

Note on non-vacuum conformal family contributions to Rényi entropy in two-dimensional CFT^{*}

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Abstract: We calculate the contributions of a general non-vacuum conformal family to Rényi entropy in two-dimensional conformal field theory (CFT). The primary operator of the conformal family can be either non-chiral or chiral, and we denote its scaling dimension by Δ . For the case of two short intervals on a complex plane, we expand the Rényi mutual information by the cross ratio x to order $x^{2\Delta+2}$. For the case of one interval on a torus with low temperature, we expand the Rényi entropy by $q = \exp(-2\pi\beta/L)$, with β being the inverse temperature and L being the spatial period, to order $q^{\Delta+2}$. To make the result meaningful, we require that the scaling dimension Δ cannot be too small. For two intervals on a complex plane we need $\Delta > 1$, and for one interval on a torus we need $\Delta > 2$. We work in the small Newton constant limit on the gravity side and so a large central charge limit on the CFT side, and find matches of gravity and CFT results.

Keywords: Rényi entropy, entanglement entropy, conformal field theory

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1 Introduction

The investigation of entanglement entropy has drawn more and more attention in the past decade, not only because it is interesting in its own right [1], but also because it opens a new angle in the investigation of AdS/CFT correspondence [2, 3]. To calculate the entanglement entropy in a quantum field theory, one can use the replica trick [4, 5]. One first calculates the general n -th Rényi entropy with $n > 1$ and being an integer, and then take the $n \rightarrow 1$ limit to get the entanglement entropy. For a conformal field theory (CFT) that has gravity dual in anti-de Sitter (AdS) spacetime [6–9], one can use the Ryu-Takayanagi formula and just calculate the area of a minimal surface on the gravity side [2, 3]. The Ryu-Takayanagi area formula of holographic entanglement entropy is the leading classical result in the limit of small Newton constant, and one can also consider quantum corrections [10–13].

In AdS₃/CFT₂ correspondence, the small Newton constant limit on the gravity side corresponds to a large central charge limit on the CFT side [14]. There are many investigations of Rényi entropy and holographic Rényi entropy in AdS₃/CFT₂ correspondence. On the gravity side, one uses the partition function of Einstein

gravity in handlebody background [15–17], and calculates the classical and one-loop parts of the holographic Rényi entropy [12, 18, 19]. On the CFT side one uses different methods to calculate Rényi entropy for the cases of two intervals on a complex plane and one interval on a torus. For the former case, one uses the operator product expansion (OPE) of twist operators [11, 20–22]. For the latter case, one uses the low temperature expansion of the density matrix [19, 23]. See Refs. [24–38] for other investigations.

In AdS/CFT correspondence, different operators in CFT are dual to different fields on the gravity side, and it is interesting to compute the contributions of some specific operators to Rényi entropy on the CFT side and compare the contributions of corresponding fields on the gravity side to holographic Rényi entropy. The cases of some specific operators have been investigated in the literature, for example the stress tensor [12, 19, 22], W operators [24, 25, 32], logarithmic partner of stress tensor [26], general scalars [27], supersymmetric partners of the stress tensor [34, 38], and current operators [38]. There are also some investigations of the contributions of a general primary operator to Rényi entropy [12, 20, 23, 25], and in this paper we generalize the results to higher orders. We consider the contributions of

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a general non-vacuum conformal family to the Rényi entropy, with the primary operator of the conformal family being non-chiral or chiral. The non-chiral primary operator with conformal weights (h, \bar{h}) has scaling dimension $\Delta = h + \bar{h}$, and the chiral primary operator with conformal weights $(h, 0)$ has scaling dimension $\Delta = h$. For the case of two short intervals on a complex plane, we expand the Rényi mutual information by the cross ratio x to order $x^{2\Delta+2}$. For the case of one interval on a torus with low temperature, we expand the Rényi entropy by $q = \exp(-2\pi\beta/L)$, with β being the inverse temperature and L being the spatial period, to order $q^{\Delta+2}$.

The rest of this paper is arranged as follows. In Section 2 we consider the Rényi mutual information of two intervals on a complex plane. In Section 3 we consider the Rényi entropy of one interval on a torus. We conclude with discussion in Section 4. In the Appendix we review some useful properties of the non-vacuum conformal family.

2 Rényi mutual information of two intervals on a complex plane

We calculate the Rényi mutual information of two short intervals on a complex plane in expansion of the small cross ratio x . On the gravity side we calculate the one-loop holographic Rényi mutual information using the method in Ref. [12], and on the CFT side we calculate the Rényi mutual information using the OPE of twist operators [11, 20, 22]. In the CFT calculation we will use some results from Ref. [26].

2.1 Non-chiral primary operator

The classical part of the holographic Rényi mutual information only depends on the graviton, but the field in gravity dual to a nonidentity primary operator \mathcal{X} changes the one-loop result. The non-chiral primary operator \mathcal{X} has conformal weights (h, \bar{h}) with $h \neq 0$ and $\bar{h} \neq 0$, and its conformal weight is $\Delta = h + \bar{h}$. The gravity Euclidean space is the quotient of global AdS_3 by a Schottky group Γ , and the one-loop partition function is multiplied by

$$Z_{\mathcal{X}}^{1\text{-loop}} = \prod_{\gamma \in \mathcal{P}} \left(1 + \frac{q_{\gamma}^h \bar{q}_{\gamma}^{\bar{h}}}{(1-q_{\gamma})(1-\bar{q}_{\gamma})} \right)^{1/2}, \quad (1)$$

with \mathcal{P} being a set of representatives of the primitive conjugacy classes of Γ . The form of q_{γ} can be found in Ref. [12]. We get the contributions to the one-loop holographic Rényi mutual information

$$I_{n, \mathcal{X}}^{1\text{-loop}} = \frac{1}{n-1} \frac{x^{2\Delta}}{2^{4\Delta+1} n^{4\Delta-1}} \left\{ f_{2\Delta} + \frac{\Delta((n^2-1)f_{2\Delta} + f_{2\Delta+1})}{n^2} x \right. \\ \left. + \frac{1}{72n^4} [\Delta((36\Delta+29)(n^2-1) + 24)(n^2-1)f_{2\Delta} \right.$$

$$+ 36\Delta(2\Delta+1)(n^2-1)f_{2\Delta+1} \\ \left. + 9(4\Delta^2 + 3\Delta + 1)f_{2\Delta+2} \right\} x^2 + O(x^3) \Big\} + O(x^{4\Delta}), \quad (2)$$

with the definition

$$f_m = \sum_{k=1}^{n-1} \frac{1}{\left(\sin \frac{\pi k}{n}\right)^{2m}}. \quad (3)$$

In (2) we only incorporated the contributions of the so called consecutively decreasing words (CDWs) of the Schottky generators [12], and the order $x^{4\Delta}$ result that is omitted is from the 2-CDWs. To make the order $x^{2\Delta+2}$ part meaningful, we need $4\Delta > 2\Delta + 2$ and so $\Delta > 1$. Using [20]

$$\lim_{n \rightarrow 1} \frac{f_m}{n-1} = \frac{\sqrt{\pi} \Gamma(m+1)}{2\Gamma(m+3/2)}, \quad (4)$$

we get the contributions to the one-loop holographic mutual information

$$I_{\mathcal{X}}^{1\text{-loop}} = \frac{\sqrt{\pi} \Gamma(2\Delta+1)x^{2\Delta}}{4^{2\Delta+1} \Gamma(2\Delta+3/2)} \left[1 + \frac{2\Delta(2\Delta+1)x}{4\Delta+3} \right. \\ \left. + \frac{(\Delta+1)(2\Delta+1)(4\Delta^2+3\Delta+1)x^2}{(4\Delta+3)(4\Delta+5)} \right. \\ \left. + O(x^3) \right] + O(x^{4\Delta}). \quad (5)$$

The holographic mutual information is in accord with the result in Ref. [27] when the primary operator \mathcal{X} is a scalar.

