Duffin-Kemmer-Petiau equation under Hartmann ring-shaped potential

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Abstract: We solve the Duffin-Kemmer-Petiau (DKP) equation in the presence of Hartmann ring-shaped potential in (3+1)-dimensional space-time. We obtain the energy eigenvalues and eigenfunctions by the Nikiforov-Uvarov (NU) method.

Key words: DKP equation, Hartmann ring-shaped potential, Nikiforov-Uvarov method, energy eigenvalue, eigenfunctions

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1 Introduction

The first order DKP equation, which describes spin-0 and spin-1 bosons [1–3], is a direct generalization of the Dirac equation in which the γ matrices are replaced with the β-matrices [4]. β matrices have three irreducible representations: the one-dimensional representation which is trivial, the five-dimensional representation that is for spin-zero particles and the ten-dimensional case which enables us to study spin-one particles [5]. In the past decade, there has been a growing interest in the study of DKP theory. Within the potential model, several efforts were devoted to considering the DKP equation under various potentials. The DKP equation with a pseudo-harmonic potential in the presence of a magnetic field in (2+1)-dimensions was solved in Ref. [6]. In Ref. [7] the authors reported the solutions of the equation in (3+1)-dimensions in the presence of coulomb and harmonic oscillator interactions [7]. The S-wave solutions of spin-one DKP equation for a deformed Hulthén potential were obtained in Ref. [8]. The equations have also been considered in the various related aspects including the quantum chromodynamics (QCD) [9], covariant Hamiltonian formalism [10], causal approach [11], in the context of five-dimensional Galilean invariance [12] and scattering of K^+- nucleus [13]. Refs. [14-16] analyze the effect of the magnetic field on the spectrum of the system. A survey on other physical terms within the framework of the equation can be found in Refs. [17–23]. In this work, we intend to solve the DKP equation for a spin-one particle in (3+1)-dimensions in the presence of Hartmann ring-shaped potential in an analytical manner. The motivation behind the present work is twofold. The first is that the spin-one DKP equation and its counterpart, i.e. the Proca equation, have not been sufficiently discussed in literature. This is not much appealing as we do require a reliable basis to study spin-one bosons. The second is the nature of the considered potential. We consider the ring-shaped Hartmann potential which enables us to study the deformation effects. This potential, as our forthcoming formulae reveals, is the more general case of the well-known Coulomb potential. The outline of this work is as follows: In Section 2, we introduce the DKP equation. In Section 3, we introduce the DKP equation in the presence of the Hartmann ring shaped potential. In Section 4, we obtain the energy eigenvalues and eigenfunctions of the radial part by the Nikiforov-Uvarov (NU) method. In Section 5, we solve the angular part of the problem and end the manuscript with the conclusions and comments on the applications of the study.

2 Basic concepts of the DKP theory

The DKP equation for free scalar and vector bosons is ($\hbar$=c=1)

$$(\mathrm{i}\beta^{\mu}\partial_{\mu}-m)\psi=0, \tag{1}$$

where β^{μ} are the DKP matrices and for a spin-one field satisfy the algebra

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda}+\beta^{\lambda}\beta^{\nu}\beta^{\mu}=g^{\mu\nu}\beta^{\lambda}+g^{\lambda\nu}\beta^{\mu}, \tag{2}$$

with

$$g^{\mu\nu}=\mathrm{diag}(1,1,1,-1), \quad (g^{\mu\nu})^2=1. \tag{3}$$

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In the case of vector bosons, the β^μ matrices are

$$\beta^0=\begin{pmatrix} 0 & \overline{0} & \overline{0} & \overline{0} \\ \overline{0}^{\mathrm{T}} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\ \overline{0}^{\mathrm{T}} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ \overline{0}^{\mathrm{T}} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{pmatrix},$$

$$\beta^i=\begin{pmatrix} 0 & \overline{0} & e_{\mathrm{i}} & \overline{0} \\ \overline{0}^{\mathrm{T}} & 0_{3\times3} & 0_{3\times3} & -\mathrm{i}S_i \\ -e_i^{\mathrm{T}} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ \overline{0}^{\mathrm{T}} & -\mathrm{i}S_i & 0_{3\times3} & 0_{3\times3} \end{pmatrix}, \qquad (4)$$

where $(S_i)_{jk}=-\mathrm{i}\varepsilon_{ijk}$ are 3×3 matrices and ε_{ijk} is 1, −1, 0 for an even permutation, an odd permutation and repeated indices, respectively. $(e_i)_{1j}=\delta_{ij}$ matrices are 1×3 ones with $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$.) The matrices, $I_{3\times3}$ and $0_{3\times3}$, represent the unit and null 3×3 matrices, respectively.

