Nonlinear description of Yang-Mills cosmology: cosmic inflation and the accompanying Hannay's angle*

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Abstract: Hannay's angle is a classical analogue of the "geometrical phase factor" found by Berry in his research on the quantum adiabatic theorem. This classical analogue is defined if closed curves of constant action variables return to the same curves in phase space after an adaibatic evolution. Adiabatic evolution of Yang-Mills cosmology, which is described by a time-dependent quartic oscillator, is investigated. Phase properties of the Yang-Mills fields are analyzed and the corresponding Hannay's angle is derived from a rigorous evaluation. The obtained Hannay's angle shift is represented in terms of several observable parameters associated with such an angle shift. The time evolution of Hannay's angle in Yang-Mills cosmology is examined from an illustration plotted on the basis of numerical evaluation, and its physical nature is addressed. Hannay's angle, together with its quantum counterpart Berry's phase, plays a pivotal role in conceptual understanding of several cosmological problems and indeed can be used as a supplementary probe for cosmic inflation.

Keywords: Yang-Mills cosmology, Hannay angle, time-dependent quartic Hamiltnian, inflation

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1 Introduction

After the advent of the inflation paradigm for the expanding universe, various cosmological models that are governed by homogeneous and isotropic fields in addition to gravity have become an active research topic in cosmology. While most traditionally accepted fields for such a paradigm are a single or several multiple scalar fields, a large number of cosmological models employ sophisticated cosmological theories based on reliable candidates for dark energy, which is responsible for the inflation. A class of potential theories of cosmology is those associated with non-minimally coupled Yang-Mills (YM) fields, suggested by Gal'tsov and his collaborators [1–5]. Extensive studies on YM or Einstein-Yang-Mills (EYM) cosmology has been carried out for the last two decades and longer, partly motivated by its suitability for explaining the origin of the current stage of accelerated inflation [4].

The aim of this paper is to analyze analytical description of YM condensates in cosmology. In particular, we investigate cosmic evolution of YM fields and find the cosmological Hannay's angle, which seems to shed light on the process of cosmological evolution. Let us recall briefly Hannay's angle, which is, for some Hamiltonians, a slight modification of the old classical adiabatic theorem which makes statements only for integrable Hamiltonian systems expressed in an action-angle variable (I,θ). It says that, in adiabatic evolution, the action variable $I(t,\overrightarrow{X}(t))$ is adiabatic invariant, while the angle can be calculated by solving for the instantaneous frequency of the system $\dot{\theta}(t,\overrightarrow{X}(t))=\dfrac{\partial H(I,\overrightarrow{X}(t))}{\partial I}$. The existence of the action variable $I(t,\overrightarrow{X}(t))$ implies that the evolution of the angle variables can be determined by making a time-dependent canonical transformation of the phase space variables to action-angle variables. Transformation to action-angle variables can be fulfilled by the generating function

$$S(q,I,\overrightarrow{X}(t))=\int^{q_t}\mathrm{d}q'p(q',I,\overrightarrow{X}(t)), \tag{1}$$

where $p=\partial S/\partial q$ and $\theta=\partial S/\partial I$. Hannay discovered, for some Hamiltonians, that an extra geometric shift $\Delta\theta$, in

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the angle variable θ that is conjugate to the action variable I, can be determined from

$$\Delta\theta = \frac{\partial}{\partial I}\int_0^t \left\langle \frac{\partial S(I,\theta,\overrightarrow{X}(t'))}{\partial t'} \right\rangle \mathrm{d}t', \tag{2}$$

where $\langle\ \rangle$ denotes the average over θ at a fixed time.

Cai and Papini [6] showed that a covariant generalization of Berry's phase which is connected to Hannay's angle leads to generating non-Abelian gauge fields in the manner adopted by Wilczek and Zee [7] with non-relativistic approximations. In fact, such covariant formalism is not only very acceptable but also naturally accompanies the theory of YM fields [6]. Moreover, through their further study, Cai and Papini [8] extended the theory of such covariant generalization to non-linearly evolving systems. Yang-Mills fields are vector fields that obey local gauge invariance. We can use them to solve the isotropy problem via an $SU(2)$ YM configuration. A special feature of the YM cosmologies driven by $SU(2)$ gauge fields is that they are described by quartic oscillators [1] that are composed of a time-dependent quadratic plus quartic Hamiltonian. In the YM cosmology description with this potential, the conformal invariance for the quadratic action of YM fields is conserved without introducing a mass scale [4], enabling us to solve the problem of vector inflation.

