Properties of the free energy density using the principle of maximum conformality *

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Abstract: We present a detailed study on the properties of the free energy density at high temperature by applying the principle of maximum conformality (PMC) scale-setting method within effective field theory. The PMC utilizes the renormalization group equation recursively to identify the occurrence and pattern of the non-conformal $\{\beta_i\}$-terms, and determines the optimal renormalization scale at each order. Our analysis shows that a more accurate free energy density up to g_s^5-order level without renormalization scale dependence can be achieved by applying the PMC. We also observe that by using a smaller factorization scale around the effective parameter m_E, the PMC prediction is consistent with the lattice QCD prediction derived at low temperature.

Keywords: perturbative QCD, free energy density, renormalization, QCD high-order corrections

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1 Introduction

At extremely high temperatures, hadronic matter is assumed to undergo a phase transition to quark-gluon plasma (QGP). The QGP might come from the early universe up to a few milliseconds after the Big Bang or from heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC), the Large Hadron Collider (LHC), etc. This system therefore behaves more like a collection of free quarks and gluons than a collection of their bound states [1].

The static equilibrium properties of the QGP at temperature T are governed by the free energy density [2]

$$F=-\frac{T}{V}\ln\mathcal{Z}_{\rm QCD}, \tag{1}$$

where V is the space volume and the partition function $\mathcal{Z}_{\rm QCD}$ is a functional integral over quark and gluon fields on a 4-dimensional Euclidean space-time, with the Euclidean time taking its values on a circle with circumference $1/T$. In the limit where the quarks are massless, the free energy density is a function of T and the strong coupling constant.

During the past decades, the free energy density, or equivalently the negative pressure, of the QGP has been calculated by using lattice gauge theory [3–23] or perturbative QCD (pQCD) theory [2, 24–31]. In the present paper, we shall focus on the situation where the QGP has a high temperature T (T is considered as a measure of the average energy of the constituents), indicating the quarks and gluons are of high energy and the strong couplings among them are small due to asymptotic freedom. Within this temperature region, pQCD is a feasible tool to study the free energy density. During the calculation, we shall resum specific diagrams such as the "ring diagrams" [2, 29], and it is helpful to expand the perturbative series by the coupling constants g_s rather than α_s. The free energy density at high temperature has been calculated up to $\mathcal{O}(g_s^2)$ [24], $\mathcal{O}(g_s^3)$ [25], $\mathcal{O}(g_s^4\ln(1/g_s))$ [26], $\mathcal{O}(g_s^4)$ [27, 28], $\mathcal{O}(g_s^5)$ [2, 29, 30], and part of $\mathcal{O}(g_s^6)$ [31], respectively. The $\mathcal{O}(g_s^0)$ term is the free energy density of the ideal gas. The $\mathcal{O}(g_s^2)$ and higher-order terms contain the corrections from the interactions among the basic particles, the screening effects from the plasma, etc. There are new nonperturbative effects entangled with the infrared divergence which emerge at the $\mathcal{O}(g_s^6)$-order [32–34], and at present, only the specific terms of the form $\mathcal{O}(g_s^6\ln g_s)$ have been achieved.

For a high-order pQCD prediction, one has to choose a renormalization scheme and a renormalization scale μ_r to finish the renormalization. The scale μ_r is usually taken as the typical momentum flow of the process or that to eliminate the large logs such as to make the pQCD series relatively steady over the scale changes. For the present case, one usually sets $\mu_r=2\pi T$, which corresponds to the energy of the first non-vanishing

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Matsubara mode [35]. However, such a simple choice of "guessed" scale leads to mis-matching of the perturbative coefficients to the strong coupling constant, resulting in the well-known scheme-and-scale ambiguities which persist at any fixed order [36–41]. By using the "guessed" scale, there are some other defects [42, 43], especially: I) the predictions for a guessed scale are incorrect for quantum electrodynamics, whose renormalization scale can be unambiguously set by the Gell-Mann–Low procedure [44]; II) the perturbative series is factorially divergent at large order – the renormalon problem [45, 46]; III) it is often argued that such scale uncertainties can be suppressed by including enough high-order terms, which however shall be diluted by the divergent renormalon terms; IV) if a poor pQCD convergence is observed for an observable, one cannot decide whether it is an intrinsic property of the pQCD series or is caused by an improper choice of scale.

