A holographic description of theta-dependent Yang-Mills theory at finite temperature*

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Abstract: Theta-dependent gauge theories can be studied using holographic duality through string theory in certain spacetimes. By this correspondence we consider a stack of N_0 dynamical D0-branes as D-instantons in the background sourced by N_c coincident non-extreme black D4-branes. According to the gauge-gravity duality, this D0-D4 brane system corresponds to Yang-Mills theory with a theta angle at finite temperature. We solve the IIA supergravity action by taking account into a sufficiently small backreaction of the Dinstantons and obtain an analytical solution for our D0-D4-brane configuration. Subsequently, the dual theory in the large N_c limit can be holographically investigated with the gravity solution. In the dual field theory, we find that the coupling constant exhibits asymptotic freedom, as is expected in QCD. The contribution of the theta-dependence to the free energy gets suppressed at high temperatures, which is basically consistent with the calculation using the Yang-Mills instanton. The topological susceptibility in the large N_c limit vanishes, and this behavior remarkably agrees with the implications from the simulation results at finite temperature. Moreover, we finally find a geometrical interpretation of the theta-dependence in this holographic system.

Keywords: AdS/CFT, gauge-gravity duality, holographic QCD, theta dependence

DOI: 10.1088/1674-1137/44/1/013103

1 Introduction

The spontaneous CP violation in Quantum Chromodynamics (QCD) has been studied for a significant amount of time, and such effects can usually be described by introducing a θ term to the four-dimensional (4d) action for gauge theories as [1],

$$S = -\frac{1}{2g_{\rm YM}^2}{\rm Tr}\int F\wedge{}^*F + {\rm i}\frac{\theta}{8\pi^2}{\rm Tr}\int F\wedge F, \tag{1}$$

where $g_{\rm YM}$ is the Yang-Mills coupling constant, and the second term defines the topological charge density with a θ angle. While the experimental value of the theta angle is stringently small $\left(|\theta|\leq 10^{-10}\right)$, the dependence of Yang-Mills theory and QCD on theta attracts great theoretical and phenomenological interests, e.g., the study of large N_c behavior [2], glueball spectrum [3], deconfinement phase transition [4, 5], and Schwinger effect [6]. Particularly, there is an open question in hadron physics, namely whether a theta vacua can be created in hot QCD. To resolve this issue, some progress was made in previous studies [7-12]. One of the most famous proposals was to search for the chiral magnet effect (CME) in heavy-ion collisions [13-16] to confirm the theta dependence at high temperature.

In contrast, the AdS/CFT correspondence, or more generally the gauge-gravity (string) duality, has rapidly become a powerful tool to investigate the strongly coupled quantum field theory (QFT) since 1997 [17-19]. In the holographic approach to study QCD or Yang-Mills theory, a concrete model was proposed by Witten [20] and Sakai and Sugimoto [21, 22] (named the WSS model), based on the IIA string theory. This model is quite successful, as it almost includes all necessary ingredients of QCD or Yang-Mills theory in a very simple manner, e.g., the fundamental quarks and mesons [21-23], baryon [24, 25], the phase diagram of hot QCD [26-30], glueball spectrum [31, 32], and the interactions of hadrons [33-

Received 29 July 2019, Revised 25 October 2019, Published online 28 November 2019

* Supported by the research startup funds of Dalian Maritime University (02502608) and the Fundamental Research Funds for the Central Universities (017192608)

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38]. Because of the non-perturbative properties of the theta dependence, it has been recognized that D-branes as D-instantons in bulk geometry play the role of the theta angle in dual theory [39-41]. By this viewpoint, the holographic correspondence of theta-dependence in QCD or Yang-Mills theory has been systematically studied using the WSS model with D0-branes as D-instantons at zero temperature or without temperature in [42-50].