On the CFT side, we use the OPE of twist operators in the n -fold CFT that is called CFT^n . The Rényi mutual information can be calculated as [22, 24, 25]

$$I_n = \frac{1}{n-1} \log \left[\sum_K \alpha_K d_K^2 x^{h_K + \bar{h}_K} {}_2F_1(h_K, h_K; 2h_K; x) {}_2F_1(\bar{h}_K, \bar{h}_K; 2\bar{h}_K; x) \right]. \quad (6)$$

with K being all the orthogonalized quasiprimary operators Φ_K in CFT^n . The coefficients α_K and d_K are, respectively, the normalization factors and OPE coefficients of Φ_K . In CFT^n , as well as the quasiprimary operators that are constructed solely by the vacuum conformal family of the original CFT, we have to consider the extra ones that are listed in Table 1. In the table we have the definitions

$$\mathcal{R}_{j_1 j_2} = \mathcal{X}_{j_1} i \partial \mathcal{X}_{j_2} - i \partial \mathcal{X}_{j_1} \mathcal{X}_{j_2}, \quad \mathcal{S}_{j_1 j_2} = \mathcal{X}_{j_1} i \bar{\partial} \mathcal{X}_{j_2} - i \bar{\partial} \mathcal{X}_{j_1} \mathcal{X}_{j_2}, \\ \mathcal{W}_{j_1 j_2} = \mathcal{X}_{j_1} \partial \bar{\partial} \mathcal{X}_{j_2} + \partial \bar{\partial} \mathcal{X}_{j_1} \mathcal{X}_{j_2} - \partial \mathcal{X}_{j_1} \bar{\partial} \mathcal{X}_{j_2} - \bar{\partial} \mathcal{X}_{j_1} \partial \mathcal{X}_{j_2}, \\ \mathcal{U}_{j_1 j_2} = \partial \mathcal{X}_{j_1} \partial \mathcal{X}_{j_2} - \frac{h}{2h+1} (\mathcal{X}_{j_1} \partial^2 \mathcal{X}_{j_2} + \partial^2 \mathcal{X}_{j_1} \mathcal{X}_{j_2}), \quad (7) \\ \mathcal{V}_{j_1 j_2} = \bar{\partial} \mathcal{X}_{j_1} \bar{\partial} \mathcal{X}_{j_2} - \frac{\bar{h}}{2\bar{h}+1} (\mathcal{X}_{j_1} \bar{\partial}^2 \mathcal{X}_{j_2} + \bar{\partial}^2 \mathcal{X}_{j_1} \mathcal{X}_{j_2}).$$

Table 1. The quasiprimary operators in CFTⁿ from the conformal family of non-chiral primary operator \mathcal{X} in the original CFT. The indices j_1, j_2, j_3 take values from 0 to $n-1$.

$L_0 + \bar{L}_0$	(L_0, \bar{L}_0)	quasiprimary operators	degeneracies
Δ	(h, \bar{h})	$\mathcal{X}_{j_1} \mathcal{X}_{j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
$\Delta+1$	$(h+1, \bar{h})$	$\mathcal{R}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
	$(h, \bar{h}+1)$	$\mathcal{S}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
$\Delta+2$	$(h+2, \bar{h})$	$\mathcal{X}_{j_1} \mathcal{Y}_{j_2}$ with $j_1 \neq j_2$	$n(n-1)$
	$(h, \bar{h}+2)$	$\mathcal{X}_{j_1} \mathcal{Z}_{j_2}$ with $j_1 \neq j_2$	$n(n-1)$
	$(h+1, \bar{h}+1)$	$\mathcal{W}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
	$(h+2, \bar{h})$	$\mathcal{U}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
	$(h, \bar{h}+2)$	$\mathcal{V}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
	$(h+2, \bar{h})$	$T_{j_1} X_{j_2} X_{j_3}$ with $j_1 \neq j_2, j_1 \neq j_3$ and $j_2 < j_3$	$\frac{n(n-1)(n-2)}{2}$
	$(h, \bar{h}+2)$	$\bar{T}_{j_1} X_{j_2} X_{j_3}$ with $j_1 \neq j_2, j_1 \neq j_3$ and $j_2 < j_3$	$\frac{n(n-1)(n-2)}{2}$
...

We get contributions to Rényi mutual information from the conformal family of \mathcal{X}

$$\begin{aligned}
 I_{n,\mathcal{X}} = & \frac{x^{2\Delta}}{n-1} \left\{ [1 - nx^2(\alpha_T d_T^2 + \alpha_{\bar{T}} d_{\bar{T}}^2)] {}_2F_1(2h, 2h; 4h; x) {}_2F_1(2\bar{h}, 2\bar{h}; 4\bar{h}; x) \sum \alpha_{\mathcal{X}\mathcal{X}} (d_{\mathcal{X}\mathcal{X}}^{j_1 j_2})^2 \right. \\
 & + x \left[{}_2F_1(2h+1, 2h+1; 4h+2; x) {}_2F_1(2\bar{h}, 2\bar{h}; 4\bar{h}; x) \sum \alpha_{\mathcal{R}} (d_{\mathcal{R}}^{j_1 j_2})^2 \right. \\
 & \left. \left. {}_2F_1(2h, 2h; 4h; x) {}_2F_1(2\bar{h}+1, 2\bar{h}+1; 4\bar{h}+2; x) \sum \alpha_{\mathcal{S}} (d_{\mathcal{S}}^{j_1 j_2})^2 \right] \right. \\
 & + x^2 \left[\sum (\alpha_{\mathcal{X}\mathcal{Y}} (d_{\mathcal{X}\mathcal{Y}}^{j_1 j_2})^2 + \alpha_{\mathcal{X}\mathcal{Z}} (d_{\mathcal{X}\mathcal{Z}}^{j_1 j_2})^2) + \sum (\alpha_{\mathcal{W}} (d_{\mathcal{W}}^{j_1 j_2})^2 + \alpha_{\mathcal{U}} (d_{\mathcal{U}}^{j_1 j_2})^2 + \alpha_{\mathcal{V}} (d_{\mathcal{V}}^{j_1 j_2})^2) \right. \\
 & \left. \left. + \sum (\alpha_{T\mathcal{X}\mathcal{X}} (d_{T\mathcal{X}\mathcal{X}}^{j_1 j_2 j_3})^2 + \alpha_{\bar{T}\mathcal{X}\mathcal{X}} (d_{\bar{T}\mathcal{X}\mathcal{X}}^{j_1 j_2 j_3})^2) \right] + O(x^3) \right\} + O(x^{3\Delta}),
 \end{aligned} \tag{8}$$

and the ranges of summations can be found in Table 1. The order $x^{3\Delta}$ result that is omitted in the above result is from contributions of the CFTⁿ operators $\mathcal{X}_{j_1} \mathcal{X}_{j_2} \mathcal{X}_{j_3}$ with $0 \leq j_1 < j_2 < j_3 \leq n-1$. We have the normalization factors [26]