The time-dependent Hamiltonian and the corresponding Langrangian for the quartic oscillator that describes the $SU(2)$ YM field will be constructed from the basic equation for the Higgs scalar. To facilitate the analysis of the system, we assume that the system evolves adiabatically in time, i.e., the parameters of the system are slowly varying. This assumption is mostly taken for granted in actual mathematical tasks that are considered to tackle dynamical systems with computational schemes. The treatment of the time-dependent quartic oscillator may be not an easy task because of the time dependence of the parameters as well as the presence of the quartic term in the Hamiltonian [9]. To simplify the problem, we perform a canonical transformation that preserves the form of the Hamiltonian equations. Based on this transformation, the time evolution of the system will be investigated on the basis of a perturbative way [10]. Phase properties of the system and Hannay's angle [11] will be derived and their physical natures will be addressed. Hannay's angle is the classical counterpart of the additional quantum phase known as Berry's phase [12–15]. Among the many areas in which Hannay's angle and Berry's phase play important roles, cosmology [6,8,16–23] and celestial mechanics [24, 25] have been actively investigated so far in connection with them. Hannay's angle (or Berry's phase) can be applied in extracting useful information associated with the theoretical interpretation of observational aspects of inflationary perturbations, such as the dynamical evolution of the quantum fluctuations [16], vacuum instability [17], the symmetry breaking of cosmological time [18], and energy conservation in a gravity-scalar system [19].

2 Yang-Mills condensates

One recently suggested potential candidate for the dark energy component is the YM condensate [26–29]. The introduction of the YM gauge boson condensate in Robertson-Walker spacetime is adequate to explain the advent of an inflationary expansion after the Planck time in the early universe [30]. The inflation of the Universe in the early epoch has been verified by the recent analysis of observational data carried out by a research group at the Harvard-Smithsonian Center for Astrophysics using the astronomical instrument BICEP2 (for details refer to their announcement on March 17, 2014, and Ref. [31]). They have discovered a faint but distinctive twisting component (B-mode waves) in the polarization of CMB radiation, which originated from primordial gravitational waves. Such a signal is regarded as the imprint of large-scale primordial cosmic inflation. However, the validity of the experimental results and the relevant cosmological consequences became controversial afterwards and they should be reconfirmed by further observations by, for example, the Keck Array [32].

Einstein-Yang-Mills-Higgs action in the gauge theory associated with spontaneous symmetry breaking with a $SU(2)$ YM field can be described in terms of $\alpha = M_{\mathrm{W}}/(gM_{\mathrm{Pl}})$ and $\beta = M_{\mathrm{H}}/M_{\mathrm{W}}$, where M_{Pl} is the Planck mass, M_{W} is the W-boson mass, and M_{H} is the Higgs mass. Gal'tsov and Davydov [1, 2] suggested the equation for the Higgs scalar q in YM cosmology as

$$\ddot{q}+3H(t)\dot{q} = -\frac{3}{2a^2(t)}q[1+f(t)]^2-\beta^2(q^2-\alpha^2)q, \tag{3}$$

where $H(t)$ is the Hubble parameter, $a(t)$ is the scale factor that appears in the FRW metrics, and $f(t)$ is a dimensionless single scalar function.

We suppose that the Higgs field varies much faster than the YM field during some stage of cosmological evolution [1, 2, 33, 34]. This allows a separate description of the Higgs field via the quartic time-dependent oscillator model described in terms of adiabatically varying parameters. According to this, we consider a universe that yields a non-trivial steady state with the parameter [1]

$$f = \frac{\sqrt{8}\beta-\sqrt{3}}{\sqrt{8}\beta+\sqrt{3}}, \tag{4}$$

which satisfies the equations for fields associated with the YM and Higgs scalar functions. Now, Eq. (3) becomes

$$\ddot{q}+3H(t)\dot{q}+\omega^2(t)q = -\beta^2 q^3, \tag{5}$$

where

$$\omega^2(t)=\frac{3}{2a^2(t)}\left(\frac{\sqrt{8}\beta-\sqrt{3}}{\sqrt{8}\beta+\sqrt{3}}+1\right)^2-\beta^2\alpha^2. \qquad (6)$$

According to the value of β, there are stable and unstable directions of cosmic evolution from a fixed point in the phase space [1]. We are interested in the points which are stable against small perturbations. The translation of the long quasistable states may terminate during the period of inflation. The time behavior of solutions of Eq. (5) is essentially characterized by the parameters α and β in this case. The evolution of the scale factor would then be mainly dominated by the Higgs potential, inducing a slow-roll regime.

