A new non-microscopic study of cluster structures in light alpha-conjugate nuclei

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Abstract: In this paper, the alpha-cluster state in light alpha-conjugate nuclei is studied and a new suitable local potential model for the α-cluster phase of these nuclei is suggested. Using the generalized Nikiforov-Uvarov (NU) method, the clusterization energy for ^{8}Be, ^{12}C, ^{16}O and ^{20}Ne nuclei is calculated. Based on the obtained results, the clustering phenomenon is more probable at energies among excited levels and it happens neither at ground state nor at excited states of light alpha-conjugate nuclei. It is found that the presented formulation for clustering phenomenon reproduces the results of previous experimental and theoretical attempts for the mentioned nuclei. The consistency of the obtained results with the previous experimental and theoretical predictions indicates the reliability of this formulation for various types of alpha-conjugate nuclei.

Keywords: light alpha-conjugate nuclei, alpha-cluster state, cluster-cluster interaction, Nikiforov-Uvarov (NU) method

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1 Introduction

The shell model has been highly successful in describing the structure of nuclei, but in the 1960s, it was found that using this model to study the even-even nuclei is accompanied by great challenges [1]. Ikeda and others speculated the existence of alpha-clusters in the light alpha-conjugate nuclei [2]. As the alpha-cluster states in these nuclei cannot be described by the shell model, the cluster model has been developed to characterize these states. In the cluster model, some basic assumptions simplify the problem, and provide a good dynamical description for the quantum system of structures composed of alpha particles. Alpha-conjugate nuclei are nuclei with neutron number equal to the atomic number and $N = Z = 2k(k = 2,3,\cdots)$ [3, 4]. The most important parameter which can affect the clustering phenomenon in these nuclei is the energy of the system. In other words, from a cluster perspective, in each light alpha-conjugate nucleus, depending on the amount of energy, one can expect a variety of configurations resembling molecular structures, such as a structure consisting of an alpha cluster and a smaller nucleus or a structure composed of several alpha clusters [4, 5].

There are two models to describe the clustering phenomenon in light alpha-conjugate nuclei, namely the microscopic [6] and the non-microscopic [1, 6, 7] models. The microscopic cluster model has been well established so far, and no phenomenological cluster-cluster potential is utilized in it. But in the non-microscopic multi cluster model, the alpha clusters are treated as structure-less particles. In this model, alpha-clusters are four spatially correlated nucleons (two protons and two neutrons) and that the phenomenological potential of the interaction between them in light alpha-conjugate nuclei includes the attractive nuclear, repulsive coulomb and centrifugal terms [6, 8]. The major purpose of this research is to attain an appropriate model in a non-microscopic approach to study the clustering phenomenon in alpha-conjugate nuclei. Various possible molecule-like structures related to each nα-cluster ($n = 2,3,\cdots,7$) configuration is schematically shown by Ikeda and his co-workers [7]. In addition, the required amount of energy for the formation of each nα-structure is approximately mentioned in Ikeda's diagram [5].

Nuclear clustering could be one of the most fruitful subjects of nuclear physics and physicists are likely to face some of the greatest challenges and opportunities in this field in future years. More experimental and theoretical works are needed to clarify the nature of the clustering phenomenon. To the best of our knowledge, despite the importance of the subject, no comprehensive non- microscopic analytical model has been proposed to investigate the nature of the clustering phenomenon in alpha-conjugate nuclei [9–13].

In this article, the non-microscopic Alpha-Cluster

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model [5, 14] is used for computational analysis of clustering phenomenon in ^{8}Be, ^{12}C, ^{16}O and ^{20}Ne nuclei as two, three, four and five alpha cluster structures, respectively. Then, a suitable potential for interaction between alpha clusters is suggested and a mathematical method is proposed for calculating the clusterization energy of these nuclei as non-relativistic multi alpha-cluster quantum systems for the first time. As mentioned before, the energy level of the system is a strategic parameter to investigate the clustering phenomenon. Therefore, this article focuses on calculating this main parameter. Based on the calculations, it is revealed that the α-cluster state is situated among excited levels. The proposed mathematical model is comprehensively discussed in the following sections and it will be demonstrated that the obtained results are consistent with the previous reported experimental and theoretical values.

2 Proposed mathematical model

As mentioned earlier, the phenomenological interaction potential between alpha clusters should be the sum of the nuclear, Coulomb and centrifugal terms [8]. Accordingly, the nuclear and Coulomb part of the cluster-cluster potential is suggested as Eq. (1), which is suitable to study the light alpha-conjugate nuclei in their α-cluster state:

$$V(r)=-\frac{b}{r}\mathrm{e}^{-ar}+\frac{c}{r}, \tag{1}$$

where b and c are nuclear attraction strength and the strength of Coulomb repulsion, respectively. a is related to the range of the potential. The first term in the proposed potential is of Yukawa type and the second term is of Coulomb type. The Yukawa potential is one of the most useful, applicable and reasonable short-range potentials in physics, and is used not only as a mathematical model in atomic physics but also as a potential model for studying bound states and scattering parameters in nuclear physics [15–17].

