Mirror symmetry, D-brane superpotentials and Ooguri–Vafa invariants of Calabi–Yau manifolds *

ZHANG Shan-Shan(张姗姗) YANG Fu-Zhong(杨富中)[1)]

School of Physical Sciences, University of Chinese Academy of Sciences, Yuquan Road 19 A, Beijing 100049, China

Abstract: The D-brane superpotential is very important in the low energy effective theory. As the generating function of all disk instantons from the worldsheet point of view, it plays a crucial role in deriving some important properties of the compact Calabi–Yau manifolds. By using the generalized GKZ hypergeometric system, we will calculate the D-brane superpotentials of two non-Fermat type compact Calabi–Yau hypersurfaces in toric varieties, respectively. Then according to the mirror symmetry, we obtain the A-model superpotentials and the Ooguri–Vafa invariants for the mirror Calabi–Yau manifolds.

Key words: D-brane, superpotential, mirror symmetry, Ooguri–Vafa invariant

PACS: 02.40.Tt, 11.15.Tk, 11.25.Tq **DOI:** 10.1088/1674-1137/39/12/121002

1 Introduction

The theory of topological strings, which is derived from the two dimensional $(N,\hat{N})=(2,2)$ superconformal field theory, has developed considerably over the past few years and has had a deep influence on mathematics. The D-brane superpotential, the generating function of correlation functions, is a particularly vital physical quantity which is a section of special holomorphic line bundles of the moduli space from the mathematical perspective. Through the superpotential we can derive a series of important properties for the CY manifolds, such as Yukawa couplings, Ooguri–Vafa invariants and so on. Therefore the calculation of the superpotential is very meaningful.

Some important properties of the moduli spaces for various Calabi–Yau manifolds [1–3] have been well studied via the mirror symmetry which was first mentioned in the local operator algebra of the $N=2$ string theory [4]. It is well known that mirror symmetry connects two different moduli spaces which are respectively parameterized by Kahler geometry deformation and complex geometry deformation in A- and B-models. In the A-model there exist contributions from the instantons while there are none in the B-model. So calculating the superpotential directly in the A-model is rather difficult. In fact only in several special cases do we know the corresponding brane configuration on mirror A-model side for a given brane configuration in the compact CY manifold [5] derived from the GKZ system in the B-model. In the GKZ system the superpotential is related to the period integral, and the Hodge theoretic approach [6] provides a useful insight into studying the period integrals for CY manifolds which satisfies the Picard–Fuchs differential equation, which is closely related to the GKZ system.

Recently, for compact CY manifolds, there have been some great developments in calculating the quantum corrected domain wall tensions on the CY threefolds via open–closed mirror symmetry [7, 8]. The properties of some compact Calabi–Yau manifolds have been studied in Refs. [9–16]. In this note, we compute the D-brane superpotentials for two non-Fermat CY threefolds in detail via mirror maps and GKZ hypergeometric system.

The structure of this paper is as follows. In Section 2 we describe the generalized GKZ hypergeometric system. The solution of the GKZ hypergeometric system is just the integral period. We also outline the approach to construct the corresponding polyhedron Δ and its mirror polyhedron Δ^* for the Calabi–Yau manifold. Then we review how to calculate the superpotential. In Section 3 we analyze two non-Fermat type compact CY manifolds in toric varieties, respectively, and compute their superpotentials as well as some disk invariants with the method referred to previously. The last section is the conclusion.

Received 9 March 2015, Revised 14 July 2015

* Supported by Y4JT01VJ01 and NSFC(11475178)

1) E-mail: fzyang@ucas.ac.cn, Corresponding Author

2 Toric geometry, relative period integrals and the GKZ system

We divide this section into two parts to review some related background.

2.1 Superpotential on the D-brane

In the presence of some background fluxes and space-filling D-branes, type II string theory compactification on the Calabi–Yau manifold gives rise to the $N=1$ low energy effective theories [13], whose effective superpotential is captured by the relative period integral of the holomorphic three-form $\Omega(z)$ over the relative cycles with boundaries wrapped by D-branes [17]. As is listed in Refs. [18–22], the above integral is derived from the action of a holomorphic Chern–Simons theory on the brane which wraps the holomorphic curves.