Notice that Eq. (5) is a kind of time-dependent quartic oscillator equation. The Hamiltonian for this oscillator can be represented in the form

$$\mathcal{H}=\frac{1}{2}\left[\mathrm{e}^{-3\int H(t)\mathrm{d}t}p^2+\omega^2(t)\mathrm{e}^{3\int H(t)\mathrm{d}t}q^2 +2\beta^2\mathrm{e}^{3\int H(t)\mathrm{d}t}q^4\right]. \qquad (7)$$

This Hamiltonian is important for treating YM condensates in cosmology [1]. From Hamilton's equation

$$\dot{q}=\frac{\partial\mathcal{H}}{\partial p}=\mathrm{e}^{-3\int H(t)\mathrm{d}t}p, \qquad (8)$$

the canonical momentum of the system becomes

$$p=\dot{q}\mathrm{e}^{3\int H(t)\mathrm{d}t}. \qquad (9)$$

From a fundamental relation for the Lagrangian, $\mathcal{L}=p\dot{q}-\mathcal{H}$, we obtain

$$\mathcal{L}=\frac{1}{2}\mathrm{e}^{3\int H(t)\mathrm{d}t}\left[\dot{q}^2-\omega^2(t)q^2-2\beta^2q^4\right]. \qquad (10)$$

In terms of a new variable Q that is defined as $Q=q\mathrm{e}^{\frac{3}{2}\int H(t')\mathrm{d}t'}$, the Lagrangian can be rewritten such that

$$\mathcal{L}(Q,\dot{Q},t)=\frac{1}{2}\left[\dot{Q}^2-3H(t)Q\dot{Q}-\Omega^2(t)Q^2 -2\beta^2\mathrm{e}^{-3\int H(t')\mathrm{d}t'}Q^4\right], \qquad (11)$$

where

$$\Omega^2(t)=\omega^2(t)-\frac{9}{4}H^2(t). \qquad (12)$$

The corresponding Hamiltonian in the transformed system is given by

$$\mathcal{H}(P,Q,t)=P\dot{Q}-\mathcal{L}(Q,\dot{Q},t). \qquad (13)$$

Considering $P=\partial\mathcal{L}/\partial\dot{Q}$, we have

$$\dot{Q}=P+\frac{3}{2}H(t)Q. \qquad (14)$$

Then, the Hamiltonian is represented to be

$$\mathcal{H}(P,Q,t)=\frac{1}{2}P^2+\frac{3}{2}H(t)PQ+\frac{1}{2}\omega^2(t)Q^2 +\beta^2Q^4\mathrm{e}^{-3\int H(t')\mathrm{d}t'}. \qquad (15)$$

For a particular case where the parameters are given by $H(t)=2\lambda(t)/3$, $\omega^2(t)=\omega_0^2$, and $\beta^2=\nu/4$, the Hamiltonian reduces to that in Eq. (30) of Ref. [10]:

$$\mathcal{H}_D(P,Q)=\frac{1}{2}P^2+\lambda PQ+\frac{1}{2}\omega_0^2Q^2+\frac{\nu}{4}\mathrm{e}^{-3\int H(t')\mathrm{d}t'}Q^4. \qquad (16)$$

which corresponds to the Hamiltonian of the damped quartic oscillator.

Since the description of the YM field in this model yields the general quartic oscillator Hamiltonian, Eq. (15), which is characterized by general time-dependent parameters, a careful mathematical treatment is necessary for a rigorous analysis of the system. From Hamilton's equations

$$\dot{Q}=\partial\mathcal{H}/\partial P, \quad \dot{P}=-\partial\mathcal{H}/\partial Q, \qquad (17)$$

we obtain the classical equation of motion as

$$\ddot{Q}+\left(\Omega^2-\frac{3}{2}\dot{H}\right)Q+4\beta^2\mathrm{e}^{-3\int H(t')\mathrm{d}t'}Q^3=0, \qquad (18)$$

which is a somewhat complicated form. The concept of effective YM condensate naturally allows a solution of the coincidence problem in the dark energy model of a flat expanding universe [27–29]. Based on the mathematical description associated with YM condensates given here, the phase properties of YM fields and the relevant Hannay's angle shift will be investigated in the subsequent sections.