To calculate the energy of a multi-cluster system from a non-relativistic perspective in the presence of such a local potential, which is only a function of the inter-particle distance (r), one should solve the time-independent Schrodinger equation. In a method that is provided to solve the Schrodinger equation for a system of N-identical particles with a hyper-spherical formalism, the part of the Schrodinger equation which is related to the hyper radius x is expressed as [18]:

$$\begin{aligned}&\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}+\frac{D-1}{x}\frac{\mathrm{d}R(x)}{\mathrm{d}x}\\&+\frac{2\mu}{\hbar^2}\left[E-V(x)-\frac{\hbar^2l(l+D-2)}{2\mu x^2}\right]R(x)=0,\end{aligned} \tag{2}$$

where $R(x)$ and $V(x)$ are called the radial part of the N-body wave function and the hyper-potential, respectively. In Eq. (2), $D=3N-3$ and μ is the reduced mass. E represents the energy of the system and h is Planck's constant, with $\hbar=\dfrac{h}{2\pi}$. l indicates the angular momentum quantum number. Thus, in the hyper-spherical formalism, the radial part of the time-independent Schrodinger equation in the presence of the proposed potential takes the following form:

$$\begin{aligned}&\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}+\frac{D-1}{x}\frac{\mathrm{d}R(x)}{\mathrm{d}x}\\&+\frac{2\mu}{\hbar^2}\left[E+\frac{b}{x}\mathrm{e}^{-ax}-\frac{c}{x}-\frac{\hbar^2l(l+D-2)}{2\mu x^2}\right]R(x)=0.\end{aligned} \tag{3}$$

In Eq. (3) the potential Eq. (1) is added to a centrifugal term and the effective potential of the system is actually the sum of the Yukawa, Coulomb and centrifugal terms [19].

Then, with a further analysis and simplification, Eq. (3) becomes:

$$\begin{aligned}&\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}+\frac{D-1}{x}\frac{\mathrm{d}R(x)}{\mathrm{d}x}\\&+\frac{1}{x^2}\left[-\varepsilon^2x^2+P_1x+P_2x\mathrm{e}^{-ax}+P_3\right]R(x)=0,\end{aligned} \tag{4}$$

where

$$\begin{cases}\varepsilon^2=\dfrac{-2\mu E}{\hbar^2}, \quad \varepsilon>0\\ P_1=\dfrac{-2\mu c}{\hbar^2},\\ P_2=\dfrac{+2\mu b}{\hbar^2},\\ P_3=-l(l+D-2).\end{cases} \tag{5}$$

Since Eq. (4) with a Yukawa potential has no exact solution, thus, similar to other authors, we use a suitable approximation for the centrifugal term [15, 19–24]:

$$\frac{1}{x^2}\approx\frac{a^2}{(1-\mathrm{e}^{-ax})^2}. \tag{6}$$

Equivalently, we can consider:

$$\frac{1}{x}\approx\frac{a}{1-\mathrm{e}^{-ax}}. \tag{7}$$

In the last two equations, a ensures the integrity of the dimensions. To see the accuracy of the approximation, in Fig. (1), it is demonstrated that for short potential ranges, Eq. (6) is a valid approximation for $y(x)=\dfrac{1}{x^2}$.

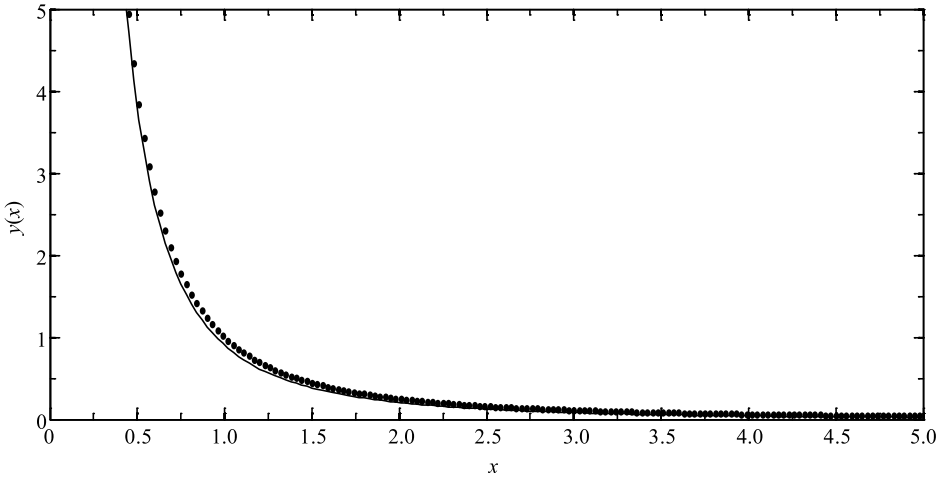

Fig. 1. (color online) The centrifugal type term (solid curve) and its approximation in Eq. (6) (dotted curve).