For the D-brane wrapping internal cycles of Calabi–Yau manifold X, the corresponding effective superpotential is [19]

$$\mathcal{W}_{\text{brane}}=\int_X \Omega\wedge\text{Tr}\left[A\wedge\bar{\partial}A+\frac{2}{3}A\wedge A\wedge A\right], \tag{1}$$

where if there are N branes, A is a holomorphic $U(N)$ gauge connection on X and Ω is the holomorphic three-form on X. For a type IIB string, the effective superpotential is a linear combination of the relative period integrals [21, 23, 25].

$$\mathcal{W}_{\text{brane}}(\varphi,\xi)=\hat{N}_{\text{a}}\hat{\Pi}^{\text{a}}(\varphi,\xi), \tag{2}$$

where $\hat{N}_{\text{a}}$ stands for the homology class, which is wrapped with the D-brane. $\hat{\Pi}^{\text{a}}(\varphi,\xi)$ represents the period integral

$$\hat{\Pi}^{\text{a}}(\varphi,\xi)=\int_{\gamma^{\text{a}}(\xi)}\Omega(\varphi). \tag{3}$$

Here, ξ and φ stand for the open- and closed-string moduli respectively. The internal background fluxes $H=H^{\text{RR}}+H^{\text{NS}}$ lead to an effective superpotential [13–15, 26–29] which is defined by

$$\mathcal{W}_{\text{Flux}}=\int_X \Omega\wedge H=\int_X \Omega\wedge(H^{\text{RR}}+\tau H^{\text{NS}}), \tag{4}$$

where Ω denotes the holomorphic three-form on the Calabi–Yau manifold, and τ denote the complex couplings for the type II string of the B-model. In this note, we only consider the RR flux, so the induced superpotential becomes

$$\mathcal{W}_{\text{flux}}=\int_X \Omega\wedge H^{\text{RR}}=\sum_\alpha N_\alpha \Pi^\alpha(\varphi). \tag{5}$$

Therefore the combined superpotential generated by the D-brane and flux is [30, 31]

$$\mathcal{W}(\varphi,\xi)=\mathcal{W}_{\text{brane}}(\varphi,\xi)+\mathcal{W}_{\text{flux}}(\varphi)=\sum N_\Sigma \Pi_\Sigma(\varphi,\xi). \tag{6}$$

Here the coefficient N_Σ denotes both the D-brane topological charge and the RR flux quantum data and $\Pi_\Sigma(\varphi,\xi)$ denotes the integral of the three-form $\Omega(\varphi)$ over the three-chains in the relative integer homology group, which is defined by

$$\Pi_\Sigma(\varphi,\xi)=\int_{\Gamma^\alpha(\xi)}\Omega(\varphi),\quad \Gamma^\alpha(\xi)\in H_3(Y,S,Z). \tag{7}$$

The relative period integral referred to previously, $\hat{\Pi}^{\text{a}}(\varphi,\xi)$, is equal to the domain wall tension $\mathcal{T}(\varphi,\xi)$ [5, 20, 21, 30, 31]. $\mathcal{T}(\varphi,\xi)$ is defined as

$$\mathcal{T}(\varphi,\xi)=\mathcal{W}(C^+_{(\varphi,\xi)})-\mathcal{W}(C^-_{(\varphi,\xi)}). \tag{8}$$

At its critical point $\xi=z$, $\mathcal{T}(\varphi,\xi)$ is identical to the on-shell domain wall tension $T=W(C^+_\varphi)–W(C^-_\varphi)$. As shown in Refs. [3, 8, 32, 33], at the critical points, the domain wall tensions are considered as a normal function from which the Abel–Jacobi invariants can be derived.

For the D-brane in the A-model, the superpotential which is expressed in terms of the flat closed/open coordinates can be calculated as the generating function of the correlation functions [21, 30, 34–36]. It is defined by

$$\mathcal{W}(t,\hat{t})=\sum_{\vec{k},\vec{m}}G_{\vec{k},\vec{m}}q^{\text{d}\vec{k}}\hat{q}^{\text{d}\vec{m}}=\sum_{\vec{k},\vec{m}}\sum_d n_{\vec{k},\vec{m}}\frac{q^{\text{d}\vec{k}}\hat{q}^{\text{d}\vec{m}}}{k^2}. \tag{9}$$

Here, $q=\text{e}^{2\pi\text{i}t}$, $\hat{q}=\text{e}^{2\pi\text{i}\hat{t}}$ and $n_{\vec{k},\vec{m}}$ is the Ooguri–Vafa invariant. Mirror symmetry, which indicates that the two superpotentials for D-branes in the A- and B-models, are related to each other by the mirror map, gives us a method to compute the Ooguri–Vafa invariant which is closely related to the open Gromov–Witten invariant $G_{\vec{k},\vec{m}}$, [5]. The superpotentials and the Ooguri–Vafa invariants are as follows [7, 37]:

$$\frac{W^{(\pm)}(z(q))}{\omega_0(z(q))}=\frac{1}{(2\pi\text{i})^2}\sum_{k\in\text{odd}}\sum_{d_1\geqslant 0,d_2\in\text{odd}}n^{(\pm)}_{d_1,d_2}\frac{q_1^{kd_1}q_2^{kd_2/2}}{k^2}, \tag{10}$$

where $q_{\text{a}}=\text{e}^{t_{\text{a}}}$ $(a=1,\ 2)$ and

$$\omega_0(z)=\sum_n c(n)z^n=\sum_n\frac{\prod_j\Gamma\left(\sum_{\text{a}}l^{(\text{a})}_{0j}n_{\text{a}}+1\right)}{\prod_i\Gamma\left(\sum_{\text{a}}l^{(\text{a})}_i n_{\text{a}}+1\right)}z^n. \tag{11}$$