3 Time evolution of the Yang-Mills fields

To know how the Universe evolved from the early epoch to today, it is necessary to study the time evolution of the Universe regarding its constituent fields and matters. To study the time behavior of the YM fields, we introduce a complex variable Z and its complex conjugate $\bar{Z}$ such that

$$Z=\sqrt{\frac{\Omega}{2}}\left[Q-\frac{\mathrm{i}}{\Omega}\left(\frac{3}{2}HQ+P\right)\right], \qquad (19)$$

$$\bar{Z}=\sqrt{\frac{\Omega}{2}}\left[Q+\frac{\mathrm{i}}{\Omega}\left(\frac{3}{2}HQ+P\right)\right]. \qquad (20)$$

Equivalently, Q and P can be read as

$$Q=\frac{1}{\sqrt{2\Omega}}\left(Z+\bar{Z}\right), \qquad (21)$$

$$P=\frac{\mathrm{i}\Omega-\frac{3}{2}H}{\sqrt{2\Omega}}Z-\frac{\mathrm{i}\Omega+\frac{3}{2}H}{\sqrt{2\Omega}}\bar{Z}. \qquad (22)$$

From the first of the Hamilton's equations given in Eq. (17), we have

$$\dot{\bar{Z}}=\frac{\dot{\Omega}}{2\Omega}\left(Z+\bar{Z}\right)+\mathrm{i}\Omega Z-\mathrm{i}\Omega\bar{Z}-\dot{Z}. \tag{23}$$

The evaluation of the second Hamilton's equation, Eq. (17), leads to

$$\dot{Z}=\mathrm{i}\Omega Z+\frac{4\mathrm{i}\beta^2}{4\Omega^2}\mathrm{e}^{-3\int H\left(t'\right)\mathrm{d}t'}\left(Z+\bar{Z}\right)^3 -\frac{\frac{3\mathrm{i}}{2}\dot{H}}{2\Omega}Z+\frac{\dot{\Omega}-\frac{3\mathrm{i}}{2}\dot{H}}{2\Omega}\bar{Z}. \tag{24}$$

This equation can be solved perturbatively using its canonical reduction to a normal form, provided that the system is weakly nonlinear $[(4\beta^2/\omega^2)Q^2\ll 1]$ in addition to the supposition that the parameters, H, ω, and β, vary adiabatically, i.e., slowly in time. The previous supposition that the Higgs field varies much faster than the YM one during some stage of cosmological evolution allows us to assume that the system is weakly nonlinear. Thus Eq. (24) can be solved perturbatively using its canonical reduction to normal form. Under these conditions, let us take advantage of the following near to identity transformation

$$Z=u+\delta\bar{u}, \tag{25}$$

where δ is sufficiently small so that $u\gg\delta\bar{u}$. Then, Eq. (24) can be rewritten in terms of u and $\bar{u}$ as

$$\begin{aligned}\dot{u}=&\mathrm{i}Yu+\mathrm{i}Y\delta\bar{u}+\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H\left(t'\right)\mathrm{d}t'}\left(u+\bar{u}\right)^2\\ &\times[\left(1+3\bar{\delta}\right)u+(1+3\delta)\bar{u}]+\frac{\dot{\Omega}-\frac{3\mathrm{i}}{2}\dot{H}}{2\Omega}\\ &\times(\bar{u}+\bar{\delta}u)-\dot{\delta}\bar{u}-\delta\dot{\bar{u}}.\end{aligned} \tag{26}$$

where $Y=\Omega-\frac{3}{2}\dot{H}/(2\Omega)$. Using the complex conjugate of the above equation:

$$\begin{aligned}\dot{\bar{u}}=&-\mathrm{i}Y\bar{u}-\mathrm{i}Y\bar{\delta}u-\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H\left(t'\right)\mathrm{d}t'}\left(u+\bar{u}\right)^2\\ &\times\left[(1+3\delta)\bar{u}+\left(1+3\bar{\delta}\right)u\right]\\ &+\frac{\dot{\Omega}+\frac{3\mathrm{i}}{2}\dot{H}}{2\Omega}\left(u+\delta\bar{u}\right)-\dot{\bar{\delta}}u-\bar{\delta}\dot{u},\end{aligned} \tag{27}$$

we can eliminate $\bar{u}$ in Eq. (26). Then, Eq. (26) becomes

$$\begin{aligned}\dot{u}=&\mathrm{i}Yu+\mathrm{i}Y\delta\bar{u}+\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H\left(t'\right)\mathrm{d}t'}\left(u+\bar{u}\right)^2\\ &\times\left[\left(1+3\bar{\delta}\right)u+(1+3\delta)\bar{u}\right]+\frac{\dot{\Omega}-\frac{3\mathrm{i}}{2}\dot{H}}{2\Omega}\left(\bar{u}+\bar{\delta}u\right)\\ &-\dot{\delta}\bar{u}+\mathrm{i}Y\delta\bar{u}+\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H\left(t'\right)\mathrm{d}t'}\delta\left(u+\bar{u}\right)^2\\ &\times\left[(1+3\delta)\bar{u}+\left(1+3\bar{\delta}\right)u\right]-\frac{\dot{\Omega}+\frac{3\mathrm{i}}{2}\dot{H}}{2\Omega}\delta u.\end{aligned} \tag{28}$$

