Krein regularization of $\lambda\phi^3$ theory

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Abstract: In this paper, the one-loop self energy of $\lambda\phi^3$ theory is calculated by using Krein regularization in four and six dimensions and the result, which is finite, is compared with the conventional result of $\lambda\phi^3$ theory in Hilbert space. The self energy is calculated in the one-loop approximation and the result is automatically regularized as a result of "Krein Regularization".

Key words: Krein space quantization, quantum metric fluctuation, Krein regularization

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1 Introduction

$\lambda\phi^3$ theory is super-renormalizable in four dimensions; in six dimensions this theory is renormalizable. Conventional methods, studying $\lambda\phi^3$ theory requires dimensional or Paui-Villars regularization and subsequent renormalization. In contrast, it is possible to show that if Krein space quantization and quantum metric fluctuation are combined, the resulting "Krein regularization" [1, 2] can be used to study the theory in a finite form with no need for a renormalization procedure. Krein space quantization has been applied to the covariant quantization of the minimally coupled scalar field in de Sitter space [3–5]. The appearance of the auxiliary negative norm state is a consequence of Krein quantization. This unphysical particle (the negative norm state) does not interact with the physical state (the positive norm state) and acts as a regulator in the theory [1, 6–8]. Krein space quantization and quantum metric fluctuations remove the ultraviolet and infrared divergent terms from the Feynman propagator. In Krein regularization for the scalar field, the propagator is [9]:

$$\mathcal{PP}\frac{m^2}{k^2(k^2-m^2)}.$$

This propagator is similar to the one used in the Pauli-Villars method [10]:

$$\mathcal{PP}\frac{(m^2-M^2)}{(k^2-M^2)(k^2-m^2)}.$$

In the Pauli-Villars method, it is assumed that a ghost particle, which has a negative norm, exists with mass M. When this mass tends to infinity the unphysical particle decouples from the theory [10]. The modified Pauli-Villars propagator behaves as $\frac{1}{k^4}$, which removes the divergent term but breaks the unitarity of the theory. Krein regularization resembles Pauli-Villars regularization because of the similarity of the propagators but in the Krein method a finite answer is gained without the application of a renormalization procedure and the unitarity is improved by the "reality condition" in which the negative norm states do not appear in the external legs of the Feynmann diagram and a renormalized definition of the S-matrix is assumed [1].

Krein regularization has been used successfully in studying the $\lambda\phi^4$ theory, Casimir effect, QED theory, removing the infrared divergence in linear gravity in de Sitter space, the effective action of $\lambda\phi^4$ and QED, in calculating values of the magnetic anomaly, Lamb shift and coupling constants of $\lambda\phi^4$ and QED [1, 7–9, 11–15].

In the next section, $\lambda\phi^3$ in Krein quantization is introduced. In Sections 3 and 4, Krein regularization is performed for $\lambda\phi^3$ theory in four and six dimensions and the one-loop self energy is calculated via the Krein regularization method and in the final section, the conclusion is presented.

2 $\lambda\phi^3$ in Krein space quantization

In the $\lambda\phi^3$ theory, for studying the self energies in four and six dimensions the negative norm state propagates in the loop and takes part in the calculation of self energy as a regulator. Because of this we expect that after applying Krein regularization, the theory will be finite and renormalization will be unnecessary. It must be noted that the possible propagation of negative norm states in the theory can be neutralized by employing a certain "reality condition", as mentioned previously, which allows

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for physical quantities only. Any negative norm states encountered in the disconnected sections of the Feynman diagram can be removed by renormalizing the relevant probability amplitudes.

Despite the use of Krein space quantization, the Feynman rules and S-matrix elements are the same as they are in Hilbert space, because the effect of unphysical fields on physical fields is zero and only the Krein regularization propagator appears in the loop. In the first loop the self energy is obtained as follows [16, 17]:

$$S_{fi}=\langle in,p'|S^{(2)}|p,in\rangle=\frac{(2\pi)^4\delta^4(p'-p)}{\sqrt{2\omega_p 2\omega'_p}}\mathrm{i}\Sigma_{\rm kr}, \qquad (1)$$

where S_{fi} describes the scattering of the i state into the f state and

$$S^{(2)}=\frac{\lambda^2}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\mathrm{T}[\varphi^3(x)\varphi^3(x')],$$

thus the self energy would be:

$$\mathrm{i}\Sigma_{\rm kr}(p)=\lambda^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\widetilde{G}_T(q)\widetilde{G}_T(p-q), \qquad (2)$$

In Eq. (2), the delta function divergence term is still present. In order to remove the divergence, it is necessary to include the quantum metric fluctuation with the Krein propagator. We therefore use $<\widetilde{G}_T(q)>$ and $<\widetilde{G}_T(p-q)>$ and Eq. (2) takes on the following form, which no longer exhibits any divergence:

$$\mathrm{i}\Sigma_{\rm kr}(p)=\lambda^2\int\frac{\mathrm{d}^dq}{(2\pi)^d}<\widetilde{G}_T(q)><\widetilde{G}_T(p-q)>. \qquad (3)$$