To analyze the theta dependence at finite temperature, several studies performed simulations, and the results imply that some large N_c behaviors are different from the situations of zero temperature or without temperature [1]. In the current status of holographic approaches, the theta dependence at finite temperature is studied mostly in the $\mathcal{N}=4$ super Yang-Mills theory by the D(-1)-D3 brane configuration, e.g., References [39, 51, 52]. On the contrary, few lectures discuss specifically QCD or Yang-Mills theory at finite temperature through the D0-D4 brane configuration. Thus, we are motivated to fill this blank by exploring a way to combine the theta-dependent Yang-Mills at finite temperature with the IIA string theory. In our setup, we adopt the gravity background sourced by a stack of N_c black non-extreme D4-branes, since the dual field theory in this background exhibits deconfinement at finite temperature [26]. Then, we introduce N_0 coincident D0-branes as D-instantons into the D4-brane background by taking into account a very small backreaction to the bulk geometry. Hence, the D-instantons are dynamical, and this setup is coincident with the bubble D0-D4 configuration in Refs. [42-50][1)]. To search for an analytical supergravity solution, we further assume that the D0-branes are homogeneously smeared in the worldvolume of the D4-branes, and this D-brane configuration is illustrated in Table 1. Solving the effective 1d gravity action, we indeed obtain a particularly analytical solution. Subsequently, we examine coupling constants and a renormalized ground-state energy by the gravity solution. The coupling constant indicates the property of asymptotic freedom, and the free energy gets suppressed at high temperature. Moreover, the topological susceptibility in the large N_c limit vanishes. We find that all these results agree with the implications of the simulation reviewed in Ref. [1], or the well-known properties of QCD and Yang-Mills theory. Therefore, our study might offer a holographic approach to study the issues proposed in Refs. [7-15].

The outline of this manuscript is as follows. In Section 2, we first discuss the general formulas of the black D0-D4 system based on IIA supergravity. Then, comparing them with the black D4-brane solution, we obtain a particular solution by including some physical constraints. In Section 3, we evaluate the coupling constant and free energy density by our gravity solution. We also provide a geometric interpretation of the theta-dependence in this D0-D4 system. The final section provides the summary and discussion. Our gravity solution, expressed in terms of the U coordinate, is summarized in the Appendix.

2 Supergravity description

2.1 General setup

In this section, we explore the holographic description based on the N_0 D0- and N_c D4-branes with the configuration illustrated in Table 1. As the gauge-gravity duality is valid in the large N_c limit, we first define the 4d Hooft coupling as $\lambda_4 = g_{\rm YM}^2 N_c$, where $g_{\rm YM}$ is the Yang-Mills coupling, and λ_4 is fixed when $N_c \to \infty$. Then, to consider a small backreaction of the N_0 D0-branes, we further require $N_0 \to \infty$, while

$$\frac{N_0}{N_c} = C \text{ fixed, } C \ll 1. \tag{2}$$

Here, C is a fixed constant in the limitation of N_c, $N_0 \to \infty$, and we note that this limit is similar as the Veneziano limit discussed in Refs. [29, 30]. Keeping this in mind, we consider the dynamics of the 10d bulk geometry, which is described by the type IIA supergravity. In a string frame, the action is given as,

$$S_{\rm IIA} = \frac{1}{2\kappa_0^2}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(\mathcal{R}+4(\partial\phi)^2\right)-\frac{1}{2}|F_2|^2-\frac{1}{2}|F_4|^2\right], \tag{3}$$

where $2\kappa_0^2 = (2\pi)^7 l_s^8$, and $l_s = \sqrt{\alpha'}$ is the string length. $F_4 = dC_3$, $F_2 = dC_1$ is the Ramond-Ramond four and Ramond-Ramond two forms sourced by N_c D4-branes and N_0 D0-branes. We used $\mathcal{R}$ and ϕ to denote the 10d scalar curvature and the dilaton field, respectively. Since D0-branes as D-instantons are extended along the x^4 direction and homogeneously smeared in the directions of $\{x^0, x^i\}$, $i = 1, 2, 3$, we may search for a possible solution using the metric ansatz, written as [26, 29, 30],

$$\begin{aligned} ds^2 =& -e^{2\tilde{\lambda}}dt^2 + e^{2\lambda}\delta_{ij}dx^i dx^j + e^{2\lambda_s}\left(dx^4\right)^2 \\ &+ l_s^2 e^{-2\varphi}d\rho^2 + l_s^2 e^{2\nu}d\Omega_4^2. \end{aligned} \tag{4}$$

Table 1. Configuration of N_0 smeared D0 and N_c black D4-branes with compactified direction x^4. "–" represents that D-branes extend along this direction, and " = " represents direction where D0-branes are smeared.