Here the mirror map t_{a} is defined as $t_{\text{a}}=\dfrac{\partial_{\text{a}}\omega_0}{\omega_0}$.

2.2 Toric geometry and the GKZ System

The generalized hypergeometric system was first introduced in Ref. [38], and soon developed quickly in mirror symmetry [6, 39–42]. Let us define a mirror pair of hypersurfaces (X, X*) in two toric ambient spaces (V, V*), respectively. The toric varieties (V, V*) are related to the fans ($\Sigma(\Delta)$, $\Sigma(\Delta^*)$) induced by the two dual polyhedra (Δ, Δ^*). The defining polynomial for the hypersurface is defined as:

$$\mathcal{P}=\sum_{i=0}^{p-1} a_i \prod_{k=1}^{4} X_k^{v_{i,k}^*}. \tag{12}$$

Or we can write the above equation in another way

$$\mathcal{P}=\sum_{i=0}^{p-1} a_i \prod_{v_j\in\Delta} x_j^{\langle v_j, v_i^*\rangle+1}. \tag{13}$$

Here a_i is complex parameter and X_k are inhomogeneous coordinates on the open torus, x_i is the homogeneous coordinates.

The general integral period is expressed as

$$\Pi(a_i)=\frac{1}{(2\pi \mathrm{i})^4}\int_{|X_k|=1}\frac{1}{P}\prod_{k=1}^{4}\frac{\mathrm{d}X_k}{X_k}. \tag{14}$$

It is shown in Refs. [39, 40] that the period can be annihilated by a GKZ hypergeometric differential system

$$\mathcal{D}_l \Pi(\mathrm{a})=0 \quad (l\in L), \quad \mathcal{Z}_i \Pi(\mathrm{a})=0 \quad (i=0,1,\cdots,p), \tag{15}$$

the operators $\mathcal{D}_l$ and $\mathcal{Z}_j$ are expressed as

$$\mathcal{D}_l = \prod_{l_i>0}\left(\frac{\partial}{\partial a_i}\right)^{l_i}-\prod_{l_j<0}\left(\frac{\partial}{\partial a_j}\right)^{-l_j}, \qquad (l\in L), \tag{16}$$

$$\mathcal{Z}_j = \sum_{i=0}^{p}\bar{v}_{i,j}^*\theta_{a_i}-\beta_j. \qquad (j=0,1,\cdots,n). \tag{17}$$

The torus invariant algebraic coordinates z_a in the large complex structure limit is [3]

$$z_\mathrm{a}=(-1)^{l_0^\mathrm{a}}\prod_j a_j^{l_j^\mathrm{a}}, \tag{18}$$

where l^a is the set of basic vectors which denote the generators of the Mori cone. Then, by $\theta_\mathrm{a}=z_\mathrm{a}\partial_{z_\mathrm{a}}$, (2.16) changes into

$$\mathcal{D}_l=\prod_{k=1}^{l_0}(\theta_0-k)\prod_{l_i>0}\prod_{k=0}^{l_i-1}(\theta_i-k)-(-)^{l_0}z_\mathrm{a}\prod_{k=1}^{-l_0}(\theta_0-k)\prod_{l_i<0}\prod_{k=0}^{-l_i-1}, \tag{19}$$

where l is the linear combination of l^a. One can refer to [32, 43–45] for more details. The result for the GKZ system is described as

$$\mathcal{B}_{\{l^\mathrm{a}\}}(z_\mathrm{a};\rho_\mathrm{a}) = \sum_{n_1,\cdots,n_\mathrm{N}\in Z_0^+}\frac{\Gamma\left(1-\sum_\mathrm{a} l_0^\mathrm{a}(n_\mathrm{a}+\rho_\mathrm{a})\right)}{\prod_{i>0}\Gamma\left(1+\sum_\mathrm{a} l_i^\mathrm{a}(n_\mathrm{a}+\rho_\mathrm{a})\right)} \times\prod_\mathrm{a} z_\mathrm{a}^{(n_\mathrm{a}+\rho_\mathrm{a})}. \tag{20}$$

3 Study of two compact non-Fermat type Calabi–Yau manifolds

In this section we will calculate the superpotentials and disk invariants for two compact CY in the weighted projective space, with the method described in Section 2.