3 Calculation of Krein regularization for the one-loop self energy in 4 dimensions

In $\lambda\phi^3$ theory for four dimensions, only the self energy diagram has a divergent term; all other diagrams are finite. Because of the delta function singularity in the propagators, the integral (2) is divergent at the ultraviolet limit, whereas the integral (3) is finite:

$$\mathrm{i}\Sigma_{\rm kr}(p) = \frac{m^4\lambda^2}{4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\mathcal{PP}\left(\frac{1}{q^2(q^2-m^2)}\right) \times\mathcal{PP}\left(\frac{1}{(p-q)^2[(p-q)^2-m^2]}\right). \qquad (4)$$

Using Feynman parameters and relation [18]:

$$\frac{1}{A_1\cdots A_n} = \int_0^1\mathrm{d}x_1\cdots\mathrm{d}x_n\delta\left(\sum_{i=1}^n x_i-1\right) \times\frac{(n-1)!}{(x_1A_1+\cdots+x_nA_n)^n}, \qquad (5)$$

equation (4) changes to:

$$\mathrm{i}\Sigma_{\rm kr}(p)=6m^4\lambda^2\int_0^1\mathrm{d}u\mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(1-u-x-y-z)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{D^4}, \qquad (6)$$

where $D=xq^2+y(q^2-m^2)+z(p-q)^2+u(p-q)^2-um^2$. The variable change $q\to l+(z+u)p$ is applied; D becomes $l^2-(z+u)^2p^2+(y+u)m^2-(z+u)p^2$ [18].

Wick rotation is applied so that if the denominator has $i\epsilon$ in it, $l_E^0=-il^0$ or if it contains $-i\epsilon$, then $l_E^0=il^0$, resulting in the following integral [18]:

$$\mathrm{i}\Sigma_{\rm kr}(p)=\frac{\mathrm{i}m^4\lambda^2}{(4\pi)^2}\int_0^1\mathrm{d}u\mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(1-u-x-y-z)\frac{1}{[(z+u)^2p^2+(y+u)m^2-(z+u)p^2]^2}, \qquad (7)$$

where relation (8) is used [18]:

$$\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{1}{|l^2-\Delta|^m}=\frac{\mathrm{i}(-1)^m}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(m-\frac{d}{2}\right)}{\Gamma(m)\Delta^{m-\frac{d}{2}}}. \qquad (8)$$

Integrating over dx in Eq. (7), we have:

$$\Sigma_{\rm kr}(p) = \frac{m^4\lambda^2}{(4\pi)^2}\int_0^1\mathrm{d}z\int_0^{1-z}\mathrm{d}u\int_0^{1-z-u}\frac{\mathrm{d}y}{[(z+u)^2p^2+(y+u)m^2-(z+u)p^2]^2}$$
$$= \frac{m^2\lambda^2}{(4\pi)^2}\int_0^1\mathrm{d}z\int_0^{1-z}\mathrm{d}u\left[\frac{1}{p^2u^2+u(2zp^2-p^2+m^2)+z^2p^2-zp^2}-\frac{1}{p^2u^2+u(2zp^2-p^2)+z^2p^2-zp^2+(1-z)m^2}\right]. \qquad (9)$$

Integrating over du and substituting the variable α^2 for $4p^2m^2z+(p^2-m^2)^2$ for the first term and $\beta^2=p^4-4p^2m^2+4p^2m^2z$

for the second term:

$$\Sigma_{\rm kr}(p) = -\frac{\lambda^2}{(4\pi)^2}\int_{m^2-p^2}^{p^2+m^2}\mathrm{d}\alpha\frac{1}{2p^2}\left\{\ln\frac{p^2+m^2-\alpha}{p^2+m^2+\alpha}-\ln\frac{\alpha^2-(p^2-m^2)^2+2m^4-2m^2p^2-2m^2\alpha}{\alpha^2-(p^2-m^2)^2+2m^4-2m^2p^2+2m^2\alpha}\right\}$$

$$+\frac{\lambda^2}{(4\pi)^2}\int_{\sqrt{p^4-4p^2m^2}}^{p^2}\mathrm{d}\beta\frac{1}{2p^2}\left\{\ln\frac{p^2-\beta}{p^2+\beta}-\ln\frac{\beta^2-p^4+2m^2p^2-2m^2\beta}{\beta^2-p^4+2m^2p^2+2m^2\beta}\right\}, \tag{10}$$

the final answer is finite without using any conventional method of regularization:

$$\Sigma_{\rm kr}(p) = \frac{\lambda^2}{32\pi^2}\left\{\sqrt{1-\frac{4m^2}{p^2}}\ln\left(\frac{1-\frac{2m^2}{p^2}+\sqrt{1-\frac{4m^2}{p^2}}}{\frac{2m^2}{p^2}-1+\sqrt{1-\frac{4m^2}{p^2}}}\right)-2\ln(2p^2)-3\ln(2p^2-2m^2)+\left(\frac{4m^2}{p^2}-1\right)\ln(2m^2)\right\}=\text{finite}. \tag{11}$$