Now, for the case that δ and its complex conjugate have the form

$$\delta=\frac{\frac{3}{2}\dot{H}+\mathrm{i}\dot{\Omega}}{4\Omega^2}\quad \bar{\delta}=\frac{\frac{3}{2}\dot{H}-\mathrm{i}\dot{\Omega}}{4\Omega^2}, \tag{29}$$

Eq. (28) becomes

$$\begin{aligned}\dot{u}=&\mathrm{i}Yu+\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H\left(t'\right)\mathrm{d}t'}\left(u+\bar{u}\right)^2\\ &\times\left[\left(1+\delta+3\bar{\delta}\right)u+(1+4\delta)\bar{u}\right].\end{aligned} \tag{30}$$

For further study of the nonlinear correction, it is necessary to introduce a second change of the variable, which is near to identity, such that [10]

$$u=v+\alpha v^3+\rho v\bar{v}^2+\gamma\bar{v}^3, \tag{31}$$

where α, ρ, and γ are time-dependent coefficients that are real. By inserting this in Eq. (30), the equation for the new variable v can be obtained as

$$\begin{aligned}&(1+3\alpha v^2+\rho\bar{v}^2)\,\dot{v}\\ =&\mathrm{i}Y(v+\alpha v^3+\rho v\bar{v}^2+\gamma\bar{v}^3)\\ &+\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H\left(t'\right)\mathrm{d}t'}\left(v+\bar{v}\right)^2[(1+\delta+3\bar{\delta})(v\\ &+\alpha v^3+\rho v\bar{v}^2+\gamma\bar{v}^3)+(1+4\delta)\left(\bar{v}+\alpha\bar{v}^3\right.\\ &\left.+\rho\bar{v}v^2+\gamma v^3\right)]-(2\rho v+3\gamma\bar{v}^2)\,\dot{\bar{v}}-\dot{\alpha}v^3\\ &-\dot{\rho}v\bar{v}^2-\dot{\gamma}\bar{v}^3.\end{aligned} \tag{32}$$

Using the complex conjugate of this equation:

$$\begin{aligned}&\dot{\bar{v}}\left(1+3\alpha\bar{v}^2+\rho v^2\right)\\ =&-\mathrm{i}Y(\bar{v}+\alpha\bar{v}^3\\ &+\rho\bar{v}v^2+\gamma v^3)-\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H\left(t'\right)\mathrm{d}t'}\left(v+\bar{v}\right)^2\\ &\times\left[\left(1+\bar{\delta}+3\delta\right)\left(\bar{v}+\alpha\bar{v}^3+\rho\bar{v}v^2+\gamma v^3\right)\right.\\ &\left.+\left(1+4\bar{\delta}\right)\left(v+\alpha v^3+\rho v\bar{v}^2+\gamma\bar{v}^3\right)\right]\\ &-\dot{v}\left(2\rho\bar{v}+3\gamma v^2\right)-\dot{\alpha}\bar{v}^3-\dot{\rho}\bar{v}v^2-\dot{\gamma}v^3,\end{aligned} \tag{33}$$

we can eliminate $\ddot{v}$ in Eq. (32). Then. Eq. (32) yields

$$\begin{aligned}&\dot{v}+\dot{\alpha}v^3+\dot{\rho}v\bar{v}^2+\dot{\gamma}\bar{v}^3\\&=\mathrm{i}Yv+\frac{3\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H(t')\mathrm{d}t'}\left[1+2\left(\delta+\bar{\delta}\right)\right]v^2\bar{v}\\&\quad+\left[2\alpha\mathrm{i}Y+\mathrm{e}^{-3\int H(t')\mathrm{d}t'}\frac{\mathrm{i}\beta^2}{\Omega^2}\left(1+\delta+3\bar{\delta}\right)\right]v^3\\&\quad+\left[2\mathrm{i}\rho Y+\mathrm{e}^{-3\int H(t')\mathrm{d}t'}\frac{3\mathrm{i}\beta^2}{\Omega^2}\left(1+3\delta+\bar{\delta}\right)\right]v\bar{v}^2\\&\quad+\left[4\mathrm{i}\gamma Y+\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H(t')\mathrm{d}t'}\left(1+4\delta\right)\right]\bar{v}^3. \qquad (34)\end{aligned}$$