	0	1	2	3	4	5(ρ)	6	7	8	9
N_0 smeared D0-branes	=	=	=	=	–					
N_c black D4-branes	–	–	–	–	–					

1) Here we emphasize that the dual theory in the approach of the bubble D0-D4 configuration is defined at zero-temperature limit, or defined without a concrete temperature. The dual theory includes a finite temperature is the distinguishing feature in our setup.

The Ramond-Ramond C_1 form and its field strength F_2 is assumed to be,

$$C_1 = [h(\rho)+H]\,\mathrm{d}x^4,$$
$$F_2 = \mathrm{d}C_1 = \partial_\rho h \mathrm{d}x^4 \wedge \mathrm{d}\rho, \tag{5}$$

where H is a constant, and $h(\rho)$ is a function to be solved. To find a static and homogeneous solution by the ansatz in Eq. (4), we further assume that the functions $\tilde{\lambda}$, λ, λ_s, φ, ν and the dilaton ϕ only depend on the holographic coordinate ρ. Hence, the action Eq. (3) could be rewritten as an effective 1d action by inserting Eqs. (4), (5) into Eq. (3), which leads to,

$$S_{\mathrm{IIA}} = \mathcal{V}\int \mathrm{d}\rho\Big[-3\dot{\lambda}^2-\dot{\lambda}_s^2-\dot{\tilde{\lambda}}^2-4\dot{\nu}^2+\dot{\varphi}^2 -\frac{1}{2}\mathrm{e}^{3\lambda+\tilde{\lambda}-\lambda_s+4\nu+\varphi}\dot{h}^2+V+\text{total derivative}\Big]. \tag{6}$$

We used "." to represent derivatives, which are w.r.t. ρ and

$$V = 12\mathrm{e}^{-2\nu-2\varphi}-Q_{\mathrm{c}}^2\mathrm{e}^{3\lambda+\lambda_s+\tilde{\lambda}-4\nu-\varphi},\quad \mathcal{V}=\frac{1}{2k_0^2}V_3V_{S^4}\beta_T\beta_4 l_s^3,$$
$$\varphi = 2\phi-3\lambda-\tilde{\lambda}-\lambda_s-4\nu,\quad Q_{\mathrm{c}}=\frac{3\pi^2 l_s}{\sqrt{2}\kappa_0}\int_{S^4}F_4. \tag{7}$$

Here, β_4, β_T refers to the size of (time) in the x^0 and x^4 direction[1)], V_3 represents the 3d spacial volume, and $V_{S^4}=\frac{8\pi^2}{3}$ is the volume of a unit S^4. Then, the solution for C_1 may be obtained as follows,

$$\dot{h}(\rho) = -q_\theta \mathrm{e}^{2\lambda_s-2\phi}, \tag{8}$$

where q_θ is an integration constant related to the θ angle, which will become more evident later. The 1d action in Eq. (6) has to be supported by the following zero-energy constraint [29, 30, 45],

$$-3\dot{\lambda}^2-\dot{\lambda}_s^2-\dot{\tilde{\lambda}}^2-4\dot{\nu}^2+\dot{\varphi}^2-\frac{1}{2}\mathrm{e}^{3\lambda+\tilde{\lambda}-\lambda_s+4\nu+\varphi}\dot{h}^2-V=0, \tag{9}$$

such that the equations of motion from the 1d effective action in Eq. (6) are coincident with those from the 10d action in Eq. (3), if the homogeneous ansatz Eq. (4) is adopted.