Many attempts have been made to improve the prediction of the free energy density [47–75]. However, in those studies, the renormalization scale uncertainty is still quite large. In the paper, we will apply the principle of maximum conformality (PMC) [76–79] to the free energy density up to $\mathcal{O}(g_s^5)$ with the goal of eliminating the renormalization scale ambiguity and achieving an accurate pQCD prediction which is independent of theoretical conventions. Because the running behavior of the coupling constant is controlled by the renormalization group equation (RGE) or the β-function, the PMC suggests using the knowledge of the $\{\beta_i\}$-terms from the known pQCD series to determine the optimal scale of a particular process. A recent review on this point can be found in Ref. [80]. If one fixes the renormalization scale of the pQCD series using the PMC, all the non-conformal $\{\beta_i\}$-terms in the perturbative series shall be resummed into the running coupling. One thus obtains a unique, scale-fixed, and scheme-independent prediction at any fixed order. Many PMC applications have been done in the literature, cf. the review in Ref. [81]. All of those examples show that due to the rapid convergence of conformal pQCD series, the residual uncertainties are highly suppressed, even for low-order predictions.

There are several typical momentum flows for the free energy density up to $\mathcal{O}(g_s^6)$, e.g. T, g_sT, g_s^2T [82–84]. The QCD hard thermal loop effective field theory (EFT) [2, 30] provides a systematic way to unravel the contributions under different energy scales. The QCD EFT is a three-dimensional one in which all the quarks and non-static bosons have been integrated out of the theory such that it reduces to purely static bosonic modes [2, 30, 82–84]. The EFT factorizes the free energy density of the hot QCD into the perturbative coefficients and the non-perturbative parts via proper matching. The PMC can be applied separately to set the renormalization scale of the free energy density within different scale regions.

The rest of the paper is organized as follows. In Section **2**, we present the calculation technique for achieving the PMC prediction on the free energy density. Numerical results and discussions are presented in Section **3**. Section **4** gives a summary.

2 Calculation technology

Using the EFT, the free energy density can be decomposed into various parts which are characterized by typical scales as T, g_sT and g_s^2T, and they are labeled as F_E, F_M, and F_G, respectively. Here the hard part F_E can be treated as a power series in $\alpha_s=g_s^2/4\pi$, the softer part F_M is a power series in g_s which begins at the g_s^3 order, and the softest part F_G is a power series in g_s which begins at the g_s^6 order. At present, the complete g_s^6-order terms are not known, so we shall concentrate our attention on the free energy density up to g_s^5 order.

Up to g_s^5 order, the free energy density F can be formulated as [2, 30],

$$F=F_E(\Lambda_E)+F_M(\Lambda_E), \tag{2}$$

where Λ_E is the factorization scale. The hard part $F_E(\Lambda_E)$ can be expressed as

$$F_E(\Lambda_E)=F_{\rm ideal}+\frac{8\pi^2}{3}T^4\widetilde{F}_E(\Lambda_E), \tag{3}$$

where $F_{\rm ideal}$ stands for the contribution of the ideal quark-gluon gas

$$F_{\rm ideal}=-\frac{8\pi^2}{45}T^4\left(1+\frac{21}{32}n_f\right), \tag{4}$$

and $\widetilde{F}_E(\Lambda_E)$ represents the "canonical" QCD part,

$$\widetilde{F}_E(\Lambda_E)=F'_E-\left[144\left(1+\frac{1}{6}n_f\right)\ln\frac{\Lambda_E}{2\pi T}\right]a_s^2(\mu_r), \tag{5}$$

where μ_r is the (arbitrary) renormalization scale. The remaining part F'_E is perturbatively calculable, which can be expressed as

$$F'_E=r^E_{1,0}a_s(\mu_r)+\left(r^E_{2,0}+r^E_{2,1}\beta_0\right)a_s^2(\mu_r), \tag{6}$$

where $a_s=\alpha_s/4\pi$, $\beta_0=11–2/3n_f$ with n_f being the number of active flavors which emerge in the α_s-renormalization. As required by the PMC, we have transformed those n_f-terms into the $\{\beta_i\}$-series. The conformal coefficients $r^E_{i,j(=0)}$ and the nonconformal ones $r^E_{i,j(\neq0)}$ under the $\overline{\rm MS}$-scheme read

$$r^E_{1,0}\ =\ 1+\frac{5}{12}n_f, \tag{7}$$

$$r^E_{2,0}\ =\ -214.54-29.15\left(1+\frac{5}{12}n_f\right), \tag{8}$$

$$r^{\rm E}_{2,1} = \left(-1.59+2\ln\frac{\mu_r}{2\pi T}\right)\left(1+\frac{5}{12}n_{\rm f}\right). \quad (9)$$

Here the $n_{\rm f}$-terms in those coefficients $r_{i,j}$ are free quark numbers in QGP, which are irrelevant to the running of the coupling constant and should be kept as conformal coefficients when applying the PMC [76–79].

After applying the PMC, the pQCD series of $F'_{\rm E}$, e.g. Eq. (6), can be improved as the following scheme-independent conformal series,

$$F'_{\rm E}=r^{\rm E}_{1,0}a_{\rm s}(Q^e_1)+r^{\rm E}_{2,0}a^2_{\rm s}(Q^e_1), \quad (10)$$

where $\ln(Q^e_1)^2/\mu^2_r = -r^{\rm E}_{2,1}/r^{\rm E}_{1,0}$. We have set the NLO PMC scale $Q^e_2=Q^e_1$ to ensure the scheme independence. Its exact value can be determined by using the NNLO terms, which are not available at present.