Hence, we can confirm that, according to the nonlinear change of variable u [Eq. (31)], the expression $\dot{u}$ given in Eq. (30) can be converted in terms of the new variable v as Eq. (34), where the time derivatives of α, ρ, and γ hold the equations

$$\dot{\alpha}=2\alpha\mathrm{i}Y+\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H(t')\mathrm{d}t'}\left(1+\delta+3\bar{\delta}\right), \qquad (35)$$

$$\dot{\rho}=2\mathrm{i}\rho Y+\frac{3\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H(t')\mathrm{d}t'}\left(1+3\delta+\bar{\delta}\right), \qquad (36)$$

$$\dot{\gamma}=4\mathrm{i}\gamma Y+\frac{\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H(t')\mathrm{d}t'}\left(1+4\delta\right). \qquad (37)$$

Then, Eq. (34) becomes

$$\dot{v}=\mathrm{i}Yv+\frac{3\mathrm{i}\beta^2}{\Omega^2}\mathrm{e}^{-3\int H(t')\mathrm{d}t'}\left[1+2\left(\delta+\bar{\delta}\right)\right]v^2\bar{v}. \qquad (38)$$

This formula will be used in the next section to investigate phase properties of the YM field and the corresponding Hannay's angle.

4 Phase properties: Hannay's angle

Phase properties of the adiabatically evolving YM field with time-dependent paramaters of the Hamiltonian will be investigated in this section. The discussion of the non-vanishing Hannay's angle in the evolution of the YM field is necessary from a theoretical point of view mainly because of its gauge-theoretical structure [2, 7]. By setting

$$v=A\mathrm{e}^{\frac{3}{2}\int H(t')\mathrm{d}t'}\mathrm{e}^{\mathrm{i}\Theta}, \qquad (39)$$

we can confirm from the use of Eq. (38) that the equations for the amplitude A and the angle Θ are read

$$\dot{A}+\frac{3}{2}HA+\mathrm{i}A\dot{\Theta}=\mathrm{i}YA+\frac{3\mathrm{i}\beta^2}{\Omega^2}\left[1+2\left(\delta+\bar{\delta}\right)\right]A^3. \qquad (40)$$

By adding this equation with its complex conjugate, which is

$$\dot{A}+\frac{3}{2}HA-\mathrm{i}A\dot{\Theta}=-\mathrm{i}YA-\frac{3\mathrm{i}\beta^2}{\Omega^2}\left[1+2\left(\delta+\bar{\delta}\right)\right]A^3, \qquad (41)$$

we have

$$\dot{A}+3HA=0. \qquad (42)$$

The subtraction of Eq. (41) from Eq. (40) gives

$$\dot{\Theta}=Y+\frac{3\beta^2}{\Omega^2}\left[1+2\left(\delta+\bar{\delta}\right)\right]A^2. \qquad (43)$$

It is possible to obtain an explicit form of $A(t)$ from Eq. (42) provided that $H(t)$ is known. Consequently, if $A(t)$ is known, we can also obtain Θ from Eq. (43). In this way, we can identify a complete solution of v given in Eq. (39).

Because

$$\delta+\bar{\delta}=\frac{\frac{3}{2}\dot{H}}{2\Omega^2}, \qquad (44)$$

the derivative of the angle given in Eq. (43) can be rewritten as

$$\dot{\Theta}=\Omega\left(1+\frac{3\beta^2}{\Omega^3}A^2\right)-\frac{3\dot{H}}{4\Omega}\left(1+\frac{6\beta^2}{\Omega^3}A^2\right). \qquad (45)$$

If we consider the fact that Eq. (42) implies that A is given by $A=A_0\mathrm{e}^{3\int_0^t H\mathrm{d}t'}$, this equation becomes

$$\dot{\Theta}=\Omega-\frac{3\dot{H}}{4\Omega}+\frac{3\beta^2}{\Omega^2}\left(1-\frac{3\dot{H}}{2\Omega^2}\right)A_0^2\mathrm{e}^{6\int_0^t H\mathrm{d}t'}. \qquad (46)$$