$$\begin{aligned}
 \alpha_{\mathcal{X}\mathcal{X}} = & i^{4s} \alpha_{\mathcal{X}}^2, \quad \alpha_{\mathcal{R}} = 4hi^{4s} \alpha_{\mathcal{X}}^2, \quad \alpha_{\mathcal{S}} = 4\bar{h}i^{4s} \alpha_{\mathcal{X}}^2, \quad \alpha_{\mathcal{X}\mathcal{Y}} = \frac{(2h+1)c + 2h(8h-5)}{2(2h+1)} i^{4s} \alpha_{\mathcal{X}}^2, \\
 \alpha_{\mathcal{X}\mathcal{Z}} = & \frac{(2\bar{h}+1)c + 2\bar{h}(8\bar{h}-5)}{2(2\bar{h}+1)} i^{4s} \alpha_{\mathcal{X}}^2, \quad \alpha_{\mathcal{W}} = 16h\bar{h}i^{4s} \alpha_{\mathcal{X}}^2, \quad \alpha_{\mathcal{U}} = \frac{4h^2(4h+1)}{2h+1} i^{4s} \alpha_{\mathcal{X}}^2, \\
 \alpha_{\mathcal{V}} = & \frac{4\bar{h}^2(4\bar{h}+1)}{2\bar{h}+1} i^{4s} \alpha_{\mathcal{X}}^2, \quad \alpha_{T\mathcal{X}\mathcal{X}} = \frac{c}{2} i^{4s} \alpha_{\mathcal{X}}^2, \quad \alpha_{\bar{T}\mathcal{X}\mathcal{X}} = \frac{c}{2} i^{4s} \alpha_{\mathcal{X}}^2,
 \end{aligned} \tag{9}$$

where the factor $i^{4s} = (-1)^{2s}$ arises from the minus sign when \mathcal{X} is an fermionic operator. Note that there is always $i^{8s} = 1$. We also have the OPE coefficients [26]

$$\begin{aligned}
 d_{\mathcal{X}\mathcal{X}}^{j_1 j_2} = & \frac{i^{2s}}{\alpha_{\mathcal{X}} (2n)^{2\Delta}} \frac{1}{s_{j_1 j_2}^{2\Delta}}, \quad d_{\mathcal{R}}^{j_1 j_2} = -d_{\mathcal{S}}^{j_1 j_2} = \frac{i^{2s}}{\alpha_{\mathcal{X}} (2n)^{2\Delta+1}} \frac{c_{j_1 j_2}}{s_{j_1 j_2}^{2\Delta+1}}, \\
 d_{\mathcal{X}\mathcal{Y}}^{j_1 j_2} = & d_{\mathcal{X}\mathcal{Z}}^{j_1 j_2} = \frac{i^{2s}(n^2-1)}{3\alpha_{\mathcal{X}} (2n)^{2\Delta+2}} \frac{1}{s_{j_1 j_2}^{2\Delta}}, \quad d_{\mathcal{W}}^{j_1 j_2} = \frac{i^{2s}}{\alpha_{\mathcal{X}} (2n)^{2\Delta+2}} \frac{c_{j_1 j_2}^2}{s_{j_1 j_2}^{2\Delta+2}},
 \end{aligned}$$

$$\begin{aligned}
 d_{\mathcal{U}}^{j_1 j_2} &= \frac{i^{2s}}{2h(4h+1)\alpha_{\mathcal{X}}(2n)^{2\Delta+2}} \frac{(2h+1)(4h+1) - 2h(n^2+4h+1)s_{j_1 j_2}^2}{s_{j_1 j_2}^{2\Delta+2}}, \\
 d_{\mathcal{V}}^{j_1 j_2} &= \frac{i^{2s}}{2\bar{h}(4\bar{h}+1)\alpha_{\mathcal{X}}(2n)^{2\Delta+2}} \frac{(2\bar{h}+1)(4\bar{h}+1) - 2\bar{h}(n^2+4\bar{h}+1)s_{j_1 j_2}^2}{s_{j_1 j_2}^{2\Delta+2}}, \\
 d_{T\mathcal{X}\mathcal{X}}^{j_1 j_2 j_3} &= \frac{i^{2s}}{\alpha_{\mathcal{X}}(2n)^{2\Delta+2}} \left(-\frac{2h}{c} \frac{1}{s_{j_1 j_2}^2 s_{j_1 j_3}^2 s_{j_2 j_3}^2} + \frac{n^2-1}{3} \frac{1}{s_{j_2 j_3}^{2\Delta}} \right), \\
 d_{\bar{T}\mathcal{X}\mathcal{X}}^{j_1 j_2 j_3} &= \frac{i^{2s}}{\alpha_{\mathcal{X}}(2n)^{2\Delta+2}} \left(-\frac{2\bar{h}}{c} \frac{1}{s_{j_1 j_2}^2 s_{j_1 j_3}^2 s_{j_2 j_3}^2} + \frac{n^2-1}{3} \frac{1}{s_{j_2 j_3}^{2\Delta}} \right). \tag{10}
 \end{aligned}$$

Here for simplicity we have defined $s_{j_1 j_2} \equiv \sin \frac{\pi(j_1 - j_2)}{n}$, $c_{j_1 j_2} \equiv \cos \frac{\pi(j_1 - j_2)}{n}$, \dots . Besides, we also need the normalization factors and OPE coefficients for the operators T_j and \bar{T}_j with $j=0, 1, \dots, n-1$

$$\alpha_T = \alpha_{\bar{T}} = \frac{c}{2}, \quad d_T = d_{\bar{T}} = \frac{n^2-1}{12n^2}. \tag{11}$$

With these coefficients and Eq. (8), in the large c limit we can reproduce the one-loop holomorphic Rényi mutual information (2).

2.2 Chiral primary operator

The case of chiral primary operator \mathcal{X} , with conformal weights $(h, 0)$ and $h \neq 0$, is similar to but a little different from the non-chiral operator case. Note that we only consider the contributions of the conformal family \mathcal{X} , and we do not count the contributions of the possible conformal family of the anti-holomorphic operator $\bar{\mathcal{X}}$ with conformal weights $(0, h)$.

Similar to (1), the one-loop partition function is multiplied by

$$Z_{\mathcal{X}}^{1\text{-loop}} = \prod_{\gamma \in \mathcal{P}} \left(1 + \frac{q_{\gamma}^h}{1 - q_{\gamma}} \right)^{1/2}. \tag{12}$$

We get the contributions to the one-loop holographic Rényi mutual information

$$\begin{aligned}
 I_{n, \mathcal{X}}^{1\text{-loop}} &= \frac{1}{n-1} \frac{x^{2h}}{2^{4h+1} n^{4h-1}} \left\{ f_{2h} + \frac{h((n^2-1)f_{2h} + f_{2h+1})}{n^2} x \right. \\
 &+ \frac{1}{144n^4} [2h((36h+29)(n^2-1) + 24)(n^2-1)f_{2h} \\
 &+ 72h(2h+1)(n^2-1)f_{2h+1} \\
 &\left. + 9(8h^2+6h+1)f_{2h+2}] x^2 + O(x^3) \right\} + O(x^{4h}), \tag{13}
 \end{aligned}$$

$$\begin{aligned}
 I_{n, \mathcal{X}} &= \frac{x^{2h}}{n-1} \left\{ [1 - x^2 n \alpha_T d_T^2] {}_2F_1(2h, 2h; 4h; x) \sum \alpha_{\mathcal{X}\mathcal{X}} (d_{\mathcal{X}\mathcal{X}}^{j_1 j_2})^2 \right. \\
 &+ x {}_2F_1(2h+1, 2h+1; 4h+2; x) \sum \alpha_{\mathcal{R}} (d_{\mathcal{R}}^{j_1 j_2})^2 \\
 &\left. + x^2 \left[\sum \alpha_{\mathcal{X}\mathcal{Y}} (d_{\mathcal{X}\mathcal{Y}}^{j_1 j_2})^2 + \sum \alpha_{\mathcal{U}} (d_{\mathcal{U}}^{j_1 j_2})^2 + \sum \alpha_{T\mathcal{X}\mathcal{X}} (d_{T\mathcal{X}\mathcal{X}}^{j_1 j_2 j_3})^2 \right] + O(x^3) \right\} + O(x^{3h}). \tag{16}
 \end{aligned}$$

as well as the one-loop holographic mutual information

$$\begin{aligned}
 I_{\mathcal{X}}^{1\text{-loop}} &= \frac{\sqrt{\pi} \Gamma(2h+1) x^{2h}}{4^{2h+1} \Gamma(2h+3/2)} \left[1 + \frac{2h(2h+1)x}{4h+3} \right. \\
 &\left. + \frac{(h+1)(2h+1)^2(4h+1)x^2}{2(4h+3)(4h+5)} + O(x^3) \right] + O(x^{4h}). \tag{14}
 \end{aligned}$$

The holographic mutual information is in accord with the result in Ref. [27].