In Hilbert space, we have [19]:

$$\Sigma_{\rm Hi}^{\rm Re}(p)=\frac{\lambda^2}{32\pi^2}\int_0^1\mathrm{d}x\ln\left(\frac{m^2-p^2x(1-x)}{m^2(1-x+x^2)}\right)=\frac{\lambda^2}{32\pi^2}\sqrt{1-\frac{4m^2}{p^2}}\ln\left(\frac{1-\frac{2m^2}{p^2}+\sqrt{1-\frac{4m^2}{p^2}}}{\frac{2m^2}{p^2}}\right). \tag{12}$$

4 Calculation of Krein regularization for the one-loop self energy in 6 dimensions

In six dimensions, the degree of divergence in the self energy diagram is two. Instead of using conventional regularization, Krein regularization is applied as follows:

$$\mathrm{i}\Pi_{\rm kr}(p) = \frac{m^4\lambda^2}{4}\int\frac{\mathrm{d}^6q}{(2\pi)^6}\mathcal{PP}\left(\frac{1}{q^2(q^2-m^2)}\right)$$

$$\times\mathcal{PP}\left(\frac{1}{(p-q)^2[(p-q)^2-m^2]}\right). \tag{13}$$

Using Feynman parameters, changing the variable q to $l+(z+u)p$ and relation (5) we have [18]:

$$\mathrm{i}\Pi_{\rm kr}(p)=6m^4\lambda^2\int_0^1\mathrm{d}u\mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(1-u-x-y-z)\int\frac{\mathrm{d}^6l}{(2\pi)^6}\frac{1}{D^4}, \tag{14}$$

where D is defined as in the previous section.

Applying Wick rotation and using relation (8), we have:

$$\mathrm{i}\Pi_{\rm kr}(p) = \frac{\mathrm{i}m^4\lambda^2}{2(4\pi)^3}\int_0^1\mathrm{d}u\mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(1-u-x-y-z)$$

$$\times\frac{1}{[(z+u)^2p^2+(y+u)m^2-(z+u)p^2]}. \tag{15}$$

Integrating over $\mathrm{d}x$ and $\mathrm{d}y$ in Eq. (15), we have:

$$\Pi_{\rm kr}(p) = \frac{m^4\lambda^2}{(4\pi)^3}\int_0^1\mathrm{d}z\int_0^{1-z}\mathrm{d}u\int_0^{1-z-u}\frac{\mathrm{d}y}{(z+u)^2p^2+(y+u)m^2-(z+u)p^2}$$

$$= \frac{m^2\lambda^2}{(4\pi)^3}\int_0^1\mathrm{d}z\int_0^{1-z}\mathrm{d}u\ln\frac{p^2u^2+p^2(2z-1)u+z^2p^2-p^2z+m^2(1-z)}{p^2u^2+u(2zp^2-p^2+m^2)+z^2p^2-zp^2}. \tag{16}$$

Integrating over $\mathrm{d}u$:

$$\Pi_{\rm kr}(p) = \frac{\lambda^2}{2(4\pi)^3}\int_0^1\mathrm{d}z\left\{A\ln\frac{A+\frac{1}{2}-\frac{m^2}{p^2}}{-A+\frac{1}{2}-\frac{m^2}{p^2}}+B\ln\frac{z-B+\frac{1}{2}-\frac{m^2}{2p^2}}{z-B-\frac{1}{2}+\frac{m^2}{2p^2}}\right\}$$

$$+\frac{\lambda^2}{2(4\pi)^3}\int_0^1\mathrm{d}z\left\{\frac{1}{2}\ln\frac{m^2}{p^2}+\left(\frac{1}{2}-z\right)\ln\left(z-\frac{m^2}{p^2}\right)-\frac{1}{2}\left(1+\frac{m^2}{p^2}\right)\ln\left(z+\frac{m^2}{p^2}\right)\right\}, \tag{17}$$

where

$$A=\sqrt{4\frac{m^2}{p^2}(z-1)+1}$$

and

$$B=\sqrt{4z^2+4\frac{m^2}{p^2}z+1+\frac{m^4}{p^4}-2\frac{m^2}{p^2}}.$$

In Hilbert space, we have [20]:

$$\Pi_{\rm Hi}(p) = \frac{\lambda^2}{2(4\pi^2)^3}\int_0^1 {\rm d}z[z(1-z)p^2 + m^2-{\rm i}\epsilon]\ln\left(\frac{m^2+p^2z(1-z)}{m^2(1-z+z^2)}\right). \quad (18)$$

5 Conclusion and outlook

The role of Krein space quantization, including the quantum metric fluctuation, is to regularize the theory. This method has been used to good effect in the elimination of the singularity in $\lambda\phi^3$, $\lambda\phi^4$ and QED theories in four dimensions without affecting the physical aspects of the theory in the one-loop approximation.

This method can easily be used for linear quantum gravity in the background field method [21], where the theory is automatically renormalized and could also be an alternative way to solve the non-renormalizability of linear quantum gravity, which would be compatible with general relativity. This paper will later be supplemented with the results of applying Krein regularization to the $\lambda\phi^5$ theory.

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