The softer part $F_{\rm M}(\Lambda_{\rm E})$ can be expressed by using the EFT parameters, $m^2_{\rm E}$ and $g^2_{\rm E}$, as [2, 30]

$$F_{\rm M}(\Lambda_{\rm E}) = -\frac{2T}{3\pi}m^3_{\rm E}\left[1-\left(0.256+\frac{9}{2}\ln\frac{\Lambda_{\rm E}}{m_{\rm E}}\right)\frac{g^2_{\rm E}}{2\pi m_{\rm E}} -27.6\left(\frac{g^2_{\rm E}}{2\pi m_{\rm E}}\right)^2\right]. \quad (11)$$

In deriving $F_{\rm M}(\Lambda_{\rm E})$, the RGE-involved fermion-loop contributions have been incorporated into the EFT parameters $m^2_{\rm E}$, $g^2_{\rm E}$, etc. [2, 30, 82–84], so it is better to apply the PMC directly to those parameters to get more accurate predictions of those EFT parameters and to avoid the double counting problem[1)].

To be consistent with the known $g^5_{\rm s}$-order prediction for the free energy density, we need to know $m^2_{\rm E}$ up to the next-to-leading order (NLO) level and $g^2_{\rm E}$ to the leading-order (LO) level. The $m^2_{\rm E}$ up to NLO level [2, 30] can be written as

$$m^2_{\rm E}=16\pi^2T^2\left[r^m_{1,0}a_{\rm s}(\mu_r)+\left(r^m_{2,0}+r^m_{2,1}\beta_0\right)a^2_{\rm s}(\mu_r)\right], \quad (12)$$

where $r^m_{i,j}$ under the $\overline{\rm MS}$-scheme read

$$r^m_{1,0} = 1+\frac{1}{6}n_{\rm f}, \quad (13)$$

$$r^m_{2,0} = 8-22.50\left(1+\frac{1}{6}n_{\rm f}\right), \quad (14)$$

$$r^m_{2,1} = \left(1.54+2\ln\frac{\mu_r}{2\pi T}\right)\left(1+\frac{1}{6}n_{\rm f}\right). \quad (15)$$

Here the $n_{\rm f}$-terms in those coefficients $r_{i,j}$ are again free quark numbers in QGP. After applying the PMC scale-setting, we obtain

$$m^2_{\rm E}=16\pi^2T^2\left[r^m_{1,0}a_{\rm s}(Q^m_1)+r^m_{2,0}a^2_{\rm s}(Q^m_1)\right], \quad (16)$$

where $\ln(Q^m_1)^2/\mu^2_r=-r^m_{2,1}/r^m_{1,0}$.

By using the LO $g^2_{\rm E}$ alone, we cannot determine its renormalization scale, and to ensure the scheme independence of $F_{\rm M}(\Lambda_{\rm E})$ at the $g^5_{\rm s}$-order level, we directly set its value as $Q^g_1=Q^m_1$.

By using the known NLO-terms for $g^2_{\rm E}$, we can determine its optimal scale by applying the PMC in the same way. For example, by using the computed $g^6_{\rm s}$-order terms from Ref. [31], we obtain $\ln(Q^g_1)^2/\mu^2_r=-r^g_{2,1}/r^g_{1,0}$ with the coefficients

$$r^g_{1,0} = 1, \quad (17)$$

$$r^g_{2,0} = -29.5, \quad (18)$$

$$r^g_{2,1} = 2.54+2\ln\frac{\mu_r}{2\pi T} \quad (19)$$

and

$$g^2_{\rm E}=16\pi^2T\left[r^g_{1,0}a_{\rm s}(Q^g_1)+r^g_{2,0}a^2_{\rm s}(Q^g_1)\right]. \quad (20)$$

As a summary, our final prediction for the free energy density F with the factorization scale $\Lambda_{\rm E}=2\pi T$ is

$$F = F_{\rm ideal}+\frac{8\pi^2}{3}T^4\left[r^{\rm E}_{1,0}a_{\rm s}(Q_1)+r^{\rm E}_{2,0}a^2_{\rm s}(Q_1)\right] -\frac{2T}{3\pi}m^3_{\rm E}\left[1-\left(0.256+\frac{9}{2}\ln\frac{2\pi T}{m_{\rm E}}\right)\frac{g^2_{\rm E}}{2\pi m_{\rm E}} -27.6\left(\frac{g^2_{\rm E}}{2\pi m_{\rm E}}\right)^2\right]. \quad (21)$$