On the CFT side, we have to consider the extra quasiprimary operators that are listed in Table 2. In the table we have the definitions

$$\begin{aligned}
 \mathcal{R}_{j_1 j_2} &= \mathcal{X}_{j_1} i \partial \mathcal{X}_{j_2} - i \partial \mathcal{X}_{j_1} \mathcal{X}_{j_2}, \\
 \mathcal{U}_{j_1 j_2} &= \partial \mathcal{X}_{j_1} \partial \mathcal{X}_{j_2} - \frac{h}{2h+1} (\mathcal{X}_{j_1} \partial^2 \mathcal{X}_{j_2} + \partial^2 \mathcal{X}_{j_1} \mathcal{X}_{j_2}). \tag{15}
 \end{aligned}$$

Table 2. The quasiprimary operators in CFTⁿ from the conformal family of chiral primary operator \mathcal{X} in the original CFT.

L_0	quasiprimary operators	degeneracies
h	$\mathcal{X}_{j_1} \mathcal{X}_{j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
$h+1$	$\mathcal{R}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
	$\mathcal{X}_{j_1} \mathcal{Y}_{j_2}$ with $j_1 \neq j_2$	$n(n-1)$
$h+2$	$\mathcal{U}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
	$T_{j_1} \mathcal{X}_{j_2} \mathcal{X}_{j_3}$ with $j_1 \neq j_2, j_1 \neq j_3$ and $j_2 < j_3$	$\frac{n(n-1)(n-2)}{2}$
\dots	\dots	\dots

We get contributions to Rényi mutual information from the conformal family \mathcal{X}

We have the normalization factors

$$\begin{aligned}\alpha_{\mathcal{X}\mathcal{X}} &= i^{4h} \alpha_{\mathcal{X}}^2, \quad \alpha_{\mathcal{R}} = 4hi^{4h} \alpha_{\mathcal{X}}^2, \\ \alpha_{\mathcal{X}\mathcal{Y}} &= \frac{(2h+1)c + 2h(8h-5)}{2(2h+1)} i^{4h} \alpha_{\mathcal{X}}^2, \\ \alpha_{\mathcal{U}} &= \frac{4h^2(4h+1)}{2h+1} i^{4h} \alpha_{\mathcal{X}}^2, \\ \alpha_{T\mathcal{X}\mathcal{X}} &= \frac{c}{2} i^{4h} \alpha_{\mathcal{X}}^2,\end{aligned}\quad (17)$$

and the OPE coefficients

$$\begin{aligned}d_{\mathcal{X}\mathcal{X}}^{j_1 j_2} &= \frac{i^{2h}}{\alpha_{\mathcal{X}}(2n)^{2h}} \frac{1}{s_{j_1 j_2}^{2h}}, \quad d_{\mathcal{R}}^{j_1 j_2} = \frac{i^{2h}}{\alpha_{\mathcal{X}}(2n)^{2h+1}} \frac{c_{j_1 j_2}}{s_{j_1 j_2}^{2h+1}}, \\ d_{\mathcal{X}\mathcal{Y}}^{j_1 j_2} &= \frac{i^{2h}(n^2-1)}{3\alpha_{\mathcal{X}}(2n)^{2h+2}} \frac{1}{s_{j_1 j_2}^{2h}}, \\ d_{\mathcal{U}}^{j_1 j_2} &= \frac{i^{2h}}{2h(4h+1)\alpha_{\mathcal{X}}(2n)^{2h+2}} \\ &\quad \cdot \frac{(2h+1)(4h+1) - 2h(n^2+4h+1)s_{j_1 j_2}^2}{s_{j_1 j_2}^{2h+2}}, \\ d_{T\mathcal{X}\mathcal{X}}^{j_1 j_2 j_3} &= \frac{i^{2h}}{\alpha_{\mathcal{X}}(2n)^{2h+2}} \left(-\frac{2h}{c} \frac{1}{s_{j_1 j_2}^2 s_{j_1 j_3}^2 s_{j_2 j_3}^{2h-2}} \right.\end{aligned}$$

$$\left. + \frac{n^2-1}{3} \frac{1}{s_{j_2 j_3}^{2h}} \right). \quad (18)$$

Using Eq. (16), we can reproduce the one-loop holomorphic Rényi mutual information (13).

3 Rényi entropy of one interval on a torus

We calculate the contributions of a non-vacuum conformal family to Rényi entropy of one interval with length ℓ on a torus in the low temperature limit. The torus has spatial period L and temporal period β , with the temperature being $1/\beta$, and at low temperature we have $\beta/L \gg 1$. On the gravity side we use the method in Refs. [12, 19], and on the CFT side we use the method in Refs. [19, 23].

3.1 Non-chiral primary operator

On the gravity side the one-loop partition function (1) still applies, and we use a different Schottky group that can be found in Refs. [12, 19]. We get the contributions of a non-chiral conformal family to the one-loop holographic Rényi entropy

$$\begin{aligned}S_{n,\mathcal{X}}^{1\text{-loop}} &= -\frac{nq^\Delta}{n-1} \left\{ \left[\frac{1}{n^{2\Delta}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2\Delta} - 1 \right] \right. \\ &\quad + \left[\frac{2}{n^{2\Delta+2}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2\Delta} \left(n^2 \Delta \cos^2 \frac{\pi\ell}{L} - n\Delta \sin \frac{2\pi\ell}{L} \cot \frac{\pi\ell}{nL} + \frac{\sin^2 \frac{\pi\ell}{L}}{\sin^2 \frac{\pi\ell}{nL}} \left(\Delta \cos^2 \frac{\pi\ell}{nL} + 1 \right) \right) - 2 \right] q \\ &\quad + \left[\frac{1}{9n^{2\Delta+4}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2\Delta} \left(n^4 \Delta \left((18\Delta+29) \cos^4 \frac{\pi\ell}{L} - 16 \cos^2 \frac{\pi\ell}{L} - 4 \right) \right. \right. \\ &\quad - 6n^3 \Delta \sin \frac{2\pi\ell}{L} \cot \frac{\pi\ell}{nL} \left((6\Delta+5) \cos^2 \frac{\pi\ell}{L} - 2 \right) \\ &\quad + 2n^2 \frac{\sin^2 \frac{\pi\ell}{L}}{\sin^2 \frac{\pi\ell}{nL}} \left(\Delta(54\Delta+19) \cos^2 \frac{\pi\ell}{L} \cos^2 \frac{\pi\ell}{nL} + (35\Delta+18) \cos^2 \frac{\pi\ell}{L} + 4\Delta \sin^2 \frac{\pi\ell}{nL} \right) \\ &\quad - 12n \frac{\sin^3 \frac{\pi\ell}{L}}{\sin^3 \frac{\pi\ell}{nL}} \cos \frac{\pi\ell}{L} \cos \frac{\pi\ell}{nL} \left(\Delta(6\Delta+1) \cos^2 \frac{\pi\ell}{nL} + 11\Delta+6 \right) \\ &\quad \left. \left. + \frac{\sin^4 \frac{\pi\ell}{L}}{\sin^4 \frac{\pi\ell}{nL}} \left(\Delta(18\Delta+5) \cos^4 \frac{\pi\ell}{nL} + 2(31\Delta+18) \cos^2 \frac{\pi\ell}{nL} - 4\Delta+27 \right) \right) - 3 \right] q^2 + O(q^3) \left. \right\} + O(q^{2\Delta}),\end{aligned}\quad (19)$$