3.1 Calabi–Yau hypersurface $X_7(1, 1, 1, 1, 3)$

$X_7(1, 1, 1, 1, 3)$ is a hypersurface in the weighted projective space $\boldsymbol{P}^4(1, 1, 1, 1, 3)$. Let X denote this hypersurface; then, its mirror manifold X^* is denoted by $X^*=\hat{X}/H$, where $\hat{X}$ represents the CY 3-fold $X_{14}(1, 2, 2, 2, 7)$ and H is defined by $(h_i^j)=\frac{1}{7}(1, 0, 6, 0, 0)$, $\frac{1}{7}(1, 6, 0, 0, 0)$. So, $X_7(1, 1, 1, 1, 3)$ is isomorphic to $X_{14}(1, 2, 2, 2, 7)$, which had been checked in Ref. [46]. The hypersurface $X_7(1, 1, 1, 1, 3)$ is defined as the zero locus of the polynomial $\mathcal{P}$.

$$\mathcal{P}=x_1^7+x_2^7+x_3^7+x_4^7+x_4x_5^2. \tag{21}$$

The weighted projective space $\boldsymbol{P}^4(1, 1, 1, 1, 3)$ is a toric variety, and the vertices of its corresponding polyhedron Δ are as follows:

$$v_1=(-1, -1, -1, 1), \quad v_2=(-1, -1, -1, -1),$$
$$v_3=(-1, -1, 1, 0), \quad v_4=(6, -1, -1, -1),$$
$$v_5=(-1, 1, -1, 0), \quad v_6=(-1, 3, -1, -1),$$
$$v_7=(-1, -1, 3, -1), \quad v_8=(0, -1, 3, -1),$$
$$v_9=(0, 3, -1, -1).$$

The vertices of the corresponding dual polyhedron Δ^* are

$$v_1^*=(-1, -1, -1, -3), \quad v_2^*=(1, 0, 0, 0), \quad v_3^*=(0, 1, 0, 0),$$
$$v_4^*=(0, 0, 1, 0), \quad v_5^*=(0, 0, 0, 1), \quad v_6^*=(0, 0, 0, -1).$$

There exists only one integral point denoted as $v_0^*=(0, 0, 0, 0)$ in Δ^*. For Δ^*, the charge vector of the Mori cone is

$$l^{(1)}=(-2, 0, 0, 0, 0, 1, 1), \quad l^{(2)}=(-1, 1, 1, 1, 1, 0, -3).$$

Consider the divisor

$$Q(D)=x_3^7+z_3x_4^7, \tag{22}$$

at the critical point $z_3=1$. Let

$$u_1=-\frac{z_1}{z_3}(1-z_3)^2, \quad u_2=z_2, \tag{23}$$

then according to (2.19) the GKZ system of the two-parameters family become

$$\mathcal{D}_1 = \tilde{\theta}_1(\tilde{\theta}_1-3\tilde{\theta}_2)-(2\tilde{\theta}_1+\tilde{\theta}_2)(2\tilde{\theta}_1+\tilde{\theta}_2-1)z_1, \tag{24}$$

$$\mathcal{D}_2 = \tilde{\theta}_2^2(7\tilde{\theta}_2-2\tilde{\theta}_1)+4\tilde{\theta}_2^2(2\tilde{\theta}_1+\tilde{\theta}_2-1)z_1 -7\prod_{i=1}^{3}(2\tilde{\theta}_2-\tilde{\theta}_1-i). \tag{25}$$

The solution to this GKZ system is written as

$$\Pi_1(u_1,u_2) = \frac{c}{2}B_{\{\tilde{l}\}}\left(u_1,u_2,0,\frac{1}{2}\right), \tag{26}$$

$$\Pi_2(u_1,u_2) = \frac{c}{2}B_{\{\tilde{l}\}}\left(u_1,u_2,\frac{1}{2},\frac{1}{2}\right). \tag{27}$$

At the critical point $z_3=1$, the on-shell superpotential satisfies $W_{\mathrm{C}}^{+}=W_{\mathrm{C}}^{-}$ according to the Z_2 symmetry. So the on-shell superpotential is described as

$$W^{\pm}(z_1,z_2)=\frac{1}{2\pi\mathrm{i}}\int_{\xi_0}^{\pm\sqrt{z_3}}\Pi(z_1,z_2,\xi^2)\frac{\mathrm{d}\xi}{\xi}|_{z_3=1}. \tag{28}$$