If choosing the factorization scale $\Lambda_{\rm E}=m_{\rm E}$, we obtain

$$F = F_{\rm ideal}+\frac{8\pi^2}{3}T^4\left[r^{\rm E}_{1,0}a_{\rm s}(Q_1)+r^{\rm E}_{2,0}a^2_{\rm s}(Q_1)\right] -\frac{8\pi^2}{3}T^4\left[144\left(1+\frac{1}{6}n_{\rm f}\right)\ln\frac{m_{\rm E}}{2\pi T}\right]a^2_{\rm s}(Q_1) -\frac{2T}{3\pi}m^3_{\rm E}\left[1-0.256\frac{g^2_{\rm E}}{2\pi m_{\rm E}}-27.6\left(\frac{g^2_{\rm E}}{2\pi m_{\rm E}}\right)^2\right]. \quad (22)$$

3 Numerical results and discussion

To do the numerical calculation, we adopt the value $\alpha_{\rm s}(1.5\ {\rm GeV}, n_{\rm f}=3)=0.336^{+0.012}_{-0.008}$ [85] as a reference point to determine the QCD asymptotic scale $\Lambda_{\overline{\rm MS}}$. By using the two-loop $\alpha_{\rm s}$-running formulae, we obtain $\Lambda^{n_{\rm f}=3}_{\overline{\rm MS}}= 0.343^{+0.018}_{-0.012}$ GeV. If not specially stated, we shall adopt $\Lambda_{\rm E}=2\pi T$ as the default value for the factorization scale. In the following, we shall set the temperature $T=10$ GeV as an example to show the basic properties of the free energy density.

Firstly, we discuss the properties of the hard part ($F_{\rm E}$) of the free energy density, which is characterized by the scale around T. We present the renormalization scale dependence of the ratio $F_{\rm E}/F_{\rm ideal}$ before and after applying the PMC in Fig. 1. We present the numerical results for the ratio $F_{\rm E}/F_{\rm ideal}$ under several typical

1) The PMC resums the $\{\beta_i\}$-terms into the running coupling and the pQCD series of the free energy density also includes partial resummation effects [27–29]. A straightforward application of PMC to the pQCD series might contain a mixing of inequivalent resummations, leading to a double counting problem.

choices of renormalization scale, $\mu_r=\pi T$, $2\pi T$ and $4\pi T$, in Table 1. After applying the PMC, $F_{\rm E}$ is independent of the choice of μ_r, while the NNLO prediction under conventional scale-setting still shows a strong scale dependence. For example, Table 1 shows that $F_{\rm E}/F_{\rm ideal}$ varies by $[-2\%,+4\%]$ for $\mu_r\in[\pi T,4\pi T]$. It is interesting to find that the typical momentum flow of $F_{\rm E}$ should be $\simeq 4\pi T$, at which the PMC and conventional scale-settings give almost the same prediction, which is about twice as high as the usually considered $2\pi T$. This condition is similar to the observation that the preferable choice of the renormalization scale for $gg\to H$ or $H\to gg$ is $m_{\rm H}/4$ [86] and the preferable one for $H\to\gamma\gamma$ is $2m_{\rm H}$ [87], different from the usually considered $m_{\rm H}$. The typical momentum flow under conventional scale-setting is usually approximated by eliminating the large log-terms of the perturbative series, while the PMC scale-setting provides a reliable way to set the exact value for the typical momentum flow for high-energy processes.

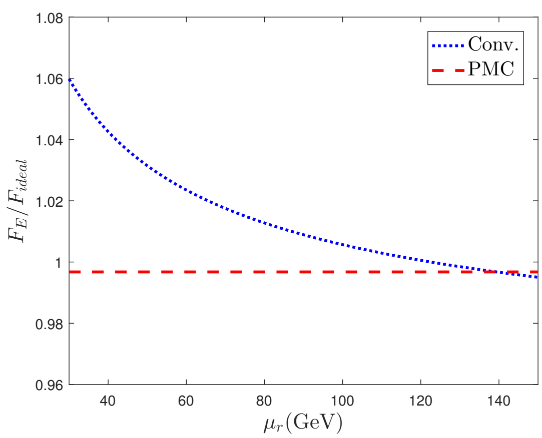

Fig. 1. (color online) The ratio $F_{\rm E}/F_{\rm ideal}$ versus the renormalization scale (μ_r) under conventional (Conv.) and PMC scale-settings. T=10 GeV.

Table 1. The ratio $F_{\rm E}/F_{\rm ideal}$ under conventional (Conv.) and PMC scale-settings. Three typical renormalization scales, $\mu_r=\pi T$, $2\pi T$ and $4\pi T$, are adopted. T=10 GeV.