with $q = e^{-2\pi\beta/L} \ll 1$. In Equation (19) the omitted order $q^{2\Delta}$ result is from the 2-CDWs [12]. To make the order $q^{\Delta+2}$ part meaningful, we need $2\Delta > \Delta+2$ and so $\Delta > 2$. Taking the $n \rightarrow 1$ limit, we get the one-loop holographic entanglement entropy

$$\begin{aligned}S_{\mathcal{X}}^{1\text{-loop}} &= \left(1 - \frac{\pi\ell}{L} \cot \frac{\pi\ell}{L} \right) 2q^\Delta (\Delta + 2(\Delta+1)) \\ &\quad q + 3(\Delta+2)q^2 + O(q^3) + O(q^{2\Delta}).\end{aligned}\quad (20)$$

Then we calculate the Rényi entropy on the CFT

side. For the vacuum conformal family we have the density matrix with

$$\rho_{\text{vac}} = |0\rangle\langle 0| + \frac{q^2}{\alpha_T} |T\rangle\langle T| + \frac{q^2}{\alpha_{\bar{T}}} |\bar{T}\rangle\langle \bar{T}| + O(q^3). \quad (21)$$

Note that we only consider the case without chemical potential. Considering the contributions of the conformal family of a non-chiral primary operator \mathcal{X} , we have the density matrix

$$\rho = \rho_{\text{vac}} + \rho_{\mathcal{X}}, \quad (22) \quad \text{On the CFT side, we get the Rényi entropy}$$

$$\begin{aligned} \rho_{\mathcal{X}} = & q^\Delta \left(\frac{1}{\alpha_{\mathcal{X}}} |\mathcal{X}\rangle\langle \mathcal{X}| + \frac{q}{\alpha_{\partial\mathcal{X}}} |\partial\mathcal{X}\rangle\langle \partial\mathcal{X}| + \frac{q}{\alpha_{\bar{\partial}\mathcal{X}}} |\bar{\partial}\mathcal{X}\rangle\langle \bar{\partial}\mathcal{X}| \right. \\ & \left. + \frac{q^2}{\alpha_{\mathcal{Y}}} |\mathcal{Y}\rangle\langle \mathcal{Y}| + \frac{q^2}{\alpha_{\mathcal{Z}}} |\mathcal{Z}\rangle\langle \mathcal{Z}| \right. \\ & \left. + \frac{q^2}{\alpha_{\partial^2\mathcal{X}}} |\partial^2\mathcal{X}\rangle\langle \partial^2\mathcal{X}| + \frac{q^2}{\alpha_{\bar{\partial}^2\mathcal{X}}} |\bar{\partial}^2\mathcal{X}\rangle\langle \bar{\partial}^2\mathcal{X}| + O(q^3) \right). \end{aligned} \quad (23)$$

$$\begin{aligned} S_{n,\mathcal{X}} = & -\frac{nq^\Delta}{n-1} \left\{ \left(\frac{\langle \mathcal{X}(\infty, \infty) \mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{X}}} - 1 \right) + \left(\frac{\langle \partial\mathcal{X}(\infty, \infty) \partial\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial\mathcal{X}}} + \frac{\langle \bar{\partial}\mathcal{X}(\infty, \infty) \bar{\partial}\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\bar{\partial}\mathcal{X}}} - 2 \right) q \right. \\ & + \left[\frac{\langle \partial^2\mathcal{X}(\infty, \infty) \partial^2\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial^2\mathcal{X}}} + \frac{\langle \partial\bar{\partial}\mathcal{X}(\infty, \infty) \partial\bar{\partial}\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial\bar{\partial}\mathcal{X}}} + \frac{\langle \bar{\partial}^2\mathcal{X}(\infty, \infty) \bar{\partial}^2\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\bar{\partial}^2\mathcal{X}}} \right. \\ & + \frac{\langle \mathcal{Y}(\infty, \infty) \mathcal{Y}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{Y}}} - n \frac{\langle T(\infty) T(0) \rangle_{\mathcal{C}^n} \langle \mathcal{X}(\infty, \infty) \mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_T \alpha_{\mathcal{X}}} \\ & + \frac{\langle \mathcal{Z}(\infty, \infty) \mathcal{Z}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{Z}}} - n \frac{\langle \bar{T}(\infty) \bar{T}(0) \rangle_{\mathcal{C}^n} \langle \mathcal{X}(\infty, \infty) \mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\bar{T}} \alpha_{\mathcal{X}}} \\ & \left. + \sum_{j=1}^{n-1} \left(\frac{\langle T(\infty) T(0) \mathcal{X}(\infty_j, \infty_j) \mathcal{X}(0_j, 0_j) \rangle_{\mathcal{C}^n}}{\alpha_T \alpha_{\mathcal{X}}} + \frac{\langle \bar{T}(\infty) \bar{T}(0) \mathcal{X}(\infty_j, \infty_j) \mathcal{X}(0_j, 0_j) \rangle_{\mathcal{C}^n}}{\alpha_{\bar{T}} \alpha_{\mathcal{X}}} \right) - 3 \right] q^2 \\ & \left. + O(q^3) \right\} + O(q^{2\Delta}). \end{aligned} \quad (24)$$

Note that in ∞_j and 0_j the subscript j denotes different replicas, and the ∞ and 0 without any subscript mean the special $j = 0$ case. The correlation functions on

the n -fold complex plane \mathcal{C}^n can be calculated by mapping it to an ordinary complex plane \mathcal{C} by a conformal transformation. The results are