As a result, Eq. (4) becomes:

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2}+\frac{(D-1)a}{(1-\mathrm{e}^{-ax})}\frac{\mathrm{d}R(x)}{\mathrm{d}x} +\frac{a^2}{(1-\mathrm{e}^{-ax})^2}\left[-\varepsilon^2\frac{(1-\mathrm{e}^{-ax})^2}{a^2} +P_1\frac{(1-\mathrm{e}^{-ax})}{a}+P_2\frac{(1-\mathrm{e}^{-ax})}{a}\mathrm{e}^{-ax}+P_3\right]R(x)=0. \tag{8}$$

By introducing a new variable as:

$$(1-\mathrm{e}^{-ax})=s(x), \tag{9}$$

Eq. (8) can be further transformed into the following form:

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}s(x)^2}+\frac{(D-1)a}{s(x)}\frac{\mathrm{d}R(x)}{\mathrm{d}s(x)}+\frac{1}{s(x)^2}\left[-(\varepsilon^2+P_2a)s(x)^2 +(P_1a+P_2a)s(x)+P_3a^2\right]R(x)=0. \tag{10}$$

The Schrodinger equation with the potential considered for the interaction between alpha clusters has been transformed to a second-order differential equation that agrees with the general form of the equation in the Nikiforov-Uvarov method. This technique can be used to solve Eq. (10). In the next part, the parametric NU method is briefly described.

3 General framework of Nikiforov-Uvarov (NU) technique

The Nikiforov-Uvarov method offers a powerful mathematical model to solve second-order differential equations. For a given potential, the Schrödinger equation is reduced to a generalized equation of hypergeometric type with an appropriate coordinate transformation $s=s(x)$ .In this method the differential equations can be written as follows [25, 26]:

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{11}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials that can be at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To find a particular solution for Eq. (11) by separation of variables, the following transformation is applied:

$$\psi_n(s)=\varphi_n(s)y_n(s), \tag{12}$$

which reduces Eq. (11) to a hyper-geometric type function:

$$\sigma(s)y_n''(s)+\tau(s)y_n'(s)+\lambda y_n(s)=0, \tag{13}$$

where $\tau(s)=\tilde{\tau}(s)+2\pi(s)$ and $\tau'(s)<0$, which means $\tau(s)$ has a negative derivative. Additionally, λ is a parameter with the following definitions:

$$\begin{cases}\lambda_n=-n\tau'(s)-\dfrac{n(n-1)}{2}\sigma''(s) \quad (n=0,1,2,\cdots)\\ \lambda=k+\pi'(s)\end{cases}, \tag{14}$$

and equality of the two parts in Eq. (14) yields the energy eigenvalues of the intended multi- particle system.

Furthermore, the function $\pi(s)$ is defined as:

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2} \pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)}, \tag{15}$$

$\pi(s)$ should be a first-degree polynomial and the expression under the square root must be the square of a polynomial. In this way, the value of $k$will be determined [27, 28].

4 Mathematical calculation and results

To present the results based on the discussed model, the NU method has been applied. To this end, Eq. (10) and (11) have been compared and the following expressions have been obtained:

$$\begin{cases} \tilde{\tau}=(D-1)a, \\ \sigma(s)=s(x), \\ \tilde{\sigma}(s)=[-(\varepsilon^2+P_2a)s(x)^2+(P_1a+P_2a)s(x)+P_3a^2]. \end{cases} \tag{16}$$

Substituting the above expressions into Eq. (15) and considering the NU method condition for $\pi(s)$, with some analysis and simplification, the following equation can be achieved:

$$\pi(s)=\frac{1-[(D-1)a]}{2}\pm\left[(\varepsilon^2+P_2a)^{\frac{1}{2}}s(x)+\left(-P_3a^2+\left[\frac{1-[(D-1)a]}{2}\right]^2\right)^{\frac{1}{2}}\right]. \tag{17}$$

Since we have the polynomial $\tau(s)=\tilde{\tau}(s)+2\pi(s)$ with a negative derivative, a suitable form has to be established for this parameter. Therefore,

$$\tau(s)=1-2\left[(\varepsilon^2+P_2a)^{\frac{1}{2}}s(x)+\left(-P_3a^2+\left[\frac{1-[(D-1)a]}{2}\right]^2\right)^{\frac{1}{2}}\right]. \tag{18}$$