$F_{\rm E}/F_{\rm ideal}$	μ_r	LO	NLO	NNLO	total
	πT	1	−0.11	0.17	1.06
Conv.	$2\pi T$	1	−0.10	0.12	1.02
	$4\pi T$	1	−0.09	0.09	1.00
PMC	$[\pi T,4\pi T]$	1	−0.09	0.08	0.99

Secondly, we consider the properties of the softer part ($F_{\rm M}$) of the free energy density, which is characterized by a softer scale around $g_{\rm s}T$.

To show how the scale uncertainty of the EFT parameters such as $m_{\rm E}^2$ change, we vary the renormalization scale from $g_{\rm s}T(\sim m_{\rm E})$ to approximately $2\pi T$. We present the scale dependence of $m_{\rm E}^2$ before and after applying the PMC in Fig. 2. It shows that the PMC prediction for $m_{\rm E}^2$ is independent of the choice of μ_r, whose value under conventional scale-setting shows a non-negligible scale dependence *. We present the scale dependence of $m_{\rm E}^2$ by using three typical scales $\pi T/2$, πT and $2\pi T$ in Table 2. Under conventional scale-setting, $m_{\rm E}^2$ varies by $[-7\%,+2\%]$ when $\mu_r\in[\pi T/2,2\pi T]$.

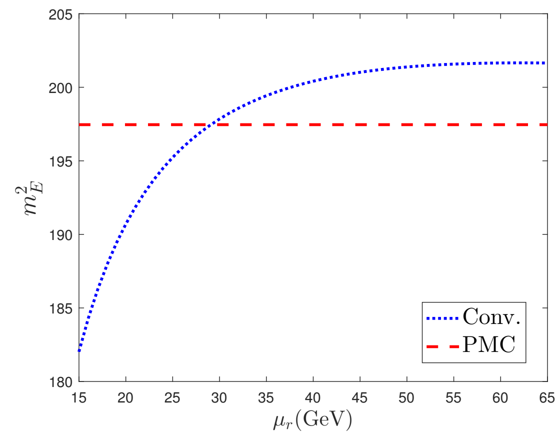

Fig. 2. (color online) The EFT parameter $m_{\rm E}^2$ versus the renormalization scale (μ_r) under conventional and PMC scale-settings. T=10 GeV.

Table 2. The results of $m_{\rm E}^2$ under conventional and PMC scale-settings. Three typical scales, $\mu_r=\pi T/2$, πT, and $2\pi T$, are adopted. $T=10$ GeV.

$m_{\rm E}^2$	μ_r	LO	NLO	total
	$\pi T/2$	271.77	−88.09	183.68
Conv.	πT	235.22	−36.85	198.37
	$2\pi T$	207.66	−6.00	201.66
PMC	$[\pi T/2,2\pi T]$	238.78	−41.32	197.46

As mentioned in Section 2, for a $\mathcal{O}(g_{\rm s}^5)$-order prediction on the free energy density, we only need a LO $g_{\rm E}^2$. However, by using the LO $g_{\rm E}^2$ alone, we cannot determine its renormalization scale. To achieve a more accurate prediction for $g_{\rm E}^2$ itself, we adopt the known NLO-terms [31] to set the scale for $m_{\rm E}^2$. The scale dependence of $g_{\rm E}^2$ up to NLO level before and after applying the PMC scale-setting is presented in Fig. 3, which shows the scale dependence can be eliminated by applying the PMC. Numerical results for $g_{\rm E}^2$ under three typical scales $\pi T/4$, $\pi T/2$ and πT are presented in Table 3. It shows that $g_{\rm E}^2$

*A similar discussion on the EFT parameter has been done by using the prototype of PMC, i.e. the Brodsky-Lepage-Mackenzie (BLM) scale-setting [41], and our corresponding PMC scales are consistent with the BLM predictions [2, 30].

under conventional scale-setting varies by [−26%,+10%] for $\mu_r\in[\pi T/4,\pi T]$.

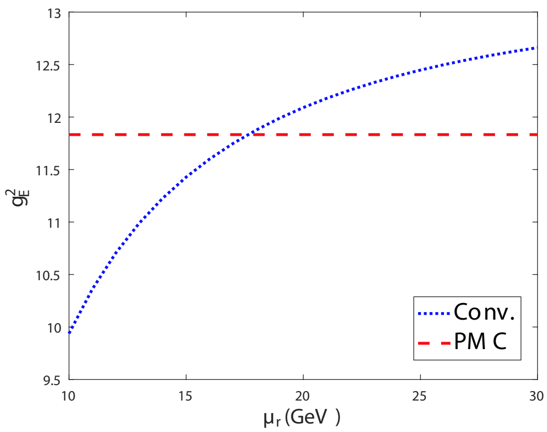

Fig. 3. (color online) The EFT parameter g_E^2 versus the renormalization scale (μ_r) under conventional and PMC scale-settings. T=10 GeV.