$$\begin{aligned} \frac{\langle \mathcal{X}(\infty, \infty) \mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{X}}} &= \frac{1}{n^{2\Delta}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2\Delta}, \\ \frac{\langle \partial\mathcal{X}(\infty, \infty) \partial\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial\mathcal{X}}} &= \frac{2}{n^{2\Delta+2}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2\Delta} \left(n^2 h \cos^2 \frac{\pi\ell}{L} - n h \sin \frac{2\pi\ell}{L} \cot \frac{\pi\ell}{nL} + \frac{\sin^2 \frac{\pi\ell}{L}}{\sin^2 \frac{\pi\ell}{nL}} \left(h \cos^2 \frac{\pi\ell}{nL} + \frac{1}{2} \right) \right), \\ \frac{\langle \bar{\partial}\mathcal{X}(\infty, \infty) \bar{\partial}\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\bar{\partial}\mathcal{X}}} &= \frac{\langle \partial\mathcal{X}(\infty, \infty) \partial\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial\mathcal{X}}} \Big|_{h \rightarrow \bar{h}}, \\ \frac{\langle \partial^2\mathcal{X}(\infty, \infty) \partial^2\mathcal{X}(0, 0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial^2\mathcal{X}}} &= \frac{1}{2h+1} \frac{1}{n^{2\Delta+4}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2\Delta} \left[n^4 h \left(2(h+1) \cos^2 \frac{\pi\ell}{L} - 1 \right)^2 \right. \\ & \quad - 2n^3 h (2h+1) \sin \frac{2\pi\ell}{L} \cot \frac{\pi\ell}{nL} \left(2(h+1) \cos^2 \frac{\pi\ell}{L} - 1 \right) \\ & \quad + 2n^2 \frac{\sin^2 \frac{\pi\ell}{L}}{\sin^2 \frac{\pi\ell}{nL}} \left(2h \cos^2 \frac{\pi\ell}{nL} + 1 \right) \left((6h^2 + 6h + 1) \cos^2 \frac{\pi\ell}{L} - h \right) \\ & \quad \left. - 4n(2h+1) \frac{\sin^3 \frac{\pi\ell}{L}}{\sin^3 \frac{\pi\ell}{nL}} \cos \frac{\pi\ell}{L} \cos \frac{\pi\ell}{nL} \left(2h^2 \cos^2 \frac{\pi\ell}{nL} + 3h + 1 \right) \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{\sin^4 \frac{\pi\ell}{L}}{\sin^4 \frac{\pi\ell}{nL}} \left(4h^3 \cos^4 \frac{\pi\ell}{nL} + 2(6h^2 + 4h + 1) \cos^2 \frac{\pi\ell}{nL} + 3h + 1 \right) \Big], \\
 \frac{\langle \bar{\partial}^2 \mathcal{X}(\infty, \infty) \bar{\partial}^2 \mathcal{X}(0, 0) \rangle_{c^n}}{\alpha_{\bar{\partial}^2 \mathcal{X}}} &= \frac{\langle \partial^2 \mathcal{X}(\infty, \infty) \partial^2 \mathcal{X}(0, 0) \rangle_{c^n}}{\alpha_{\partial^2 \mathcal{X}}} \Big|_{h \rightarrow \bar{h}}, \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \partial \bar{\partial} \mathcal{X}(\infty, \infty) \partial \bar{\partial} \mathcal{X}(0, 0) \rangle_{c^n}}{\alpha_{\partial \bar{\partial} \mathcal{X}}} &= \frac{\langle \partial \mathcal{X}(\infty, \infty) \partial \mathcal{X}(0, 0) \rangle_{c^n}}{\alpha_{\partial \mathcal{X}}} \frac{\langle \bar{\partial} \mathcal{X}(\infty, \infty) \bar{\partial} \mathcal{X}(0, 0) \rangle_{c^n}}{\alpha_{\bar{\partial} \mathcal{X}}} \frac{\alpha_{\mathcal{X}}}{\langle \mathcal{X}(\infty, \infty) \mathcal{X}(0, 0) \rangle_{c^n}}, \\
 \frac{\langle \mathcal{Y}(\infty, \infty) \mathcal{Y}(0, 0) \rangle_{c^n}}{\alpha_{\mathcal{Y}}} &= \frac{1}{n^{2\Delta+4}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2\Delta} \left[\frac{(n^2 - 1)^2}{18(2h + 1)} \sin^4 \frac{\pi\ell}{L} \left((2h + 1)c + 2h(8h - 5) \right) + \frac{\sin^4 \frac{\pi\ell}{L}}{\sin^4 \frac{\pi\ell}{nL}} \right], \\
 \frac{\langle \mathcal{Z}(\infty, \infty) \mathcal{Z}(0, 0) \rangle_{c^n}}{\alpha_{\mathcal{Z}}} &= \frac{\langle \mathcal{Y}(\infty, \infty) \mathcal{Y}(0, 0) \rangle_{c^n}}{\alpha_{\mathcal{Y}}} \Big|_{h \rightarrow \bar{h}} \\
 \frac{\langle T(\infty) T(0) \mathcal{X}(\infty_j, \infty_j) \mathcal{X}(0_j, 0_j) \rangle_{c^n}}{\alpha_T \alpha_{\mathcal{X}}} &= \frac{1}{n^{2\Delta+4}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2\Delta} \sin^4 \frac{\pi\ell}{L} \left[\frac{(n^2 - 1)^2 c}{18} + \frac{1}{\sin^4 \frac{\pi\ell}{nL}} \right. \\
 & \quad \left. + \frac{(n^2 - 1)h}{3} \sin^2 \frac{\pi\ell}{nL} \frac{\cos \frac{2\pi j}{n} \cos \frac{2\pi\ell}{nL} - 1}{\left(\cos \frac{2\pi j}{n} - \cos \frac{2\pi\ell}{nL} \right)^2} \right] + O\left(\frac{1}{c}\right), \\
 \frac{\langle \bar{T}(\infty) \bar{T}(0) \mathcal{X}(\infty_j, \infty_j) \mathcal{X}(0_j, 0_j) \rangle_{c^n}}{\alpha_{\bar{T}} \alpha_{\mathcal{X}}} &= \frac{\langle T(\infty) T(0) \mathcal{X}(\infty_j, \infty_j) \mathcal{X}(0_j, 0_j) \rangle_{c^n}}{\alpha_T \alpha_{\mathcal{X}}} \Big|_{h \rightarrow \bar{h}}.
 \end{aligned}$$

In the last two correlation functions we have omitted the $O(1/c)$ terms in the large c limit. Also, we have to use [19]

$$-\frac{n \cos \frac{\pi\ell}{L} \cos \frac{\pi\ell}{nL}}{\sin \frac{\pi\ell}{L} \sin^3 \frac{\pi\ell}{nL}} + \frac{5 \cos^2 \frac{\pi\ell}{nL} + 4}{6 \sin^4 \frac{\pi\ell}{nL}}, \tag{27}$$

$$\begin{aligned}
 \frac{\langle T(\infty) T(0) \rangle_{c^n}}{\alpha_T} &= \frac{\langle \bar{T}(\infty) \bar{T}(0) \rangle_{c^n}}{\alpha_{\bar{T}}} \\
 &= \frac{c(n^2 - 1)^2}{18n^4} \sin^4 \frac{\pi\ell}{L} + \frac{1}{n^4} \frac{\sin^4 \frac{\pi\ell}{L}}{\sin^4 \frac{\pi\ell}{nL}}. \tag{26}
 \end{aligned}$$

Given the summation formula

$$\sum_{j=1}^{n-1} \frac{\cos \frac{2\pi j}{n} \cos \frac{2\pi\ell}{nL} - 1}{\sin^2 \frac{\pi j}{n} \left(\cos \frac{2\pi j}{n} - \cos \frac{2\pi\ell}{nL} \right)^2} = \frac{n^2 \left(\cos^2 \frac{\pi\ell}{L} - 4 \right)}{6 \sin^2 \frac{\pi\ell}{L} \sin^2 \frac{\pi\ell}{nL}}$$

we reproduce the one-loop gravity result (19) in the large c limit.

