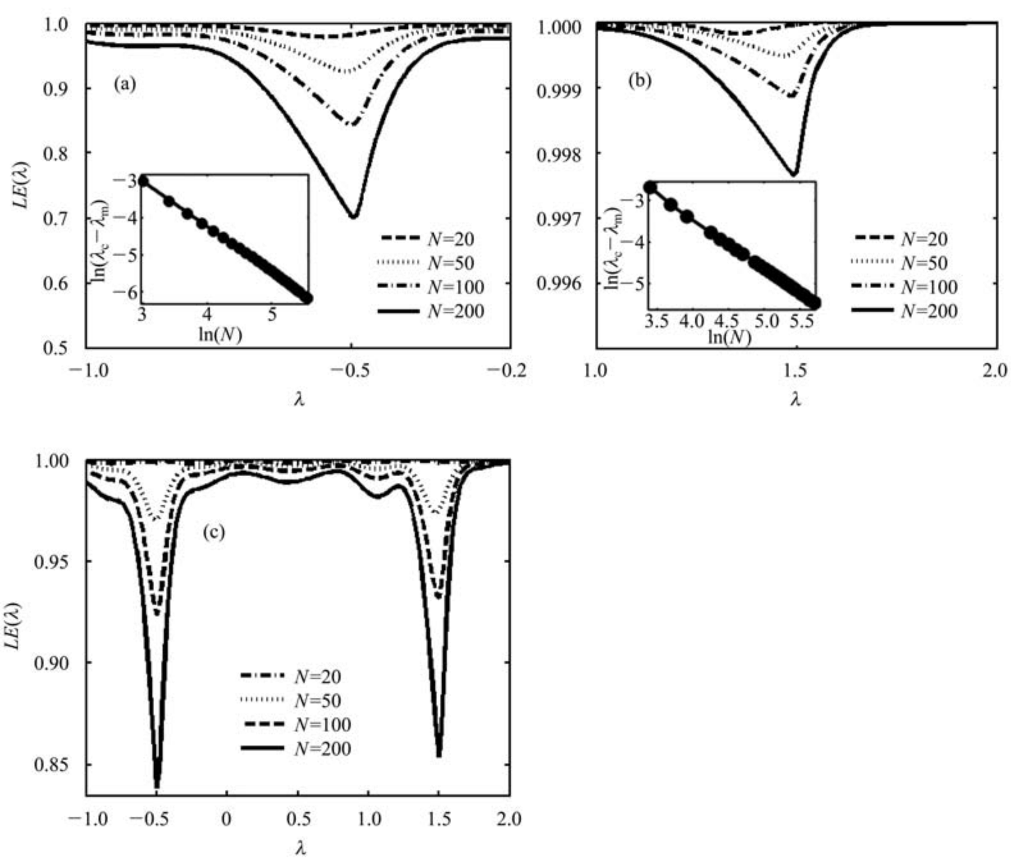

Fig. 4. LE as a function of λ for γ=0.5 with different N in the neighborhood of the critical values $\lambda_{\rm c}=\lambda_1=-0.5$ (a) and $\lambda_{\rm c}=\lambda_2$=1.5 (b). The insets show the fitting curves ln $(\lambda_{\rm c}-\lambda_{\rm m})\sim\ln(N)$. (c) LE as a function of λ for γ=1.0 with different N. Here, α=0.5, g=0.01, t=10.

4 Conclusions

The effect of the three-site interaction on the critical behaviors of the XY spin chain is explored in terms of the LE. After introducing the three-site interaction, the critical lines $|\lambda|=1$ shift to $|\lambda-\alpha|=1$. The anisotropy parameter has no effect on the critical lines. The QPT of the XY spin chain with three-site interaction can be well characterized by the LE.

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