Analysis of undulator radiation with an electromagnet undulator

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Abstract: In this paper we discuss the theory of undulator radiation in an electromagnet undulator. We discuss the spectral properties of undulator radiation when electrons are injected off the undulator axis. This paper highlights the distinctive features of the radiation spectrum from electromagnet undulators, as compared to PPM undulators.

Key words: undulator, electromagnet undulator, radiation spectrum, relativistic electron beam

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1 Introduction

In recent years there has been much interest in short wavelength free electron lasers (FEL) in pure and applied research, in both physical and biological sciences. In a FEL, a relativistic electron beam travels along the length of an undulator and emits undulator radiation. The spectral properties of undulator radiation is a diagnostic tool in the design and performance studies of a FEL. One can extract several useful pieces of information on the quality of the electron and radiation beam from the undulator radiation. The spectral properties of the undulator radiation depend on a large number of effects associated with the undulator design and fabrication. For this reason, several FEL experimental configurations have been modelled from the results of undulator radiation.

The radiation emitted by an FEL depends on the undulator wavelength, the field strength, and the beam energy. In a Pure Permanent Magnet (PPM) based undulators, the undulator period is limited due to the physical size and properties of the magnet. Attempts have been made to design PPM undulators with short periods using a periodic multi-domain structure [1] and by using magnetic recording technology [2]. However, the most attractive proposal is the fabrication of an electromagnet undulator, where the magnetic field is controlled by the external current in the electromagnet winding [3–7]. In this paper, we analyse the undulator radiation in a case where the relativistic electron beam propagates off-axis in an electromagnet undulator constructed with a conducting strip of copper foils in between ferromagnetic strips [3–5]. In our analysis, we use analytical means to evaluate the distortion induced in the central emission lines of the radiation spectrum when the electrons are injected off the electromagnet undulator axis. An essential approach in our calculation is the use of GBF (Generalized Bessel Functions) to derive the underlying physics and gain a transparent understanding of the problem. In Section 2, we study the theory of emission of radiation in electromagnet undulators. The results and discussion are given in Section 3.

2 Undulator radiation

We assume that the electrons are moving in an electromagnet undulator with N periods and with a magnetic field that, very near to the undulator axis, is given by [3]:

$$B_y(y,z)=-\frac{2\mu_0 I}{\pi h}\frac{\sin(\pi h/\lambda_{\mathrm{u}})}{\sinh(\pi\delta/\lambda_{\mathrm{u}})}\sin\left(\frac{2\pi z}{\lambda_{\mathrm{u}}}\right)\cosh\left(\frac{2\pi y}{\lambda_{\mathrm{u}}}\right), \tag{1}$$

where μ_0 is the permeability of free space, h is the thickness of the conductor, δ is the length of the gap between two undulator magnets, I is the current passing through the magnets, and λ_{u} is the period of the undulator magnets. The peak magnetic field B_0 is given by:

$$B_0=-\frac{2\mu_0 I}{\pi h}\frac{\sin(\pi h/\lambda_{\mathrm{u}})}{\sinh(\pi\delta/\lambda_{\mathrm{u}})}. \tag{2}$$

By defining

$$k_{\mathrm{u}}=\frac{2\pi}{\lambda_{\mathrm{u}}},\quad \Omega_{\mathrm{u}}=k_{\mathrm{u}}c,$$

Eq. (1) can be rewritten as

$$B_y=B_0\sin(k_{\mathrm{u}}z)\cosh(k_{\mathrm{u}}y). \tag{3}$$

For the on-axis equation, where $y=0$, the trajectory of the electron is described by the Lorentz equation. This gives

$$x(t)=-\frac{Kc}{\gamma\Omega_{\mathrm{u}}}\sin(\Omega_{\mathrm{u}}t), \tag{4}$$

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where

$$K=\frac{eB_0}{m_0c\Omega_{\rm u}},$$

and

$$z(t)=\beta^{*}ct-\frac{K^2c}{8\gamma^2\Omega_{\rm u}}\sin(2\Omega_{\rm u}t), \tag{5}$$

where

$$\beta^{*}=1-\frac{1}{2\gamma^2}\left(1+\frac{K^2}{2}\right).$$

K defines the undulator parameter. The representation of the field in Eq. (3) does not satisfy the Maxwell equation, as Eq. (3) reads:

$$\begin{aligned}B_y&=B_0\sin(k_{\rm u}z)\cosh(k_{\rm u}y),\\ B_z&=B_0\cos(k_{\rm u}z)\sinh(k_{\rm u}y).\end{aligned} \tag{6}$$