For this model the on-shell superpotentials are expressed as

$$W_1^{\pm}=\mp\frac{c}{8}\sum_{n_1,n_2\geqslant 0}\frac{\Gamma\left(2n_1+n_2+\frac{3}{2}\right)z_1^{n_1}z_2^{n_2+\frac{1}{2}}}{\Gamma(n_1+1)\Gamma\left(n_1-3n_2-\frac{1}{2}\right)\Gamma\left(n_2+\frac{3}{2}\right)^4}, \tag{29}$$

$$W_2^{\pm}=\mp\frac{c}{8}\sum_{n_1,n_2\geqslant 0}\frac{\Gamma\left(2n_1+n_2+\frac{5}{2}\right)z_1^{n_1+\frac{1}{2}}z_2^{n_2+\frac{1}{2}}}{\Gamma\left(n_1+\frac{3}{2}\right)\Gamma(n_1-3n_2)\Gamma\left(n_2+\frac{3}{2}\right)^4}. \tag{30}$$

The flat coordinates in the A- and B-models are connected via the mirror map $t_{\mathrm{a}}=\frac{\partial_{\mathrm{a}}\omega_0}{\omega_0}$. The mirror map is as follows:

$$\begin{aligned} t_1 &= \log(z_1)+2z_1+3z_1^2+\frac{20}{3}z_1^3+68z_1^2z_2-10z_1z_2 \\ &\quad +2z_2-15z_2^2+66z_1z_2^2+\frac{560}{3}z_2^3+o(z^3), \end{aligned} \tag{31}$$

$$\begin{aligned} t_2 &= \log(z_2)-6z_2+45z_2^2-560z_2^3-198z_2^2z_1+30z_2z_1 \\ &\quad +9z_1-204z_2z_1^2+\frac{43}{2}z_1^2+62z_2^3+o(z^3), \end{aligned} \tag{32}$$

and the corresponding inverse mirror map is

$$z_1 = q_1-2q_1^2+3q_1^3-2q_1q_2+5q_1q_2^2+36q_1^2q_2+o(q^4), \tag{33}$$

$$z_2 = q_2+6q_2^2+9q_2^3-9q_1q_2-120q_1q_2^2+37q_1^2q_2+o(q^4). \tag{34}$$

According to (2.10) we can derive the Ooguri–Vafa invariants from the on-shell superpotentials. The results are listed in the following Tables (Table 1, Table 2).

3.2 Calabi–Yau hypersurface $X_9(1,\ 1,\ 2,\ 2,\ 3)$

$X_9(1,\ 1,\ 2,\ 2,\ 3)$ is a hypersurface defined in the weighted projective space $\boldsymbol{P}^4(1,\ 1,\ 2,\ 2,\ 3)$. Let X denote the corresponding Calabi–Yau three-fold; then, its mirror manifold X^* is denoted by $X^*=\hat{X}/H$, where $\hat{X}$ represents the CY 3-fold $X_{12}(1,\ 1,\ 3,\ 3,\ 4)$ and H is defined by $(h_i^j)=\frac{1}{9}(1,\ 8,\ 0,\ 0,\ 0)$, $\frac{1}{4}(1,\ 0,\ 3,\ 0,\ 0)$. So, $X_9(1,\ 1,\ 2,\ 2,\ 3)$ is isomorphic to $X_{12}(1,\ 1,\ 3,\ 3,\ 4)$. The polyhedron Δ for this model has the vertices

$$\begin{gathered} v_1=(-1,\ -1,\ -1,\ 2),\ \ v_2=(-1,\ -1,\ -1,\ -1), \\ v_3=(-1,\ -1,\ 2,\ 0),\ \ v_4=(8,\ -1,\ -1,\ -1), \\ v_5=(-1,\ 2,\ -1,\ 0),\ \ v_6=(-1,\ 3,\ -1,\ -1), \\ v_7=(-1,\ -1,\ 3,\ -1),\ \ v_8=(0,\ -1,\ 3,\ -1), \\ v_9=(0,\ 3,\ -1,\ -1) \end{gathered}$$

then the dual polyhedron Δ^* has vertices

$$v_1^*=(-1,\ -2,\ -2,\ -3),\ \ v_2^*=(1,\ 0,\ 0,\ 0),\ \ v_3^*=(0,\ 1,\ 0,\ 0),$$

$$v_4^*=(0,\ 0,\ 1,\ 0),\ \ v_5^*=(0,\ 0,\ 0,\ 1),\ \ v_6^*=(0,\ -1,\ -1,\ -1).$$

There exist no points inside the polyhedron Δ^* but the original point $v_0^*=(0,0,0,0)$. We define the hypersurface $X_9(1,1,2,2,3)$ as the zero locus of the above polynomial $\mathcal{P}$.