In particular, for the limit that t is sufficiently small, we can express $\dot{\Theta}$, under the assumption of the adiabatic change of the parameters H and β, in the form

$$\dot{\Theta}=\Omega\left(1+\frac{3\tilde{\beta}^2}{\Omega^3}A_0^2\right)-\frac{3\dot{H}}{4\Omega}\left(1+\frac{6\tilde{\beta}^2}{\Omega^3}A_0^2\right), \qquad (47)$$

where $\tilde{\beta}$ is a renormalized parameter of the form $\tilde{\beta}=\beta\left(1+3\int_0^t H\mathrm{d}t'\right)$. The second term in Eq. (47) is the time-dependent geometric one, i.e., Hannay's part of $\dot{\Theta}$, whereas the first term is the dynamical part. Thus, Hannay's angle is obtained to be

$$\Theta_{\mathrm{H}}(t)=-\int_0^t\frac{3\dot{H}}{4\Omega}\left(1+\frac{6\tilde{\beta}^2}{\Omega^3}A_0^2\right)\mathrm{d}t+\Theta_{\mathrm{H}}(0), \qquad (48)$$

where $\Theta_{\mathrm{H}}(0)$ is the initial Hannay's angle. If we think that it is difficult to know the amount of $\Theta_{\mathrm{H}}(0)$, the difference in Hannay's angle between arbitrary two times t_0 and t ($t_0<t$) may be more useful:

$$\Delta\Theta_{\mathrm{H}}(t)=-\int_{t_0}^t\frac{3\dot{H}}{4\Omega}\left(1+\frac{6\tilde{\beta}^2}{\Omega^3}A_0^2\right)\mathrm{d}t. \qquad (49)$$

The appearance of this angle holonomy is of purely geometrical origin. Figures 1-4 represent simple time evolutions of Eq. (49) and their time derivatives where we have taken $H = \dot{a}(t)/a(t)$ and $a(t) = (t/a_0)^{a_1}$. Here, a_0 and a_1 are constants. In these figures, Hannay's angle slowly increases at first, but its increment becomes gradually faster as time goes by. This is an extra angle shift emerging via the angle variables in a classical system along adiabatic or non-adiabatic changes of the parameters through a closed path in the parameter space. Figure 1 shows that the accumulation of Hannay's angle over time becomes small as a_0 increases. On the other hand, from Figs. 2-4, we can conclude that the accumulation of Hannay's angle becomes large as the parameters such as a_1, α, and β increase. From the time derivatives of Hannay's angle in the figures, we can confirm the exact ratio of the increment of geometric angle per unit time.

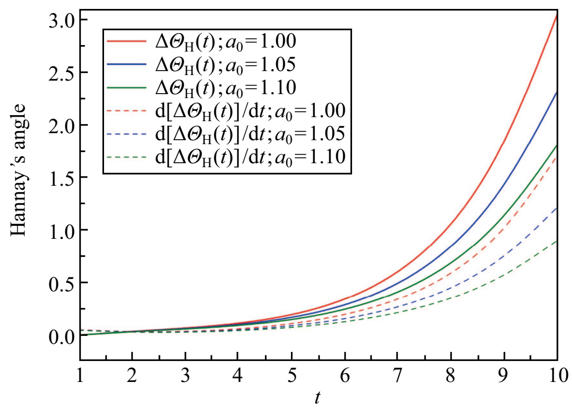

Fig. 1. (color online) Hannay's angle shifts for several different values of a_0 and their time derivatives represented as a function of t. This is plotted on the basis of numerical evaluation of Eq. (49) with the choice of $t_0 = 1$, where all values are taken to be dimensionless for the sake of convenience (This convention will also be applied to all the subsequent figures.). The values we used here are $a_1 = 1$, $A_0 = 1$, $\alpha = 1$, and $\beta = 1$.

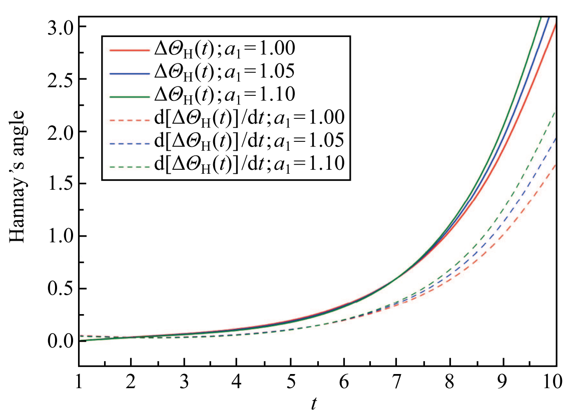

Fig. 2. (color online) Hannay's angle shifts for several different values of a_1 and their time derivatives represented as a function of t. The values we used here are $a_0 = 10$, $A_0 = 1$, $\alpha = 1$, and $\beta = 1$.