Finally, considering the notations of Eq. (5) and (14), the equation of energy for such an N-particle system of alpha clusters can be obtained:

$$E=ba-\left[\left(\frac{2\mu}{\hbar^2}\right)\left(\frac{ba-ca}{2n-\left[2\left(l(l+D-2)a^2+\left[\frac{1-[(D-1)a]}{2}\right]^2\right)^{\frac{1}{2}}-1\right]}\right)^2\right]. \tag{19}$$

Having achieved this important equation, we can calculate the clusterization energy for light alpha-conjugate nuclei in their alpha-cluster state by assigning the appropriate values to coefficients of the potential.

In our non-microscopic model, the alpha clusters are assumed to be structure-less particles. Thus, we are dealing with two clusters in ^{8}Be , three clusters in ^{12}C, four clusters in ^{16}O and five clusters in^{20}Ne, interacting via Eq. (1). In Table (1), the obtained results are compared with the previous theoretical and experimental values [4, 13, 29].

In this paper, binding energy is the energy of a bound system consisting of A nucleons, but clusterization energy is the energy of a bound system consisting of k alpha-clusters. (A is the number of nucleons and k is the number of alpha-clusters). As shown in Table (1), the binding energy is more negative than the clusterization energy, meaning that if the nucleonic system receives the appropriate amount of energy, it will change to a clustered system [14, 30]. This reminder corresponds to

Table 1. Binding energy (at ground state); energy of ^{8}Be, ^{12}C, ^{16}O and ^{20}Ne in their alpha-cluster state; and comparison of the obtained results with previous experimental and theoretical attempts [4, 13, 29, 30].

nucleus	coefficients of potential	binding energy/MeV	energy of the clustered system in our work/MeV	other experimental attempts/MeV	other theoretical attempts/MeV
^{8}Be (as a 2α- structure)	$a=0.5$ fm^{-1}, $b=41.3$ MeV·fm, $c=1$ MeV·fm	−56.500	−56.49	−56.5	−55
^{12}C (as a 3α-structure)	$a=0.56$ fm^{-1}, $b=150$ MeV·fm, $c=1$ MeV·fm	−92.163	−84.73	−84.89	−82.5
^{16}O (as a 4α-structure)	$a=0.6$ fm^{-1}, $b=322$ MeV·fm, $c=1$ MeV·fm	−127.621	−112.70	−113.181	−110
^{20}Ne (as a 5α-structure)	$a=0.62$ fm^{-1}, $b=558$ MeV·fm, $c=1$ MeV·fm	−160.647	−141.21	−140.687	−137.5

Ikeda's achievements in 1968 [5]. Additionally, by studying the ground state and excited levels of the mentioned nuclei and considering the clusterization energies, we find that the cluster states in these nuclei are hybrid states and the clustering phenomenon occurs neither at ground state nor at excited levels of light alpha-conjugate nuclei [1, 4]. Furthermore, it should be noted that, for each of the nuclei listed in Table 1, the potential coefficients are determined by fitting to the clusterization energies mentioned in Ikeda's diagram [5]. There is a significant increment in the value of parameter b, which could be due to the increase of the power of attraction between alpha-clusters with the increment of the number of clusters.

The clusterization energies, calculated with the proposed potential, are in good agreement with the experimental and theoretical results for the four intended nuclei, which are mentioned as preliminary examples. Therefore, the proposed mathematical model can be reliable for other light alpha-conjugate nuclei such as ^{24}Mg, ^{28}Si, ^{32}S and so on, with $N = 6, 7, 8, \cdots$ in Eq. (2) and fitting the potential parameters for each nucleus.

5 Conclusions

The principal focus of this research is on the introduction of a unified method to utilize the alpha-cluster model for all light alpha-conjugate nuclei. In this way, the molecule-like picture of ^{8}Be, ^{12}C, ^{16}O and ^{20}Ne nuclei as two, three, four and five alpha cluster structures, respectively is studied by a proper non-microscopic approach. The suitable potential which is assumed for the interaction between alpha clusters in this paper is a composition of nuclear, Coulomb and centrifugal terms. The radial part of the time-independent Schrödinger equation with this potential is analytically solved. The energy equation for an N-alpha cluster system is obtained and the clusterization energies for these four nuclei are calculated. The results are consistent with previous experimental and theoretical attempts.

Finally, it is concluded that the non-microscopic approach and the mathematical model proposed in this paper for the clustering phenomenon can be suitable for all other light alpha-conjugate nuclei. Under these interpretations, more theoretical works are required to clarify the application of cluster models to various properties of light alpha-conjugate nuclei.

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