$$\mathcal{P}=x_1^9+x_2^9+x_1x_3^4+x_2x_4^4+x_5^3, \tag{35}$$

Table 1. $n_{d_1,d_2}^{(1,+)}$

$d_1\diagdown d_2$	1	3	5	7	9	11
0	−2	2	−10	84	−858	13820
1	28	−28	252	−2828	36400	−729130
2	70	112	−2702	42910	−714140	17644120
3	0	252	16716	−391580	8645280	−262033434
4	0	6832	−83286	2543170	−74112654	2741674588
5	0	84364	315980	−13445796	496350736	−22527070394

Table 2. $n_{d_1,d_2}^{(2,+)}$

$d_1\diagdown d_2$	1	3	5	7	9	11
1	0	0	0	0	0	0
3	−70	0	0	0	0	137970
5	−28	56	−140	896	−8008	−5747938
7	2	−792	3164	−27608	315050	108056554
9	0	−38962	−32424	391104	−5784408	−1230725514
11	0	−84364	358288	−3814860	68819744	10291361734

$l^{(1)}=(-3, -1, -1, 1, 1, 0, 3)$, $l^{(2)}=(-1, 1, 1, 0, 0, 1, -2)$

are the generators of the Mori cone related to this model.

Consider the divisor

$$Q(D)=x_1^9+z_3x_2^9, \tag{36}$$

at the critical point $z_3=1$. Let

$$u_1=-\frac{z_1z_3}{(1-z_3)^2}, \quad u_2=-\frac{z_2}{z_3}(1-z_3)^2, \tag{37}$$

according to (2.16) the GKZ system for the two-parameters family become

$$\mathcal{D}_1=\tilde{\theta}_2(\tilde{\theta}_2-\tilde{\theta}_1)^2-(3\tilde{\theta}_1+\tilde{\theta}_2)(3\tilde{\theta}_1-2\tilde{\theta}_2+2)(3\tilde{\theta}_1-2\tilde{\theta}_2+1)z_2, \tag{38}$$

$$\begin{aligned}\mathcal{D}_2 = &(\tilde{\theta}_2-\tilde{\theta}_1)^2-(\tilde{\theta}_2-\tilde{\theta}_1)(3\tilde{\theta}_1-2\tilde{\theta}_2)+4\tilde{\theta}_1(3\tilde{\theta}_1-2\tilde{\theta}_2)\\ &+3z_1(3\tilde{\theta}_1-2\tilde{\theta}_2)(3\tilde{\theta}_1-2\tilde{\theta}_2-1)-16z_1(\tilde{\theta}_2-\tilde{\theta}_1)^2\\ &-48z_1z_2(3\tilde{\theta}_1+\tilde{\theta}_2+1)(3\tilde{\theta}_1+\tilde{\theta}_2+2)\\ &-48z_1z_2(3\tilde{\theta}_1+\tilde{\theta}_2+1)(3\tilde{\theta}_1-2\tilde{\theta}_2).\end{aligned} \tag{39}$$

The solution to this GKZ system is written as

$$\Pi_1(u_1,u_2)=\frac{c}{2}B_{\{\tilde{l}\}}\left(u_1,u_2,\frac{1}{2},0\right), \tag{40}$$

$$\Pi_2(u_1,u_2)=\frac{c}{2}B_{\{\tilde{l}\}}\left(u_1,u_2,\frac{1}{2},\frac{1}{2}\right). \tag{41}$$

Similarly, in this model the on-shell superpotentials satisfy $W_C^+=W_C^-$ according to the Z_2 symmetry. So the superpotentials are described as

$$W^{\pm}(z_1,z_2)=\frac{1}{2\pi i}\int_{\xi_0}^{\pm\sqrt{z_3}}\Pi(z_1,z_2,\xi^2)\frac{d\xi}{\xi}|_{z_3=1}. \tag{42}$$

At the critical point $z_3=1$, on-shell superpotentials are expressed as

$$\begin{aligned}W_1^{\pm} = &\mp\frac{c}{8}\sum_{n_1,n_2\geqslant 0}\frac{\Gamma\left(3n_1+n_2+\frac{5}{2}\right)}{\Gamma\left(-n_1+n_2+\frac{1}{2}\right)^2\Gamma\left(3n_1-2n_2+\frac{5}{2}\right)}\\ &\times\frac{1}{\Gamma(n_2+1)\Gamma\left(n_1+\frac{3}{2}\right)^2}z_1^{n_1+\frac{1}{2}}z_2^{n_2},\end{aligned} \tag{43}$$

$$\begin{aligned}W_2^{\pm} = &\mp\frac{c}{8}\sum_{n_1,n_2\geqslant 0}\frac{\Gamma(3n_1+n_2+3)}{\Gamma(-n_1+n_2+1)^2\Gamma\left(3n_1-2n_2+\frac{3}{2}\right)}\\ &\times\frac{1}{\Gamma\left(n_2+\frac{3}{2}\right)\Gamma\left(n_1+\frac{3}{2}\right)^2}z_1^{n_1+\frac{1}{2}}z_2^{n_2+\frac{1}{2}}.\end{aligned} \tag{44}$$