Table 3. The results of g_E^2 under conventional and PMC scale-settings. Three typical scales, $\mu_r=\pi T/4$, $\pi T/2$, and πT, are adopted. T=10 GeV.

g_E^2	μ_r	LO	NLO	total
	$\pi T/4$	21.53	−12.94	8.59
Conv.	$\pi T/2$	18.12	−6.57	11.55
	πT	15.68	−2.98	12.70
PMC	$[\pi T/4,\pi T]$	17.65	−5.82	11.83

Table 4. The ratio $F_M/F_{\rm ideal}$ under conventional and PMC scale-settings. Four typical scales, $\mu_r=\pi T/4$, $\pi T/2$, πT and $2\pi T$, are adopted. The LO g_E^2 with $Q_1^g=0.93\pi T$ (PMC-I) or $Q_1^g=0.56\pi T$ (PMC-II) is adopted for a $\mathcal{O}(g_s^5)$-order prediction. T=10 GeV.

$F_M/F_{\rm ideal}$	μ_r	LO	NLO	NNLO	total
	$\pi T/4$	0.07	−0.15	−0.16	−0.24
	$\pi T/2$	0.10	−0.15	−0.13	−0.18
Conv.	πT	0.12	−0.14	−0.10	−0.12
	$2\pi T$	0.12	−0.13	−0.08	−0.09
PMC-I	$[1/4\pi T,2\pi T]$	0.11	−0.14	−0.10	−0.13
PMC-II	$[1/4\pi T,2\pi T]$	0.11	−0.16	−0.12	−0.17

As a summary, by substituting the EFT parameters m_E^2 and g_E^2 into Eq. (11), we obtain the PMC predictions for the ratio $F_M/F_{\rm ideal}$, which are presented in Table 4. Summing the F_E and F_M together, by taking $Q_1^g\equiv Q_1^m$=$0.93\pi T$ to calculate the LO g_E^2, we obtain

$$\left.\frac{F}{F_{\rm ideal}}\right|_{T=10\ {\rm GeV}}=0.866^{+0.003}_{-0.002}. \tag{23}$$

where the uncertainty is for $\Delta\alpha_s(1.5\ {\rm GeV})=\binom{+0.012}{-0.008}$. If we take Q_1^m=$0.56\pi T$, determined from the known NLO g_E^2-term, to calculate the LO g_E^2, we obtain

$$\left.\frac{F}{F_{\rm ideal}}\right|_{T=10\ {\rm GeV}}=0.827^{+0.004}_{-0.003}. \tag{24}$$

4 Summary

In the paper, we have studied the properties of the free energy density up to g_s^5 order at high temperature T by applying the PMC within the EFT framework. The PMC provides a systematic method to set the renormalization scale of the high-energy process. Its predictions are free of renormalization scale dependence even for low-order predictions. As shown by Tables 1 and 4, our predictions for the free energy density up to g_s^5 order confirm this observation.

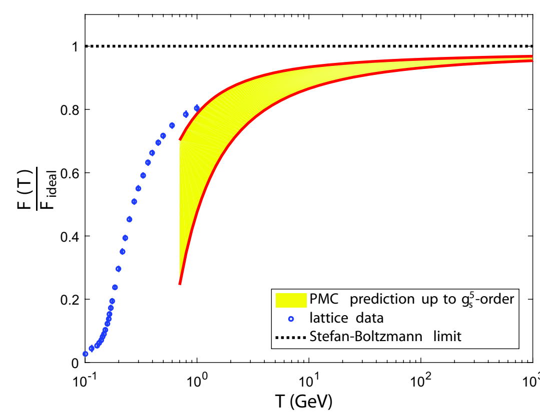

Fig. 4. (color online) The PMC prediction of the free energy density up to g_s^5 order versus the temperature T with free quark numbers in QGP $n_f=3$. The upper edge of the band corresponds to $\Lambda_E=m_E$ and the lower edge of the band corresponds to $\Lambda_E=2\pi T$. The lattice data with pion mass $m_\pi=160$ GeV [13] and the Stefan-Boltzmann limit of the ideal gas are presented as a comparison.

It is noted that the determination of the factorization scale is a completely separate issue from the renormalization scale setting problem, since it is presented even for a conformal theory with $\beta=0$. With the help of Eqs. (21) and (22), we present a prediction for the factorization scale dependence on the ratio $F/F_{\rm ideal}$ up to g_s^5 order as a function of T in Fig. 4. The factorization scale uncertainty is discussed by taking the range $g_sT\sim m_E<\Lambda_E<2\pi T$ [61]. The dashed line indicates the Stefan-Boltzmann limit of the ideal gas. The lattice data for the case $n_f=2+1$ [13] is adopted for a comparison. The upper edge of the band corresponds to Λ_E=m_E and the lower edge of the band corresponds to Λ_E=$2\pi T$. Figure 4 shows that when Λ_E=m_E, the free energy density agrees with the lattice data even for low temperature T around 1 GeV, indicating a smaller factorization scale is preferable.