The more general form of the interaction is considered as

$$U=S_1(r)+PS_2(r)+\beta^0V_1(r)+\beta^0PV_2(r), \qquad (5)$$

and the equation takes the form

$$(\mathrm{i}\beta^\mu\partial_\mu-m-U)\psi=0, \qquad (6)$$

or

$$(\mathrm{i}\beta^\mu\partial_\mu-m-S_1(r)-PS_2(r)-\beta^0V_1(r)-\beta^0PV_2(r))\psi=0, \qquad (7)$$

where the projection operator is

$$P=\mathrm{diag}(1,1,1,1,0,0,0,0,0,0). \qquad (8)$$

We write the wave function as

$$\psi=(\mathrm{i}\phi,F^1,F^2,F^3,W^1,W^2,W^3,X^1,X^2,X^3)^{\mathrm{T}}. \qquad (9)$$

and therefore the equation can be expanded as

$$\left[\begin{pmatrix} -m & 0 & 0 & 0 & \mathrm{i}\partial_1 & \mathrm{i}\partial_2 & \mathrm{i}\partial_3 & 0 & 0 & 0 \\ 0 & -m & 0 & 0 & \mathrm{i}\partial_0 & 0 & 0 & 0 & -\mathrm{i}\partial_3 & \mathrm{i}\partial_2 \\ 0 & 0 & -m & 0 & 0 & \mathrm{i}\partial_0 & 0 & \mathrm{i}\partial_3 & 0 & -\mathrm{i}\partial_1 \\ 0 & 0 & 0 & -m & 0 & 0 & \mathrm{i}\partial_0 & -\mathrm{i}\partial_2 & \mathrm{i}\partial_1 & 0 \\ -\mathrm{i}\partial_1 & \mathrm{i}\partial_0-\mathrm{V}_2 & 0 & 0 & -m & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{i}\partial_2 & 0 & \mathrm{i}\partial_0-\mathrm{V}_2 & 0 & 0 & -m & 0 & 0 & 0 & 0 \\ -\mathrm{i}\partial_3 & 0 & 0 & \mathrm{i}\partial_0-\mathrm{V}_2 & 0 & 0 & -m & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\partial_3 & \mathrm{i}\partial_2 & 0 & 0 & 0 & -m & 0 & 0 \\ 0 & \mathrm{i}\partial_3 & 0 & -\mathrm{i}\partial_1 & 0 & 0 & 0 & 0 & -m & 0 \\ 0 & -\mathrm{i}\partial_2 & \mathrm{i}\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & -m \end{pmatrix}\right.$$

$$\left.-\begin{pmatrix} S_1+S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S_1+S_2 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S_1+S_2 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_1+S_2 & 0 & 0 & V_1 & 0 & 0 & 0 \\ 0 & V_1 & 0 & 0 & S_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_1 & 0 & 0 & S_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_1 & 0 & 0 & S_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_1 \end{pmatrix}\right]\begin{pmatrix} \mathrm{i}\phi \\ F^1 \\ F^2 \\ F^3 \\ W^1 \\ W^2 \\ W^3 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}=0. \qquad (10)$$

After some algebra, we find the coupled equations

$$\mathrm{i}(-m-S_1-S_2)\phi+\mathrm{i}\partial_1W^1+\mathrm{i}\partial_2W^2+\mathrm{i}\partial_3W^3=0, \qquad (11a)$$

$$(-m-S_1-S_2)F^1+(\mathrm{i}\partial_0-V_1)W^1-\mathrm{i}\partial_3X^2+\mathrm{i}\partial_2X^3=0, \qquad (11b)$$

$$(-m-S_1-S_2)F^2+(\mathrm{i}\partial_0-V_1)W^2+\mathrm{i}\partial_3X^1-i\partial_1X^3=0, \qquad (11c)$$

$$(-m-S_1-S_2)F^3+(\mathrm{i}\partial_0-V_1)W^3-\mathrm{i}\partial_2X^1+\mathrm{i}\partial_1X^2=0, \qquad (11d)$$