3.2 Chiral primary operator

Using the one-loop partition function (12) and the Schottky group in Refs. [12, 19], we get the contributions of the fields dual to the conformal family of a chiral primary operator to the one-loop holographic Rényi entropy

$$\begin{aligned}
 S_{n, \mathcal{X}}^{1\text{-loop}} &= -\frac{nq^h}{n-1} \left\{ \left[\frac{1}{n^{2h}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2h} - 1 \right] \right. \\
 & \quad + \left[\frac{2}{n^{2h+2}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2h} \left(n^2 h \cos^2 \frac{\pi\ell}{L} - nh \sin \frac{2\pi\ell}{L} \cot \frac{\pi\ell}{nL} + \frac{\sin^2 \frac{\pi\ell}{L}}{\sin^2 \frac{\pi\ell}{nL}} \left(h \cos^2 \frac{\pi\ell}{nL} + \frac{1}{2} \right) \right) - 1 \right] q \\
 & \quad + \left[\frac{1}{9n^{2h+4}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2h} \left(n^4 h \left((18h + 29) \cos^4 \frac{\pi\ell}{L} - 16 \cos^2 \frac{\pi\ell}{L} - 4 \right) \right. \right. \\
 & \quad \left. \left. - 6n^3 h \sin \frac{2\pi\ell}{L} \cot \frac{\pi\ell}{nL} \left((6h + 5) \cos^2 \frac{\pi\ell}{L} - 2 \right) \right) \right. \\
 & \quad \left. + 2n^2 \frac{\sin^2 \frac{\pi\ell}{L}}{\sin^2 \frac{\pi\ell}{nL}} \left(h(54h + 19) \cos^2 \frac{\pi\ell}{L} \cos^2 \frac{\pi\ell}{nL} + (26h + 9) \cos^2 \frac{\pi\ell}{L} + 4h \sin^2 \frac{\pi\ell}{nL} \right) \right] \tag{28}
 \end{aligned}$$

$$\begin{aligned}
 & -12n \frac{\sin^3 \frac{\pi\ell}{L}}{\sin^3 \frac{\pi\ell}{nL}} \cos \frac{\pi\ell}{L} \cos \frac{\pi\ell}{nL} \left(h(6h+1) \cos^2 \frac{\pi\ell}{nL} + 8h+3 \right) \\
 & + \frac{\sin^4 \frac{\pi\ell}{L}}{\sin^4 \frac{\pi\ell}{nL}} \left(h(18h+5) \cos^4 \frac{\pi\ell}{nL} + 2(22h+9) \cos^2 \frac{\pi\ell}{nL} - 4h+9 \right) - 1 \Big] q^2 + O(q^3) \Big\} + O(q^{2h}),
 \end{aligned}$$

with $q = e^{-2\pi\beta/L} \ll 1$. Taking the $n \rightarrow 1$ limit, we get the one-loop holographic entanglement entropy

$$\begin{aligned}
 S_{\mathcal{X}}^{1\text{-loop}} &= \left(1 - \frac{\pi\ell}{L} \cot \frac{\pi\ell}{L} \right) 2q^h (h + (h+1)q + (h+2) \\
 & q^2 + O(q^3)) + O(q^{2h}). \quad (29)
 \end{aligned}$$

Then we calculate the Rényi entropy on the CFT side. For the vacuum conformal family we have the holomorphic part of the density matrix

$$\rho_{\text{vac}} = |0\rangle\langle 0| + \frac{q^2}{\alpha_T} |T\rangle\langle T| + O(q^3). \quad (30)$$

Considering the contributions of the conformal family of a chiral primary operator \mathcal{X} we have the density matrix

$$\rho = \rho_{\text{vac}} + \rho_{\mathcal{X}}, \quad (31)$$

with

$$\begin{aligned}
 \rho_{\mathcal{X}} &= q^h \left(\frac{1}{\alpha_{\mathcal{X}}} |\mathcal{X}\rangle\langle \mathcal{X}| + \frac{q}{\alpha_{\partial\mathcal{X}}} |\partial\mathcal{X}\rangle\langle \partial\mathcal{X}| + \frac{q^2}{\alpha_{\mathcal{Y}}} |\mathcal{Y}\rangle\langle \mathcal{Y}| \right. \\
 & \left. + \frac{q^2}{\alpha_{\partial^2\mathcal{X}}} |\partial^2\mathcal{X}\rangle\langle \partial^2\mathcal{X}| + O(q^3) \right). \quad (32)
 \end{aligned}$$

Then we get the Rényi entropy

$$\begin{aligned}
 S_{n,\mathcal{X}} &= -\frac{nq^h}{n-1} \left[\left(\frac{\langle \mathcal{X}(\infty)\mathcal{X}(0) \rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{X}}} - 1 \right) + \left(\frac{\langle \partial\mathcal{X}(\infty)\partial\mathcal{X}(0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial\mathcal{X}}} - 1 \right) q + \left(\frac{\langle \partial^2\mathcal{X}(\infty)\partial^2\mathcal{X}(0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial^2\mathcal{X}}} \right. \right. \\
 & \left. \left. + \frac{\langle \mathcal{Y}(\infty)\mathcal{Y}(0) \rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{Y}}} - n \frac{\langle T(\infty)T(0) \rangle_{\mathcal{C}^n} \langle \mathcal{X}(\infty)\mathcal{X}(0) \rangle_{\mathcal{C}^n}}{\alpha_T \alpha_{\mathcal{X}}} + \sum_{j=1}^{n-1} \frac{\langle T(\infty)T(0)\mathcal{X}(\infty_j)\mathcal{X}(0_j) \rangle_{\mathcal{C}^n}}{\alpha_T \alpha_{\mathcal{X}}} - 1 \right) q^2 \right. \\
 & \left. + O(q^3) \right] + O(q^{2h}). \quad (33)
 \end{aligned}$$

We need the correlation functions

$$\begin{aligned}
 \frac{\langle \mathcal{X}(\infty)\mathcal{X}(0) \rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{X}}} &= \frac{1}{n^{2h}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2h}, \\
 \frac{\langle \partial\mathcal{X}(\infty)\partial\mathcal{X}(0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial\mathcal{X}}} &= \frac{2}{n^{2h+2}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2h} \left(n^2 h \cos^2 \frac{\pi\ell}{L} - nh \sin \frac{2\pi\ell}{L} \cot \frac{\pi\ell}{nL} + \frac{\sin^2 \frac{\pi\ell}{L}}{\sin^2 \frac{\pi\ell}{nL}} \left(h \cos^2 \frac{\pi\ell}{nL} + \frac{1}{2} \right) \right), \\
 \frac{\langle \partial^2\mathcal{X}(\infty)\partial^2\mathcal{X}(0) \rangle_{\mathcal{C}^n}}{\alpha_{\partial^2\mathcal{X}}} &= \frac{1}{2h+1} \frac{1}{n^{2h+4}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2h} \left[n^4 h \left(2(h+1) \cos^2 \frac{\pi\ell}{L} - 1 \right)^2 \right. \\
 & - 2n^3 h(2h+1) \sin \frac{2\pi\ell}{L} \cot \frac{\pi\ell}{nL} \left(2(h+1) \cos^2 \frac{\pi\ell}{L} - 1 \right) \\
 & + 2n^2 \frac{\sin^2 \frac{\pi\ell}{L}}{\sin^2 \frac{\pi\ell}{nL}} \left(2h \cos^2 \frac{\pi\ell}{nL} + 1 \right) \left((6h^2 + 6h + 1) \cos^2 \frac{\pi\ell}{L} - h \right) \\
 & - 4n(2h+1) \frac{\sin^3 \frac{\pi\ell}{L}}{\sin^3 \frac{\pi\ell}{nL}} \cos \frac{\pi\ell}{L} \cos \frac{\pi\ell}{nL} \left(2h^2 \cos^2 \frac{\pi\ell}{nL} + 3h + 1 \right) \\
 & \left. + \frac{\sin^4 \frac{\pi\ell}{L}}{\sin^4 \frac{\pi\ell}{nL}} \left(4h^3 \cos^4 \frac{\pi\ell}{nL} + 2(6h^2 + 4h + 1) \cos^2 \frac{\pi\ell}{nL} + 3h + 1 \right) \right], \\
 \frac{\langle \mathcal{Y}(\infty)\mathcal{Y}(0) \rangle_{\mathcal{C}^n}}{\alpha_{\mathcal{Y}}} &= \frac{1}{n^{2h+4}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2h} \left[\frac{(n^2 - 1)^2}{18(2h+1)} \sin^4 \frac{\pi\ell}{L} \left((2h+1)c + 2h(8h-5) \right) + \frac{\sin^4 \frac{\pi\ell}{L}}{\sin^4 \frac{\pi\ell}{nL}} \right],
 \end{aligned}$$