References

1 K. Kajantie, M. Laine, K. Rummukainen, and Y. Schroder, Phys. Rev. Lett., **86**: 10 (2001)
2 E. Braaten and A. Nieto, Phys. Rev. D, **53**: 3421 (1996)
3 G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lutgemeier, and B. Petersson, Nucl. Phys. B, **469**: 419 (1996)
4 M. Okamoto et al (CP-PACS Collaboration), Phys. Rev. D, **60**: 094510 (1999)
5 C. Bernard et al (MILC Collaboration), Phys. Rev. D, **71**: 034504 (2005)
6 Y. Aoki, Z. Fodor, S. D. Katz, and K. K. Szabo, JHEP, **0601**: 089 (2006)
7 C. Bernard et al, Phys. Rev. D, **75**: 094505 (2007)
8 M. Cheng et al, Phys. Rev. D, **77**: 01451 (2008)
9 G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, PoS LATTICE, **2007**: 228 (2007)
10 F. Di Renzo, M. Laine, Y. Schroder, and C. Torrero, JHEP, **0809**: 061 (2008)
11 A. Hietanen, K. Kajantie, M. Laine, K. Rummukainen, and Y. Schroder, Phys. Rev. D, **79**: 045018 (2009)
12 A. Bazavov et al, Phys. Rev. D, **80**: 014504 (2009)
13 S. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo, JHEP, **1011**: 077 (2010)
14 S. Borsányi, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabó, PoS Lattice, **2010**: 171 (2014)
15 S. Borsanyi, G. Endrodi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo, JHEP, **1208**: 053 (2012)
16 S. Borsanyi, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, JHEP, **1207**: 056 (2012)
17 S. Borsanyi et al, JHEP, **1208**: 126 (2012)
18 A. Bazavov et al (HotQCD Collaboration), Phys. Rev. D, **86**: 034509 (2012)
19 A. Bazavov et al, Phys. Rev. Lett., **109**: 192302 (2012)
20 O. Philipsen, Prog. Part. Nucl. Phys., **70**: 55 (2013)
21 S. Borsanyi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo, Phys. Rev. Lett., **111**: 062005 (2013)
22 A. Bazavov et al, Phys. Rev. Lett., **111**: 082301 (2013)
23 U. Gursoy, Acta Phys. Polon. B, **47**: 2509 (2016)
24 E. V. Shuryak, Sov. Phys. JETP, **47**: 212 (1978)
25 J. I. Kapusta, Nucl. Phys. B, **148**: 461 (1979)
26 T. Toimela, Phys. Lett., **124B**: 407 (1983)
27 P. B. Arnold and C. X. Zhai, Phys. Rev. D, **50**: 7603 (1994)
28 P. B. Arnold and C. X. Zhai, Phys. Rev. D, **51**: 1906 (1995)
29 C. X. Zhai and B. M. Kastening, Phys. Rev. D, **52**: 7232 (1995)
30 E. Braaten and A. Nieto, Phys. Rev. Lett., **76**: 1417 (1996)
31 K. Kajantie, M. Laine, K. Rummukainen, and Y. Schroder, Phys. Rev. D, **67**: 105008 (2003)
32 A. D. Linde, Rept. Prog. Phys., **42**: 389 (1979)
33 A. D. Linde, Phys. Lett., **96B**: 289 (1980)
34 D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys., **53**: 43 (1981)
35 T. Matsubara, Prog. Theor. Phys., **14**: 351 (1955)
36 W. Celmaster and R. J. Gonsalves, Phys. Rev. D, **20**: 1420 (1979)
37 L. F. Abbott, Phys. Rev. Lett., **44**: 1569 (1980)
38 A. J. Buras, Rev. Mod. Phys., **52**: 199 (1980)
39 G. Grunberg, Phys. Lett., **95B**: 70 (1980)
40 P. M. Stevenson, Phys. Rev. D, **23**: 2916 (1981)
41 S. J. Brodsky, G. P. Lepage, and P. B. Mackenzie, Phys. Rev. D, **28**: 228 (1983)
42 X. G. Wu, S. J. Brodsky, and M. Mojaza, Prog. Part. Nucl. Phys., **72**: 44 (2013)
43 X. G. Wu, Y. Ma, S. Q. Wang, H. B. Fu, H. H. Ma, S. J. Brodsky, and M. Mojaza, Rept. Prog. Phys., **78** (2015) 126201.
44 M. Gell-Mann and F. E. Low, Phys. Rev., **95**: 1300 (1954)