$$\partial_1\phi+(\mathrm{i}\partial_0-V_2-V_1)F^1+(-m-S_1)W^1=0, \qquad (11e)$$

$$\partial_2\phi+(\mathrm{i}\partial_0-V_2-V_1)F^2+(-m-S_1)W^2=0, \qquad (11f)$$

$$\partial_3\phi+(\mathrm{i}\partial_0-V_2-V_1)F^3+(-m-S_1)W^3=0, \qquad (11g)$$

$$-\mathrm{i}\partial_3F^2+\mathrm{i}\partial_2F^3+(-m-S_1)X^1=0, \qquad (11h)$$

$$\mathrm{i}\partial_3 F^1 - \mathrm{i}\partial_1 F^3 + (-m - S_1)X^2 = 0, \tag{11i}$$

$$-\mathrm{i}\partial_2 F^1 + \mathrm{i}\partial_1 F^2 + (-m - S_1)X^3 = 0. \tag{11j}$$

Thus, we have

$$\overrightarrow{\nabla}\cdot\overrightarrow{W} = (m + S_1 + S_2)\phi, \tag{12a}$$

$$(E - V_1)\overrightarrow{W} + \mathrm{i}(\overrightarrow{\nabla}\times\overrightarrow{X}) = (m + S_1 + S_2)\overrightarrow{F}, \tag{12b}$$

$$\overrightarrow{\nabla}\phi + (E - V_2 - V_1)\overrightarrow{F} = (m + S_1)\overrightarrow{W}, \tag{12c}$$

$$\mathrm{i}(\overrightarrow{\nabla}\times\overrightarrow{F}) = (m + S_1)\overrightarrow{X}. \tag{12d}$$

To our best knowledge, the latter has not been solved in the general case and the existing papers are restricted to special cases.

3 DKP equation in some special cases

It is clear that by choosing the $S_1 = S_2 = V_1 = V_2 = 0$, we have

$$(E^2_{n,l,\lambda} - m^2)\overrightarrow{F} + \nabla^2\overrightarrow{F} = 0, \tag{13a}$$

$$\nabla^2 = -p^2, \tag{13b}$$

$$(E^2_{n,l,\lambda} - m^2 - p^2)\overrightarrow{F} = 0. \tag{13c}$$

For the case of $S_1 = S_2 = V_1 = 0$, we may write

$$(E_{n,l,\lambda}(E_{n,l,\lambda} - V_2) - m^2)\overrightarrow{F} + \nabla^2\overrightarrow{F} = 0. \tag{14}$$

The Hartmann ring-shaped potential is a special case of the non-central potentials originally introduced in quantum chemistry to explain the ring-shaped molecules like benzene and has the form [24, 25]

$$V_2(r,\theta) = \frac{A}{r} - \frac{B}{r^2\sin^2\theta}, \tag{15a}$$

with

$$A = 2\eta\sigma^2 a_0\varepsilon_0, \tag{15b}$$

and

$$B = \eta^2\sigma^2 a_0^2\varepsilon_0. \tag{15c}$$

The dimensionless parameters η and σ are positive and real. For the wave function, we introduce

$$\overrightarrow{F} = \overrightarrow{F}(\vec{r},\theta,\varphi) = R(\vec{r})Q(\theta)\mathrm{e}^{\mathrm{i}\lambda\varphi}, \tag{16}$$

and rewrite the wave equation as

$$\nabla^2\overrightarrow{F}(\vec{r},\theta,\varphi) - E_{n,l,\lambda}V_2\overrightarrow{F}(\vec{r},\theta,\varphi) + (E^2_{n,l,\lambda} - m^2)\overrightarrow{F}(\vec{r},\theta,\varphi) = 0. \tag{17}$$

4 Energy eigenvalues and eigenfunctions

Expanding Eq. (17), we find

$$\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\right]\overrightarrow{F} - E_{n,l,\lambda}\left(\frac{A}{r} - \frac{B}{r^2\sin^2\theta}\right)\overrightarrow{F} + (E^2_{n,l,\lambda} - m^2)\overrightarrow{F} = 0. \tag{18}$$