With a little argument expansion, Eq. (6) reads,

$$\begin{aligned}B_y&=B_0\left(1+\frac{k_{\rm u}^2y^2}{2}\right)\sin(k_{\rm u}z)\\ B_z&=B_0k_{\rm u}y\cos(k_{\rm u}z).\end{aligned} \tag{7}$$

The equation of motion can now be written with Eq. (7) as:

$$\begin{aligned}\ddot{x}&=\frac{e}{m_0\gamma c}\left[B_0\dot{z}\left(1+\frac{k_{\rm u}^2y^2}{2}\right)\sin(k_{\rm u}z)\right.\\ &\left.\quad-B_0\dot{y}k_{\rm u}y\cos(k_{\rm u}z)\right],\\ \ddot{y}&=\frac{e}{m_0\gamma c}B_0\dot{x}k_{\rm u}y\cos(k_{\rm u}z).\end{aligned} \tag{8}$$

We assume that the motion can be decomposed as $x=x_{\rm R}+x_1$ and $y=y_{\rm R}+y_1$, where $x_{\rm R}$ and $y_{\rm R}$ are the reference trajectories, and x_1 and y_1 are additional motions arising due to the additional off-axis field in Eq. (8). By extracting the additional betatron motion, we find that:

$$\frac{{\rm d}^2x_1}{{\rm d}t}=0,\quad \frac{{\rm d}^2y_1}{{\rm d}t^2}+\Omega_1^2y_1=0. \tag{9}$$

The betatron oscillations are described by

$$\Omega_1^2=\frac{K^2\Omega_{\rm u}^2}{2\gamma^2}.$$

The solution to Eq. (9) gives:

$$\begin{aligned}x_1&=x_1(0)+\dot{x}(0)t,\\ y_1&=y_1(0)\cos(\Omega_1t)+\frac{\dot{y}_1(0)}{\Omega_1}\sin(\Omega_1t),\end{aligned} \tag{10}$$

where $x_1(0)$ and $y_1(0)$ are the initial off axis transverse positions, and $\dot{x}_1(0)$ and $\dot{y}_1(0)$ describe the injection angles. The solution of Eq. (10) gives the off-axis position of the betatron oscillation in the electromagnet undulator. The transverse motion is directly coupled to the longitudinal motion $z(t)$ because the energy is constant.

$$\begin{aligned}z(t)&=\beta^{**}ct-\frac{K^2c}{8\gamma^2\Omega_{\rm u}}\sin(2\Omega_{\rm u}t)+\frac{K}{\gamma\Omega_{\rm u}}\dot{x}_1(0)\sin(\Omega_{\rm u}t)\\ &\quad+\frac{1}{8\Omega_1c}(y_1^2(0)\Omega_1^2-\dot{y}_1^2(0))\sin(2\Omega_1t)\end{aligned} \tag{11}$$

where

$$\beta^{**}=1-\frac{1}{2\gamma^2}-\frac{K^2}{4\gamma^2}-\frac{1}{4}\left[\frac{2\dot{x}_1^2(0)}{c^2}+\frac{+y_1^2(0)\Omega_1^2}{c^2}+\frac{\dot{y}_1^2(0)}{c^2}\right].$$

The brightness, which is the energy radiated per unit solid angle per unit frequency interval by an electron in an undulator field, is calculated from the Lienard-Wiechert integral [8],

$$\frac{{\rm d}^2P}{{\rm d}\omega{\rm d}\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\int_0^{\infty}{\rm d}t[\hat{n}\times(\hat{n}\times\beta)]\,\exp\left\{{\rm i}\omega\left(t-\frac{\hat{n}\cdot r}{c}\right)\right\}\right|^2, \tag{12}$$

where $\hat{n}$ is the unit observation vector. Within the small angle approximation, we can write

$$\hat{n}=\left(\psi\cos\phi,\psi\sin\phi,1-\frac{1}{2}\psi^2\right),$$

where ψ is the observation angle and ϕ is the azimuthal angle. The cross product can be evaluated as follows:

$$\begin{aligned}[\hat{n}\times(\hat{n}\times\beta)]_x&=-\frac{\dot{x}_1(0)}{c}+\frac{K}{\gamma}\cos(\Omega_{\rm u}t)+\psi\beta^{**}\\ &\quad+\frac{\psi\dot{x}_1(0)K}{\gamma}\cos(\Omega_{\rm u}t)\\ &\quad-\frac{\psi\dot{y}_1^2(0)}{4c^2}\cos(2\Omega_1t)+\frac{\psi^2\dot{x}_1(0)}{c},\\ [\hat{n}\times(\hat{n}\times\beta)]_y&=\frac{1}{c}[y_1(0)\Omega_1\sin(\Omega_1t)-\dot{y}_1(0)\cos(\Omega_1t)],\\ [\hat{n}\times(\hat{n}\times\beta)]_z&=\frac{\psi K}{\gamma}\cos(\Omega_{\rm u}t)-\psi^2\beta^{**}+\frac{\psi^2\dot{y}_1^2}{4c^2}\cos(2\Omega_1t),\end{aligned} \tag{13}$$

where the integration is carried out over the electromagnet undulator length and ω is the emission frequency of the source. The oscillating part in Eq. (11) can be written:

$$\begin{aligned}\exp\left\{{\rm i}\omega\left(t-\frac{\hat{n}\cdot r}{c}\right)\right\}&=\sum_{m,n=-\infty}^{\infty}\exp[{\rm i}\nu t]\\ &\quad\times J_m(\xi_1,\xi_2)D_n(\xi_3,0;\xi_4,\xi_5),\end{aligned} \tag{14}$$

where

$$\xi_1=\frac{K\omega}{\gamma c\Omega_{\rm u}}\dot{x}_1(0),\quad \xi_2=-\left(\frac{K\omega\psi\cos\phi}{\gamma\Omega_{\rm u}}+\frac{K^2\omega}{8\gamma^2\Omega_{\rm u}}\right),$$

$$\xi_3=-\frac{\omega\psi\cos\phi y_1(0)}{c},\quad \xi_4=\frac{\omega\psi\cos\phi\dot{y}_1(0)}{c\Omega_{\rm u}},$$

and

$$\xi_5=-\left(\frac{\omega\Omega_1y_1^2(0)}{8c^2}-\frac{\omega\dot{y}_1^2(0)}{8c^2\Omega_1}+\frac{\omega\psi^2\dot{y}_1^2(0)}{16c^2\Omega_1}\right).$$

Also,

$$\nu=\left(\frac{\omega}{\omega_1}-m\Omega_{\mathrm{u}}-n\Omega_1\right),$$

$$\omega_1=\frac{2\gamma^2}{1+\dfrac{K^2}{2}+\dfrac{\gamma^2}{2}\left\{\dfrac{2\dot{x}_1^2(0)}{c^2}+\dfrac{\Omega_1^2y_1^2(0)}{c^2}+\dfrac{\dot{y}_1^2(0)}{c^2}\right\}}.$$

$J_m(\xi_1,\ \xi_2)$, $D_n(\xi_3,\ 0;\ \xi_4,\ \xi_5)$ are the Generalized Bessel Functions (GBF) of order m and n. The GBF functions are defined as [9, 10]:

$$\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\theta}J_m(x,y)=\exp[\mathrm{i}(x\sin\theta+y\sin2\theta)],$$

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n\theta}D_n(x,y;z,u)\ =\ \exp\{x\cos\theta+y\cos2\theta+[\mathrm{i}(z\sin\theta+u\sin2\theta)]\}.$$

It can easily be shown that the intensity can be expressed as

$$\frac{\mathrm{d}^2P}{\mathrm{d}\omega\mathrm{d}\Omega}=\frac{e^2\omega^2T^2}{4\pi^2c}[\mathrm{sinc}(\nu T/2)]^2\left[|T_x|^2+|T_y|^2+|T_z|^2\right],\quad(15)$$