Afterwards, the complete equations of motion can be obtained by varying the 1d action in Eq. (6), which are given as

$$\ddot{\lambda}-\frac{Q_{\mathrm{c}}^2}{2}\mathrm{e}^{6\lambda+2\lambda_s+2\tilde{\lambda}-2\phi}=\frac{q_\theta^2}{4}\mathrm{e}^{2\lambda_s-2\phi},$$
$$\ddot{\lambda}_s-\frac{Q_{\mathrm{c}}^2}{2}\mathrm{e}^{6\lambda+2\lambda_s+2\tilde{\lambda}-2\phi}=-\frac{q_\theta^2}{4}\mathrm{e}^{2\lambda_s-2\phi},$$
$$\ddot{\tilde{\lambda}}-\frac{Q_{\mathrm{c}}^2}{2}\mathrm{e}^{6\lambda+2\lambda_s+2\tilde{\lambda}-2\phi}=\frac{q_\theta^2}{4}\mathrm{e}^{2\lambda_s-2\phi},$$
$$\ddot{\nu}+\frac{Q_{\mathrm{c}}^2}{2}\mathrm{e}^{6\lambda+2\lambda_s+2\tilde{\lambda}-2\phi}-3\mathrm{e}^{6\lambda+2\lambda_s+2\tilde{\lambda}-4\phi+6\nu}=\frac{q_\theta^2}{4}\mathrm{e}^{2\lambda_s-2\phi},$$
$$\ddot{\phi}-\frac{Q_{\mathrm{c}}^2}{2}\mathrm{e}^{6\lambda+2\lambda_s+2\tilde{\lambda}-2\phi}=\frac{3q_\theta^2}{4}\mathrm{e}^{2\lambda_s-2\phi}. \tag{10}$$

To find a solution for Eq. (10), let us introduce new variables γ, p, χ, defined as

$$\gamma=6\lambda+2\lambda_s+2\tilde{\lambda}-2\phi,\ p=6\lambda+2\lambda_s+2\tilde{\lambda}-4\phi+6\nu,\ \chi=2\lambda_s-2\phi. \tag{11}$$

Hence, Eq. (10) reduces to three simple equations,

$$\ddot{\gamma}-4Q_{\mathrm{c}}^2\mathrm{e}^{\gamma}=0,\quad \ddot{p}-18\mathrm{e}^{p}=0,\quad \ddot{\chi}+2q_\theta^2\mathrm{e}^{\chi}=0. \tag{12}$$

Moreover, the solution for Equations in (12) could be analytically obtained as

$$\gamma = -2\log\left[a_1-\mathrm{e}^{-a_2\rho}\right]-a_2\rho+\log\left[\frac{a_1a_2^2}{2Q_{\mathrm{c}}^2}\right],$$
$$p = -2\log\left[a_3-\mathrm{e}^{-a_4\rho}\right]-a_4\rho+\log\left[\frac{a_3a_4^2}{9}\right],$$
$$\chi = -2\log\left[a_5+\mathrm{e}^{-a_6\rho}\right]-a_6\rho+\log\left[\frac{a_5a_6^2}{q_\theta^2}\right], \tag{13}$$

where $a_{1,2,3,4,5,6}$ are integration constants. According to Eq. (10), in contrast, we have

$$\lambda-\tilde{\lambda}=b_1\rho+b_2,$$
$$\lambda-\lambda_s-\phi+\tilde{\lambda}=b_3\rho+b_4, \tag{14}$$

where $b_{1,2,3,4}$ are additional integration constants. Altogether with Eqs. (13) and (14), we could obtain the full solution for Eq. (4) as,

$$\lambda=\frac{1}{8}(\gamma-\chi)+\frac{1}{4}(b_2+b_1\rho),$$
$$\lambda_s=\frac{1}{8}(\gamma+\chi)-\left(\frac{b_1}{4}+\frac{b_3}{2}\right)\rho-\frac{b_2}{4}-\frac{b_4}{2},$$
$$\tilde{\lambda}=\frac{1}{8}(\gamma-\chi)-\frac{3b_2}{4}-\frac{3b_1}{4}\rho,$$
$$\phi=\frac{1}{8}(\gamma-3\chi)-\left(\frac{b_1}{4}+\frac{b_3}{2}\right)\rho-\frac{b_2}{4}-\frac{b_4}{2},$$
$$\nu=\frac{p}{6}-\frac{1}{8}(\gamma+\chi)-\left(\frac{b_1}{12}+\frac{b_3}{6}\right)\rho-\frac{b_2}{12}-\frac{b_4}{6}. \tag{15}$$