The separation of variable yields

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R(r)}{\mathrm{d}r} + \left[\frac{(E^2_{n,l,\lambda} - m^2)r^2 - E_{n,l,\lambda}Ar - l(l+1)}{r^2}\right]R(r) = 0, \tag{19}$$

where

$$\begin{aligned}
\alpha_1 &= 2,\ \alpha_2 = 0,\ \alpha_3 = 0,\ \xi_1 = m^2 - E^2_{n,l,\lambda},\ \xi_2 = -E_{n,l,\lambda}A,\\
\xi_3 &= l(l+1),\ \alpha_4 = -\frac{1}{2},\ \alpha_5 = 0,\ \alpha_6 = m^2 - E^2_{n,l,\lambda},\\
\alpha_7 &= E_{n,l,\lambda}A,\ \alpha_8 = \frac{1}{4} + l(l+1),\ \alpha_9 = m^2 - E^2_{n,l,\lambda},\\
\alpha_{10} &= 1 + 2\sqrt{\frac{1}{4} + l(l+1)},\ \alpha_{11} = 2\sqrt{m^2 - E^2_{n,l,\lambda}},\\
\alpha_{12} &= -\frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)},\ \alpha_{13} = -\sqrt{m^2 - E^2_{n,l,\lambda}}.
\end{aligned} \tag{20}$$

Compared with the appendix, the energy can be simply found as

$$E_{n,l,\lambda} = \pm\frac{m\left\{(2n+1) + 2\sqrt{\left(l + \frac{1}{2}\right)^2}\right\}}{\sqrt{A^2 + \left\{(2n+1) + 2\sqrt{\left(l + \frac{1}{2}\right)^2}\right\}^2}}. \tag{21}$$

5 The angular section

Let us now start to solve the angular section with the governing equation

$$\frac{\mathrm{d}^2Q}{\mathrm{d}\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}Q}{\mathrm{d}\theta} + \frac{E_{n,l,\lambda}B - \lambda^2}{\sin^2\theta}Q(\theta) + l(l+1)Q(\theta) = 0. \tag{22}$$

By choosing the change of variable $s = \cos\theta$, Eq. (22) appears as

$$\frac{\mathrm{d}^2Q}{\mathrm{d}s^2} + \frac{-2s}{1 - s^2}\frac{\mathrm{d}Q}{\mathrm{d}s} + \left(\frac{E_{n,l,\lambda}B - \lambda^2 + l(l+1) - l(l+1)s^2}{1 + s^4 - 2s^2}\right)Q(s) = 0, \tag{23}$$

after a new transformation of the form $z=\frac{1}{2}-\frac{1}{2}s$, becomes

$$\frac{\mathrm{d}^2Q}{\mathrm{d}z^2}+\frac{1-2z}{-z^2+z}\frac{\mathrm{d}Q}{\mathrm{d}z}+\frac{\frac{1}{4}(E_{n,l,\lambda}B-\lambda^2)-l(l+1)z^2+l(l+1)z}{z^4-2z^3+z^2}Q(z)=0. \quad (24)$$

By a simple comparison with the appendix, we find the requisite parameters as

$$\begin{aligned}
\alpha_1 &= 1,\ \alpha_2=2,\ \alpha_3=1,\ \xi_1=l(l+1),\ \xi_2=l(l+1),\\
\xi_3 &= \frac{1}{4}(\lambda^2-E_{n,l,\lambda}B),\ \alpha_4=0,\alpha_5=0,\ \alpha_6=l(l+1),\\
\alpha_7 &= -l(l+1),\ \alpha_8=\frac{1}{4}(\lambda^2-E_{n,l,\lambda}B),\ \alpha_9=\frac{1}{4}(\lambda^2-E_{n,l,\lambda}B),\\
\alpha_{10} &= 1+\sqrt{(\lambda^2-E_{n,l,\lambda}B)},\ \alpha_{11}=2+2\sqrt{(\lambda^2-E_{n,l,\lambda}B)},\\
\alpha_{12} &= \frac{1}{2}\sqrt{(\lambda^2-E_{n,l,\lambda}B)},\ \alpha_{13}=-\sqrt{(\lambda^2-E_{n,l,\lambda}B)}. \quad (25)
\end{aligned}$$