with

$$\begin{aligned}T_x\ =\ &\left(-\frac{\dot{x}_1(0)}{c}+\frac{\psi\dot{x}_1(0)}{c}+\psi^2\beta^{**}\right)J_m(\xi_1,\xi_2)\\&\times D_n(\xi_3,0;\xi_4,\xi_5)+\left(\frac{K}{2\gamma}+\frac{\psi\dot{x}_1(0)}{2\gamma}\right)\\&\times[J_{m+1}(\xi_1,\xi_2)+J_{m-1}(\xi_1,\xi_2)]\\&\times D_n(\xi_3,0;\xi_4,\xi_5)-\frac{\psi\dot{y}_1^2(0)}{8c^2}[D_{n+1}(\xi_3,0;\xi_4,\xi_5)\\&+D_{n-1}(\xi_3,0;\xi_4,\xi_5)]\times J_m(\xi_1,\xi_2),\\T_y\ =\ &\frac{\Omega_1y_1(0)}{2\mathrm{i}c}[D_{n+1}(\xi_3,0;\xi_4,\xi_5)-D_{n-1}(\xi_3,0;\xi_4,\xi_5)]\\&\times J_m(\xi_1,\xi_2)-\frac{\dot{y}_1(0)}{2c}\times[D_{n+1}(\xi_3,0;\xi_4,\xi_5)\\&+D_{n-1}(\xi_3,0;\xi_4,\xi_5)]J_m(\xi_1,\xi_2),\\T_z\ =\ &\left(\frac{\psi\dot{x}_1(0)}{c}+\psi^2\beta^{**}\right)J_m(\xi_1,\xi_2)D_n(\xi_3,0;\xi_4,\xi_5)\\&-\frac{K\psi}{2\gamma}[J_{m+1}(\xi_1,\xi_2)+J_{m-1}(\xi_1,\xi_2)]D_n(\xi_3,0;\xi_4,\xi_5)\\&+\frac{\psi\dot{y}_1^2(0)}{8c^2}[D_{n+1}(\xi_3,0;\xi_4,\xi_5)+D_{n-1}(\xi_3,0;\xi_4,\xi_5)]\\&\times J_m(\xi_1,\xi_2).\end{aligned}\quad(16)$$

For ψ=0 and on-axis emission ($\phi=0$), we get from Eq. (16):

$$\begin{aligned}T_x\ =\ &-\frac{\dot{x}_1(0)}{c}J_m(\xi_1,\xi_{2a})J_n(0,\xi_{5a})+\frac{K}{2\gamma}[J_{m+1}(\xi_1,\xi_{2a})\\&+J_{m-1}(\xi_1,\xi_{2a})]J_n(0,\xi_{5a}),\\T_y\ =\ &\frac{\Omega_1y_1(0)}{2\mathrm{i}c}[J_{n+1}(0,\xi_{5a})-J_{n-1}(0,\xi_{5a})]J_m(\xi_1,\xi_{2a})\\&-\frac{\dot{y}_1(0)}{2c}[J_{n+1}(0,\xi_{5a})+J_{n-1}(0,\xi_{5a})]J_m(\xi_1,\xi_{2a}).\end{aligned}\quad(17)$$

where $\xi_{2a}=-\left(\dfrac{K^2\omega}{8\gamma^2\Omega_{\mathrm{u}}}\right)$ and

$$\xi_{5a}=-\left(\frac{\omega\Omega_1y_1^2(0)}{8c^2}-\frac{\omega\dot{y}_1^2(0)}{8c^2\Omega_1}\right).$$

Furthermore, when $y_1(0)=\dot{y}_1(0)=0$, and for $\dot{x}_1(0)\neq0$, Eq. (17) reduces to:

$$T_x=-\frac{\dot{x}_1(0)}{c}J_m(\xi_1,\xi_{2a})+\frac{K}{2\gamma}[J_{m+1}(\xi_1,\xi_{2a})+J_{m-1}(\xi_1,\xi_{2a})].\quad(18)$$

When $\dot{x}_1(0)=\dot{y}_1(0)=0$, we get:

$$T_x=\frac{K}{2\gamma}[J_{m+1}(0,\xi_{2a})+J_{m-1}(0,\xi_{2a})]\times J_n(0,\xi_{5b}).\quad(19)$$

In

$$T_y=\frac{\Omega_1y_1(0)}{2\mathrm{i}c}[J_{n+1}(0,\xi_{5b})-J_{n-1}(0,\xi_{5b})]\times J_m(0,\xi_{2a}),$$

this case $\xi_{5b}=-\dfrac{\omega\Omega_1y_1^2(0)}{8c^2}$.

3 Results & discussion

In this paper we have derived the expression for undulator radiation with an electromagnet undulator. In this case, the undulator parameter, which is a function of the geometry, is given by,

$$K=\frac{e|B_0|}{m_0c\Omega_{\mathrm{u}}}=\frac{e}{m_0c\Omega_{\mathrm{u}}}\frac{2\mu_0I}{\pi h}\frac{\sin(\pi h/\lambda_{\mathrm{u}})}{\sinh(\pi\delta/\lambda_{\mathrm{u}})}.$$

For an electromagnet undulator with parameters h=2.38 mm, δ=8 mm and λ_{u}=10 mm, the value of the undulator parameter is $K=6.957\times10^{-4}$ for I=20 A. In electromagnet undulators, the undulator parameter depends on the winding current. For comparison, a PPM-based undulator with $K\approx1$ requires a winding current of I=28 kA.