$$\frac{\langle T(\infty)T(0)\mathcal{X}(\infty_j)\mathcal{X}(0_j)\rangle_{c^n}}{\alpha_T\alpha_{\mathcal{X}}} = \frac{1}{n^{2h+4}} \left(\frac{\sin \frac{\pi\ell}{L}}{\sin \frac{\pi\ell}{nL}} \right)^{2h} \sin^4 \frac{\pi\ell}{L} \left[\frac{(n^2-1)^2 c}{18} + \frac{1}{\sin^4 \frac{\pi\ell}{nL}} + \frac{(n^2-1)h}{3} \sin^2 \frac{\pi\ell}{nL} \frac{\cos \frac{2\pi j}{n} \cos \frac{2\pi\ell}{nL} - 1}{\left(\cos \frac{2\pi j}{n} - \cos \frac{2\pi\ell}{nL} \right)^2} \right] + O\left(\frac{1}{c}\right). \quad (34)$$

Taking the large c limit we reproduce the one-loop gravity result (28).

4 Discussion and conclusions

In this paper we have considered the contributions of a general non-vacuum conformal family to the Rényi mutual information of two intervals on a complex plane and the Rényi entropy of one interval on a torus in two-dimensional CFT. The primary operator of the conformal family can be either non-chiral or chiral. We got the results to orders higher than those in the previous literature, and found matches of gravity and CFT results.

We have only considered the contributions of one non-vacuum conformal family, and this is not complete for a concrete CFT. The algebra of the operators in the vacuum conformal family and one non-vacuum family is not close. For example at level 2Δ there may be a new con-

formal family with primary operator

$$\mathcal{O} = (\mathcal{X}\mathcal{X}) + \dots. \quad (35)$$

To make the result in this paper meaningful, we have to require that the scaling dimension Δ of the primary operator cannot be too small. For two intervals on a complex plane we need $\Delta > 1$, and for one interval on a torus we need $\Delta > 2$. For the contributions of primary operators with a smaller scaling dimension and the contributions of more than two non-vacuum conformal families, further investigations are needed.

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Appendix A

Review of non-vacuum conformal family

In this Appendix we review some properties of the non-vacuum conformal family that are useful for this paper, including the conformal family of a non-chiral primary operator and the conformal family of a chiral primary operator. Details can be found in Refs. [39, 40], or can be easily derived from the results therein.

1 Non-chiral primary operator

The one-loop partition function of the vacuum conformal family is

$$Z_{\text{vac}} = \prod_{k=2}^{+\infty} \frac{1}{(1-q^k)(1-\bar{q}^k)}. \quad (A1)$$

Considering the contributions of the conformal family of a non-chiral primary operator \mathcal{X} , one has to multiply the result (A1) by

$$Z_{\mathcal{X}} = 1 + \frac{q^h \bar{q}^{\bar{h}}}{(1-q)(1-\bar{q})}. \quad (A2)$$

The non-chiral primary operator \mathcal{X} has conformal weights (h, \bar{h}) with $h \neq 0$ and $\bar{h} \neq 0$. One has the scaling dimension $\Delta = h + \bar{h}$ and the spin $s = h - \bar{h}$. As usual, we require that s is an integer or a half integer. In the conformal family of \mathcal{X} , the operators can be written as quasiprimary operators and their derivatives. At level $(h+2, \bar{h})$ and level $(h, \bar{h}+2)$ there

are quasiprimary operators, respectively,

$$\mathcal{Y} = (T\mathcal{X}) - \frac{3}{2(2h+1)} \partial^2 \mathcal{X}, \quad \mathcal{Z} = (\bar{T}\mathcal{X}) - \frac{3}{2(2\bar{h}+1)} \bar{\partial}^2 \mathcal{X}, \quad (A3)$$

with the normalization factors being

$$\alpha_{\mathcal{Y}} = \frac{(2h+1)c + 2h(8h-5)}{2(2h+1)} \alpha_{\mathcal{X}},$$

$$\alpha_{\mathcal{Z}} = \frac{(2\bar{h}+1)c + 2\bar{h}(8\bar{h}-5)}{2(2\bar{h}+1)} \alpha_{\mathcal{X}}. \quad (A4)$$

Note that $\alpha_{\mathcal{X}}$ is the normalization factor of \mathcal{X} , and that we consider a CFT with equaling holomorphic and anti-holomorphic central charges $c = \bar{c}$. Under a general conformal transformation $z \rightarrow f(z)$, $\bar{z} \rightarrow \bar{f}(\bar{z})$, the primary operator \mathcal{X} transforms as

$$\mathcal{X}(z, \bar{z}) = f'^h \bar{f}'^{\bar{h}} \mathcal{X}(f, \bar{f}), \quad (A5)$$

and the quasiprimary operators \mathcal{Y} , \mathcal{Z} transform as

$$\mathcal{Y}(z, \bar{z}) = f'^{h+2} \bar{f}'^{\bar{h}} \mathcal{Y}(f, \bar{f})$$

$$+ \frac{(2h+1)c + 2h(8h-5)}{12(2h+1)} s f'^h \bar{f}'^{\bar{h}} \mathcal{X}(f, \bar{f}),$$

$$\mathcal{Z}(z, \bar{z}) = f'^h \bar{f}'^{\bar{h}+2} \mathcal{Z}(f, \bar{f})$$

$$+ \frac{(2\bar{h}+1)c + 2\bar{h}(8\bar{h}-5)}{12(2\bar{h}+1)} \bar{s} f'^h \bar{f}'^{\bar{h}} \mathcal{X}(f, \bar{f}), \quad (A6)$$

with the Schwarzian derivatives

$$s(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2, \quad \bar{s}(\bar{z}) = \frac{\bar{f}'''(\bar{z})}{\bar{f}'(\bar{z})} - \frac{3}{2} \left(\frac{\bar{f}''(\bar{z})}{\bar{f}'(\bar{z})} \right)^2. \quad (\text{A7})$$

2 Chiral primary operator

For the conformal family of a chiral primary operator \mathcal{X} that has conformal weights $(h, 0)$ with $h \neq 0$, one has to multiply Eq. (A1) by

$$Z_{\mathcal{X}} = 1 + \frac{q^h}{1-q}. \quad (\text{A8})$$

The scaling dimension is $\Delta = h$, and the spin $s = h$ is an inte-

ger or a half integer. At level $(h+2, 0)$ there is a quasiprimary operator and its normalization factor,

$$\mathcal{Y} = (T\mathcal{X}) - \frac{3}{2(2h+1)} \partial^2 \mathcal{X}, \quad \alpha_{\mathcal{Y}} = \frac{(2h+1)c + 2h(8h-5)}{2(2h+1)} \alpha_{\mathcal{X}}. \quad (\text{A9})$$

Under a general conformal transformation $z \rightarrow f(z)$ the primary operator \mathcal{X} and quasiprimary operator \mathcal{Y} transform as

$$\begin{aligned} \mathcal{X}(z) &= f^h \mathcal{X}(f), \quad \mathcal{Y}(z) = f^{h+2} \mathcal{Y}(f) \\ &+ \frac{(2h+1)c + 2h(8h-5)}{12(2h+1)} s f^h \mathcal{X}(f). \end{aligned} \quad (\text{A10})$$

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