Therefore, the angular part of the wave function is calculated as

$$\begin{aligned}
Q(\theta) &= \left(\frac{1-\cos\theta}{2}\right)^{\alpha_{12}}\left(1-\alpha_3\left(\frac{1-\cos\theta}{2}\right)\right)^{-\alpha_{12}-\left(\frac{\alpha_{13}}{\alpha_3}\right)}\\
&\quad\times P_n^{\left(\alpha_{10}-1,\left(\frac{\alpha_{11}}{\alpha_3}\right)-\alpha_{10}-1\right)}\left(1-2\alpha_3\left(\frac{1-\cos\theta}{2}\right)\right), \quad (26)
\end{aligned}$$

where

$$P_n^{\left(\alpha_{10}-1,\left(\frac{\alpha_{11}}{\alpha_3}\right)-\alpha_{10}-1\right)}\left(1-2\alpha_3\left(\frac{1-\cos\theta}{2}\right)\right)$$

represents the Jacobi polynomials. Combining the above equations, the energy spectrum can be determined as

$$\begin{aligned}
l(l+1) &= n'^2+2\sqrt{(\lambda^2-E_{n,l,\lambda}B)}n'+n'+(\lambda^2-E_{n,l,\lambda}B\\
&\quad+\sqrt{\lambda^2-E_{n,l,\lambda}B}). \quad (27)
\end{aligned}$$

Therefore,

$$l=n'+\sqrt{\lambda^2-E_{n,l,\lambda}B}. \quad (28)$$

Finally, we can write the total form of the considered component as

$$\begin{aligned}
&F_{n,l,\lambda}(\vec{r},\theta,\varphi)\\
&= r^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)}\right)}\mathrm{e}^{(-\sqrt{m^2-E^2_{n,l,\lambda}})r}L_n^{\left(2\sqrt{\frac{1}{4}+l(l+1)}\right)}\\
&\quad\times\left((2\sqrt{m^2-E^2_{n,l,\lambda}})r\right)\times\left(\frac{1-\cos\theta}{2}\right)^{\left(\frac{1}{2}\sqrt{(\lambda^2-E_{n,l,\lambda}B)}\right)}\\
&\quad\times\left(\frac{1+\cos\theta}{2}\right)^{\left(\frac{1}{2}\sqrt{(\lambda^2-E_{n,l,\lambda}B)}\right)}\\
&\quad\times P_n^{\left(\sqrt{(\lambda^2-E_{n,l,\lambda}B)},\sqrt{(\lambda^2-E_{n,l,\lambda}B)}\right)}(\cos\theta)\mathrm{e}^{\mathrm{i}\lambda\varphi}. \quad (29)
\end{aligned}$$

6 Conclusion

In the present work, we have considered the DKP equation in the presence of Hartmann ring-shaped potential in (3+1)-dimensions for spin-one particles. The eigenvalues and the eigenfunctions are calculated via the NU technique. The eigenfunctions and energy eigenvalues, after proper facts and modifications are done, can be as well used in meson spectroscopy, in the study of equilibrium separation between the nuclei, decay properties of the wave function, cross sections, interference patterns, charge transfer, excitation effects, static multiple polarizabilities of the interacting particles and various static properties of mesons.

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Appendix A

We consider the following second-order differential equation whose form represents a general Schrödinger-type equation to obtain the parametric generalization of the NU method [26, 27],

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{1}{[s(1-\alpha_3 s)]^2}[-\xi_1 s^2+\xi_2 s-\xi_3]\right\}\times\psi_n(s)=0. \tag{A1}$$

According to the NU method, the eigenfunctions are

$$\psi_n(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}\times P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s). \tag{A2}$$

Where the Jacobi polynomial is,

$$P_n^{(c,d)}(z) = 2^{-n}\sum_{p=0}^{n}\begin{pmatrix} n+c \\ P\end{pmatrix}\begin{pmatrix} n+d \\ n-p\end{pmatrix}(1-z)^{n-p}(1+z)^p$$

$$P_n^{(c,d)}(z) = \frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\begin{pmatrix} n \\ r\end{pmatrix}\times\frac{\Gamma(n+c+d+r+1)}{\Gamma(r+c+1)}\left(\frac{z-1}{2}\right)^r. \tag{A3}$$

Where

$$\begin{pmatrix} n \\ r\end{pmatrix}=\frac{n!}{r!(n-r)!}=\frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}.$$