The main thrust of our calculation is in Eq. (9)and Eq. (10). These equations apply when the electrons are injected off the undulator axis. Unlike the PPM undulator, Eq. (10) is focused in the y-direction and defocused in the x-direction. This suggests that, when the electrons are injected off-axis with both position and angular offset along the y-direction, there is a correction in the resonance condition. However, an axial offset in the

x-direction does not bring any change in the resonance condition. However, the resonance condition does change when there is an angular offset in the x-direction. When $\nu=0$, the radiation frequency is:

$$\omega=\frac{2\gamma^2(m\Omega_{\rm u}+n\Omega_1)}{1+\dfrac{K^2}{2}+\dfrac{\gamma^2}{2}\left\{\dfrac{2\dot{x}_1^2(0)}{c^2}+\dfrac{\Omega_1^2y_1^2(0)}{c^2}+\dfrac{\dot{y}_1^2(0)}{c^2}\right\}}.$$

The electrons execute additional one-dimensional simple harmonic motion in the y-direction at betatron frequency, which gives rise to emission at these frequencies. Both the even and odd harmonics of the betatron frequency contribute. The intensity expression is modified by the Bessel function in these expressions. Eqs. (17–19) highlight the intensity of electron radiation in an electromagnet undulator. For on-axis radiation, $\psi=0$, Eq. (17) says that the radiation is both horizontally and vertically polarized. The vertical contributions are entirely due to imperfect injection along the y-axis. Eq. (18) gives the intensities when the imperfections along the y-direction are removed. For angular offsets along the x-axis there are no additional harmonics, although the original odd harmonic intensities are modified.

The resonant frequency gives a transparent understanding of the undulator radiation physics. For an example, let us consider the case for which $\gamma = 20$, $y_1(0)=\dot{y}_1(0)=0$. The radiation frequency now reads:

$$\omega_{m,n}=\frac{2\gamma^2(m\Omega_{\rm u}+n\Omega_1)}{1+\dfrac{K^2}{2}+\gamma^2\left\{\dfrac{\dot{x}_1(0)}{c}\right\}^2}.$$

For an undulator with $\lambda_{\rm u}=10$ mm, $\Omega_{\rm u}=18.84\times10^{10}$ rad/s and $\Omega_1=46\times10^5$ rad/s. For $\dot{x}_1/c=0$,

$$\omega_{m,n}=(m\Omega_{\rm u}+n\Omega_1)\ \text{rad/s}.$$

The radiation frequency gets shifted to $799.679(m\Omega_{\rm u}+n\Omega_1)$ rad/s and $794.912(m\Omega_{\rm u}+n\Omega_1)$ rad/s for off axis injection angles of 1 mrad and 4 mrad, respectively. This accounts for a shift of 0.04% and 0.6%, respectively, in the resonance frequency. The undulator emits at $\omega_m=800(m\Omega_{\rm u})$ rad/s for the perfect injection condition. The betatron frequency gives a substructure in the spectrum at 46×10^5 rad/s. This implies that a central emission frequency of 24 THz will show a substructure emission at 0.7 MHz.

We have studied the radiation spectrum of undulator radiation for an electron moving in an electromagnet undulator with imperfect initial conditions. Several points are observed from the analysis. First, the electromagnet undulator is focused in the y-direction and defocused in the x-direction.There are additional betatron oscillations due to off-axis and angular injection along the y-direction of the undulator. The additional betatron oscillations give rise to intensity modification due to the generation of new harmonics, and the resonance condition is also modified. Second, the undulator radiation is modified when there is an angular offset in the electron direction along the x-direction. In this case, both the resonance condition and the intensity are modified. However, the radiation spectrum is independent of axial offset along the x-direction. There are no additional harmonics of emission due to x-axial misalignments. Third, electromagnet undulators are self-biharmonic. For on-axis emission ($y=0$), the undulator is a single-frequency undulator. However, for off-axis emission close to the electromagnet surface, there are additional oscillations at harmonics of the fundamental [3–5]. This is a distinctive feature of electromagnet undulators in comparison with PPM based undulators. In the case of PPM undulators, vanadium permendur strips are used to modify the PPM undulator for higher harmonic lasing [11–17]. In the case of electromagnet undulators, higher harmonic lasing is arguably possible if the electron beam is shifted up and away from the axis.

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