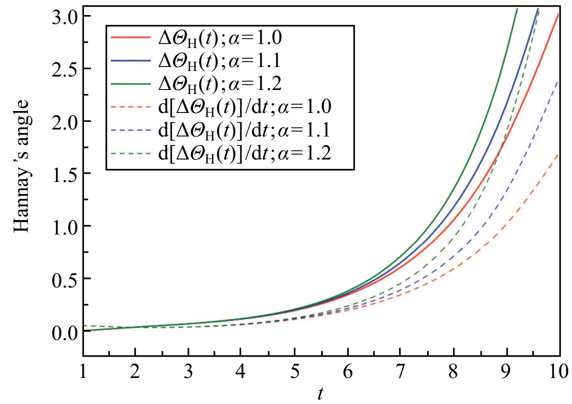

Fig. 3. (color online) Hannay's angle shifts for several different values of α and their time derivatives represented as a function of t. The values we used here are $a_0 = 10$, $a_1 = 1$, $A_0 = 1$, and $\beta = 1$.

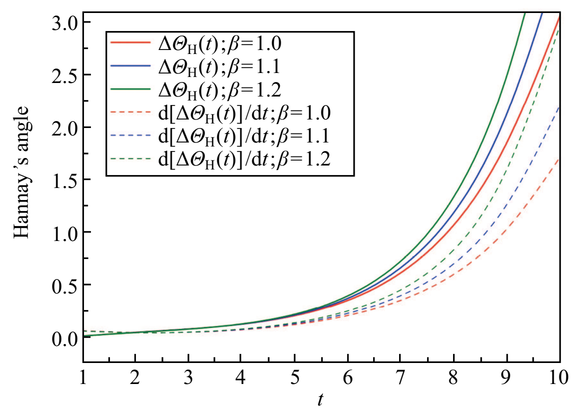

Fig. 4. (color online) Hannay's angle shifts for several different values of β and their time derivatives represented as a function of t. The values we used here are $a_0 = 10$, $a_1 = 1$, $A_0 = 1$, and $\alpha = 1$.

Through a comparative study via the first order approximation of perturbation theory [10], we can easily confirm that the damped quartic oscillator is canonically equivalent to this generalized quartic oscillator with a renormalized parameter $\tilde{\beta}$. Hence, both systems have an identical Hannay's angle. The Hannay's angle shift given in Eq. (49) provides fertile ground for applications in analyzing the evolution of the Universe.

Because the amount of phase shift originating from Hannay's angle is in general very small compared to the dynamical phase shift, the detection of Hannay's angle may be a somewhat cumbersome task and requires high precision measurements or observations [35]. A simple observation of Hannay's angle is possible for an elliptically orbiting object in celestial mechanics by averaging initial angles of the motion over the torus [35–38]. Such measurability of Hannay's angle can be extended to more general and complicated systems, such as circular three-body gravitational systems [24, 25, 39], non-Abellian gauge fields systems [40], interacting many-particle boson systems [41], and systems undergoing noncyclic evolution [42]. In cosmology associated with the theory developed here, the evolution of Hannay's angle given in Eq. (49) is represented in terms of observable parameters. One can claim that the observation of Hannay's angle can serve as a demonstration of the validity of a given cosmological theory. This is the physical significance of analyzing the cosmological Hannay's angle regarding its theoretical and observational aspects.

5 Conclusion

The time behavior of YM cosmology has been investigated under the adiabatic evolution of time-dependent parameters. Novel features of phase transition in the early epoch can be explained by YM condensates which accompany the description of dark energy responsible for the inflation of the Universe. A rigorous evaluation leads the equation for the YM field to that of a particle confined in the quartic potential. Although the usual YM action is conformally invariant, YM fields associated with a vector field for inflation violate conformal invariance [3]. The phase properties of the system were analyzed and Hannay's angle was derived. Hannay's angle, given in Eq. (48), is equivalent to that of a damped quartic oscillator. We can confirm that the system acquires a supplement angular shift as the system evolves in phase space according to the adiabatic theorem. The smooth dependence of the Hamiltonian associated with the YM field on time-dependent parameters is responsible for the emergence of this additional change of the angle. Hannay's angle under a semiclassical approximation exhibits a well-known natural relation with Berry's phase in a quantum system [13], which is necessary to understand the quantum characteristics of the system. Hannay's angle and Berry's phase play important roles in analyzing several conceptual problems relevant to anomalies and their related problems [17] and indeed can be used as a supplementary measure to probe the inflation that is typically introduced in inflationary cosmologies [16]. An indirect route for estimating scalar or tensor spectral indices and other observable parameters therefrom can be achieved by measuring Hannay's angle or Berry's phase of the cosmological perturbations. During slow roll inflation, the overall phase or angle accumulated by the adiabatic limit of each mode along sub-Hubble oscillations is a new parameter made of the spectral indices of the corresponding scalar or tensor.

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