The mirror map is

$$\begin{aligned}t_1 = &\log(z_1)+30z_1z_2-3z_2+252z_1z_2^2-10z_2^3\\ &+927z_1^2z_2^2+288z_1z2^3-\frac{105}{4}z_2^4+o(z^4),\end{aligned} \tag{45}$$

$$\begin{aligned}t_2 = &\log(z_2)+2z_2+3z_2^2+\frac{20}{3}z_2^3+74z_1z_2-168z_1z_2^2\\ &-192z_1z_2^3+8853z_1^2z_2^2+\frac{35}{2}z_2^4+o(z^4),\end{aligned} \tag{46}$$

and the corresponding inverse mirror map is

$$z_1=q_1+3q_1q_2+3q_1q_2^2+q_1q_2^3-30q_1^2q_2-594q_1^2q_2^2+o(q^4), \tag{47}$$

$$z_2=q_2-2q_2^2+3q_2^3-4q_2^4-74q_1q_2^2+390q_1q_2^3+o(q^4). \tag{48}$$

Analogous to computing the disk invariants of the $X_7(1, 1, 1, 1, 3)$, we have the results listed in the following Tables (Table 3, Table 4).

Table 3. $n_{d_1,d_2}^{(1,+)}$

$d_1\diagdown d_2$	0	1	2	3	4	5
1	2	18	2	2	2	2
3	0	0	1584	−710	−626	−616
5	0	0	0	38018	208244	49382
7	0	0	0	0	1745190	111219514
9	0	0	0	0	0	90081018
11	0	0	0	0	0	−94519326

Table 4. $n_{d_1,d_2}^{(2,+)}$

$d_1\diagdown d_2$	1	3	5	7	9	11
1	16	0	−2	−2	−2	46
3	0	−80	2208	608	584	26332
5	0	0	720	158784	−4120	3039542
7	0	0	0	−8848	15904928	94908448
9	0	0	0	0	126608	3473058320
11	0	0	0	0	0	−434510208

4 Conclusion

The D-brane superpotential plays a crucial role in both physics and mathematics. From the physical point of view, it determines the vacuum of the low energy $N=1$ effective theory. From the A-model worldsheet viewpoint, it is the generating function of the Ooguri–Vafa invariants of the Calabi–Yau manifold and the submanifold which is wrapped by the D-branes in the A-model. These Ooguri–Vafa invariants are closely related to the number of BPS states. From the mirror geometric viewpoint, it is the integral period which is the solution to the generalized GKZ system. It is very hard to calculate directly the D-brane superpotential for compact Calabi–Yau manifolds in the A-model because these superpotentials are essentially non-perturbative, and are impossible to obtain in the perturbative or localization ways which are important methods to compute the D-brane superpotential in non-compact Calabi–Yau manifolds in the A-model. An effective approach to obtain the D-brane superpotential is using the blown-up geometry of target space along the submanifold wrapped by the D-branes [9, 47]. The alternative approach to compute the superpotential of the D-brane in compact Calabi–Yau manifolds in the A-model is via the algebraic geometric method and mirror symmetry.

In this paper, we extend the generalized GKZ system in a Fermat Calabi–Yau three-folds to the compact non-Fermat Calabi–Yau three-folds which have been less studied so far in contrast to the Fermat type Calabi–Yau three-folds. We first constructed the generalized GKZ system for the compact non-Fermat type Calabi–Yau manifolds, then worked out the corresponding D-brane superpotential in the mirror B-model by the algebraic geometric method. The superpotential in the A-model was obtained according to mirror symmetry. Finally the Ooguri–Vafa invariants were extracted from the A-model superpotential.

These superpotentials have potential phenomenological applications. Furthermore, according to the type II string/M-theory/F-theory duality, in the weak decoupling limit $g_s \to 0$, these superpotentials of Type II strings give the Gukov–Vafa–Witten superpotentials $\mathcal{W}_{\rm GVW}$ of F-theory compactified on the dual fourfold. On the other hand, since there is not yet a systematic mathematical method to compute them, it is difficult to get from other approaches, those Ooguri–Vafa invariants predicted in this paper. Those Ooguri–Vafa invariants provide some concrete data which could potentially be checked by an independent mathematical calculation.