And the eigenenergies satisfy

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0. \tag{A4}$$

Where

$$\begin{aligned}
\alpha_4 &= \frac{1}{2}(1-\alpha_1),\quad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\quad \alpha_6=\alpha_5^2+\xi_1,\\
\alpha_7 &= 2\alpha_4\alpha_5-\xi_2,\quad \alpha_8=\alpha_4^2+\xi_3,\quad \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\\
\alpha_{10} &= \alpha_1+2\alpha_4+2\sqrt{\alpha_8},\quad \alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),\\
\alpha_{12} &= \alpha_4+\sqrt{\alpha_8},\quad \alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}).
\end{aligned} \tag{A5}$$

Furthermore in some cases we can use

$$\psi_n(s) = s^{\alpha_{12}^*}(1-\alpha_3 s)^{-\alpha_{12}^*-\frac{\alpha_{13}^*}{\alpha_3}}\times P_n^{\left(\alpha_{10}^*-1,\frac{\alpha_{11}^*}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s), \tag{A6}$$

$$\alpha_{10}^*=\alpha_1+2\alpha_4-2\sqrt{\alpha_8},$$

$$\alpha_{11}^*=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}-\alpha_3\sqrt{\alpha_8}),$$

$$\alpha_{12}^*=\alpha_4-\sqrt{\alpha_8},$$

$$\alpha_{13}^*=\alpha_5-(\sqrt{\alpha_9}-\alpha_3\sqrt{\alpha_8}).$$

Also for this problem when

$$\lim_{\alpha_3\to 0}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-\alpha_3 s)=L_n^{\alpha_{10}-1}(\alpha_{11}s), \tag{A7}$$

And $\lim_{\alpha_3\to 0}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}=\mathrm{e}^{\alpha_{13}s}$.

Thus the solution given in Eq. (A4) becomes

$$\psi_n(s)=s^{\alpha_{12}}\mathrm{e}^{\alpha_{13}s}L_n^{\alpha_{10}-1}(\alpha_{11}s). \tag{A8}$$

References

1 Petiau G. (Ph. D. Thesis). University of Paris, 1936; Acad. R. Belg. Cl. Sci. Mem. Collect 8, 16, 1936
2 Duffin R J. Phys. Rev. Lett., 1938, **54**: 1114
3 Kemmer N. Proc. Roy. Soc. A, 1939, **173**: 91
4 Unal N. In: Concepts of Physics., 2005, **2**: 273
5 Umezawa H. Quantum Field Theory. Amsterdam: North-Holland, 1956
6 Hassanabadi H et al. Found. Phys., 2013, **43**: 225–235
7 Hassanabadi H et al. Adv. High Energy Phys., 2012, 489641
8 Molaee Z et al. Eur. Phys. J. Plus., 2012, **127**: 116
9 Friedman E, Kalbermann G. Phys. Rev. C, 1986, **34**: 2244
10 Clark B C et al. Phys. Rev. Lett., 1985, **55**: 592
11 Gribov V. Eur. Phys. J. C, 1999, **10**: 71
12 Kanatchikov I V. Rep. Math. Phys., 2000, **46**: 107
13 Casana R et al. Class. Quant. Grav., 2003, **20**: 2457
14 Nedjadi Y, Barrett R C. J. Phys. A, 1994, **27**: 4301
15 Nedjadi Y, Barrett R C. J. Math. Phys., 1994, **35**: 4517
16 Bolivar A O. Physica A, 2002, **315**: 601
17 Boumali A, Chetouani L. Phys. Lett. A, 2005, **346**: 261–268
18 Chetouani L et al. Int. J. Theor. Phys., 2004, **43**: 4
19 Hassanabadi H et al. Phys. Rev. C, 2011, **84**: 064003
20 Oudi R et al. Commun. Theor. Phys., 2012, **57**: 15–18
21 Hassanabadi H et al. Can. J. Phys., 2012, **90**: 3
22 Yasuk F et al. I Phys. Scr., 2008, **78**: 045010
23 Falek M, Merad M. Comm. Theor. Phys., 2008, **50**: 587
24 Hartmann H. Theor. Chim. Acta, 1972, **24**: 201
25 Hartmann H, Schuch D. Int. J. Quant. Chem., 1980, **18**: 125
26 Nikiforov A F, Uvarof V B. Special Functions of Mathematical Physics. (Birkhauser, Bassel, 1988)
27 Tezcan C, Sever R. Int. J. Theor. Phys., 2009, **48**: 337