45 M. Beneke, Phys. Rept., **317**: 1 (1999)
46 E. Gardi and G. Grunberg, Phys. Lett. B, **517**: 215 (2001)
47 F. Karsch, A. Patkos, and P. Petreczky, Phys. Lett. B, **401**: 69 (1997)
48 S. Chiku and T. Hatsuda, Phys. Rev. D, **58**: 076001 (1998)
49 J. O. Andersen, E. Braaten, and M. Strickland, Phys. Rev. D, **61**: 074016 (2000)
50 J. O. Andersen, E. Braaten, and M. Strickland, Phys. Rev. Lett., **83**: 2139 (1999)
51 J. O. Andersen, E. Braaten, and M. Strickland, Phys. Rev. D, **63**: 105008 (2001)
52 J. O. Andersen, L. E. Leganger, M. Strickland, and N. Su, Phys. Lett. B, **696**: 468 (2011)
53 J. O. Andersen, L. E. Leganger, M. Strickland, and N. Su, JHEP, **1108**: 053 (2011)
54 N. Haque, M. G. Mustafa, and M. Strickland, Phys. Rev. D, **87**: no. 10, 105007 (2013)
55 N. Haque, A. Bandyopadhyay, J. O. Andersen, M. G. Mustafa, M. Strickland, and N. Su, JHEP, **1405**: 027 (2014)
56 M. Strickland, J. O. Andersen, A. Bandyopadhyay, N. Haque, M. G. Mustafa, and N. Su, Nucl. Phys. A, **931**: 841 (2014)
57 N. Haque, PoS ICPAQGP, **2015**: 057 (2017)
58 B. M. Kastening, Phys. Rev. D, **56**: 8107 (1997)
59 T. Hatsuda, Phys. Rev. D, **56**: 8111 (1997)
60 G. Cvetic and R. Kogerler, Phys. Rev. D, **66**: 105009 (2002)
61 G. Cvetic and R. Kogerler, Phys. Rev. D, **70**: 114016 (2004)
62 T. Hatsuda and T. Kunihiro, Phys. Rept., **247**: 221 (1994)
63 K. Fukushima, Phys. Rev. D, **68**: 045004 (2003)
64 K. Fukushima, Phys. Lett. B, **591**: 277 (2004)
65 C. Ratti, M. A. Thaler, and W. Weise, Phys. Rev. D, **73**: 014019 (2006)
66 S. Mukherjee, M. G. Mustafa, and R. Ray, Phys. Rev. D, **75**: 094015 (2007)
67 A. Bhattacharyya, P. Deb, S. K. Ghosh, and R. Ray, Phys. Rev. D, **82**: 014021 (2010)
68 A. Bhattacharyya, P. Deb, A. Lahiri, and R. Ray, Phys. Rev. D, **83**: 014011 (2011)
69 M. Bluhm and B. Kampfer, Phys. Rev. D, **77**: 034004 (2008)
70 V. M. Bannur, JHEP, **0709**: 046 (2007)
71 V. M. Bannur, Phys. Rev. C, **78**: 045206 (2008)
72 F. G. Gardim and F. M. Steffens, Nucl. Phys. A, **825**: 222 (2009)
73 B. J. Schaefer, M. Wagner, and J. Wambach, PoS CPOD, **2009**: 017 (2009)
74 B. J. Schaefer, M. Wagner, and J. Wambach, Phys. Rev. D, **81**: 074013 (2010)
75 V. Skokov, B. Friman, and K. Redlich, Phys. Rev. C, **83**: 054904 (2011)
76 S. J. Brodsky and X. G. Wu, Phys. Rev. D, **85**: 034038 (2012)
77 S. J. Brodsky and X. G. Wu, Phys. Rev. Lett., **109**: 042002 (2012)
78 M. Mojaza, S. J. Brodsky, and X. G. Wu, Phys. Rev. Lett., **110**: 192001 (2013)
79 S. J. Brodsky, M. Mojaza, and X. G. Wu, Phys. Rev. D, **89**: 014027 (2014)
80 X. G. Wu, J. M. Shen, B. L. Du, and S. J. Brodsky, arXiv:1802.09154 [hep-ph].
81 X. G. Wu, S. Q. Wang, and S. J. Brodsky, Front. Phys., **11**: 111201 (2016)
82 P. H. Ginsparg, Nucl. Phys. B, **170**: 388 (1980)
83 T. Appelquist and R. D. Pisarski, Phys. Rev. D, **23**: 2305 (1981)
84 S. Nadkarni, Phys. Rev. D, **27**: 917 (1983)
85 A. Bazavov, N. Brambilla, X. Garcia i Tormo, P. Petreczky, J. Soto, and A. Vairo, Phys. Rev. D, **907**: 074038 (2014)
86 S. Q. Wang, X. G. Wu, S. J. Brodsky, and M. Mojaza, Phys. Rev. D, **94**: 053003 (2016)
87 S. Q. Wang, X. G. Wu, X. C. Zheng, G. Chen, and J. M. Shen, J. Phys. G, **41**: 075010 (2014)