References

1 Hosono S, Klemm A, Theisen S et al. Nucl. Phys. B, 1995, **433**: 501–554
2 Berglund P, Katz P, Klemm A. Nucl. Phys. B, 1995, **456**: 153–204
3 Alim M, Hecht M, Mayr P et al. JHEP, 2011, **1106**: 103
4 Lerche W, Vafa C, Warner N. Nucl. Phys. B, 1989, **324**: 427
5 Jockers H, Soroush M. Commun. Math. Phys., 2009, **290**: 249–290
6 Hosono S, lian B H, Yau S T et al. Commun. Math. Phys., 1996, **182**: 535–578
7 Walcher J. Commun. Math. Phys., 2007, **276**: 671–689
8 Morrison D R, Walcher J. arXiv: hepth/0709.4028
9 Grimm T W, Ha T W, Klemm A et al. Nucl. Phys. B, 2009, **816**: 139–184
10 Candelas P, Font A, Katz S et al. Nucl. Phys. B, 1994, **429**: 626–674
11 Douglas M R, Govindarajan S, Jayaraman T et al. Commun. Math. Phys., 2004, **248**: 85–118
12 Aspinwall P S. arXiv: hepth/0403166
13 XU Feng-Jun, YANG Fu-Zhong. Chin. Phys. C, 2014, **38**(3): 033103
14 CHENG Shi, XU Feng-Jun, YANG Fu-Zhong. Modern Physics Letters A, 2014, **29**(12): 1450061
15 XU Feng-Jun, YANG Fu-Zhong. Modern Physics Letters A, 2014, **29**(13): 1450062
16 XU Feng-Jun, YANG Fu-Zhong, Chin. Phys. C, 2015, **39**(4): 043102
17 Fuji H, Nakayama S, Shimizu M et al. arXiv: hepth/1011.2347
18 Baumgartl M, Brunner L, Gaberdiel M R. JHEP, 2007, **0707**: 061
19 Witten E. Prog. Math., 1995, **133**: 637–678
20 Witten E. Nucl. Phys. B, 1997, **507**: 658–690
21 Aganagic M, Vafa C. arXiv: hepth/0012041
22 YANG Fu-Zhong. Commun. Theor. Phys., 2003, **40**: 537–542
23 Aganagic M, Klemm A, Vafa C. Z. Naturforsch. A, 2002, **57**: 1–28
24 Witten E. Nucl. Phys. B, 1993, **403**: 159–222
25 Mayr P. Adv. Theor. Math. Phys., 2002, **5**: 213–242
26 Dauglas M R, Kachru S. Rev. Mod. Phys., 2007, **79**: 733–796
27 Haack M, Louis J. Phys. Lett. B, 2001, **507**: 296–304
28 Grimm T W, Ha T W, Klemm A et al. arXiv: hepth/0909.2025
29 Gukov S, Vafa C, Witten E. Nucl. Phys. B, 2000, **584**: 69–108
30 Lerche W, Mayr P, Warner N. arXiv: hepth/0207259
31 Lerche W, Mayr P, Warner N. arXiv: hepth/0208039
32 Si-Li, Lian B H, Yau S T. arXiv: hepth/0910.4215
33 Clemens H. arXiv: hepth/0206219
34 Baumgartl M, Brunner L, Soroush M. Nucl. Phys. B, 2011, **843**: 602–637
35 Alim M, Hecht M, Mayr P et al. arXiv:0901.2937
36 Ooguri H, Vafa C. Nucl. Phys. B, 2000, **577**: 419–438
37 Walcher J. arXiv: hepth/0904.4905
38 Gel'fand I M, Zelevinski A, Kapranov M. Funct. Anal. Appl., 1989, **28**: 94
39 Batyrev V V. arXiv: hepth/9310003
40 Batyrev V V, Straten D V. Commun. Math. Phys., 1995, **168**: 493
41 Hosono S, Klemm A, Theisen S et al. Commun. Math. Phys., 1995, **167**: 301–350
42 Hosono S. GKZ System. arXiv: hepth/9707003
43 Alim M, Hecht M, Jockers H et al. Nucl. Phys. B, 2010, **841**: 303–338
44 Aganagic M, Beem C. JHEP, 2011, **1112**: 060
45 Aganagic M, Klemm A, Marino M et al. Commun. Math. Phys., 2005, **254**: 452–478
46 Candelas P, Ossa X, Katz S. Nucl. Phys. B, 1995, **450**: 267–292
47 Grimm T W, Klemm, Klevers D. JHEP, 2011, **1105**: 113