Moreover, the zero-energy constraint Eq. (9) reduces to the following relation,

$$-3a_2^2+8a_4^2-3a_6^2-20b_1^2-8b_1b_3-8b_3^2=0. \tag{16}$$

While all the integration constants should be further determined by some additional physical conditions, we note that these could depend on q_θ, which is the only parameter in the solution.

2.2 A particular solution

In this section, we discuss a particular solution to fix

1) Since we would consider a 4d dual field theory at finite temperature, the x^0 and x^4 directions have to be compactified on S^1 in this model. And β_4, $\beta_T\to\infty$ corresponds to the decompactified limit.

the integration constants in the supergravity solution obtained in the last section. Since $|\theta|$ is usually very small in Yang-Mills theory, we consider a sufficiently small backreaction of the D-instantons (D0-branes) in the black D4 configuration. Therefore, we require that the solution to Eq. (15) must be able to return to the pure black D4-brane solution if $q_\theta \to 0$, i.e., no D0-branes. Hence, the black D4-brane solution corresponds to the situation of $C_1 = 0$[1)] in Eq. (3), and in the near-horizon limit the solution is given as

$$\begin{aligned}
\mathrm{d}s^2 =& \left(\frac{U}{R}\right)^{3/2}\left[-f_T(U)\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \left(\mathrm{d}x^4\right)^2\right] \\
&+ \left(\frac{R}{U}\right)^{3/2}\left[\frac{\mathrm{d}U^2}{f_T(U)} + U^2\mathrm{d}\Omega_4^2\right], \\
f_T(U) =& 1 - \frac{U_T^3}{U^3}, \quad \mathrm{e}^\phi = g_s\left(\frac{U}{R}\right)^{3/4}, \\
F_4 =& 3R^3 g_s^{-1}\omega_4, \quad R^3 = \pi g_s N_\mathrm{c} l_s^3,
\end{aligned} \tag{17}$$

where g_s, ω_4 represents the string coupling constant and the volume form of S^4. Accordingly, we identify the solution Eq. (17) as the zero-th order solution of Eq. (13), and rewrite it in terms of γ, p, χ, defined as in Eq. (11),

$$\begin{aligned}
\gamma_0 =& -2\log\left[1-\mathrm{e}^{-3a\rho}\right] - 3a\rho + \log\left[\frac{U_T^6}{g_s^2R^6}\right], \\
p_0 =& -2\log\left[1-\mathrm{e}^{-3a\rho}\right] - 3a\rho + \log\left[\frac{U_T^6}{g_s^4 l_s^6}\right], \\
\chi_0 =& -2\log\left[g_s\right].
\end{aligned} \tag{18}$$

This yields the relation of ρ and the usually employed U coordinate in Eq. (17) as

$$\rho = -\frac{b_\theta}{3a}\log\left[1-\frac{U_T^3}{U^3}\right], \; a = \frac{\sqrt{2}Q_\mathrm{c}U_T^3}{3R^3g_s} = \frac{U_T^3}{l_s^3g_s^2}, \; Q_\mathrm{c} = \frac{3\pi N_\mathrm{c}}{\sqrt{2}}. \tag{19}$$

Here, b_θ is another constant dependent on θ, which is required as $b_\theta \to 1$ if $q_\theta \to 0$. Comparing Eq. (18) with Eq. (13), this implies that in the limit of $q_\theta \to 0$ there must be $a_{1,3} \to 1$, $a_{2,4} \to 3a$, $a_5 a_6^2 \to \frac{4q_\theta^2}{g_s^2}$, $a_6 \to q_\theta$ so that γ, p, χ consistently returns to γ_0, p_0, χ_0. In this sense, we could in particular choose $a_5 = 1$, $a_6 = 2|q_\theta|g_s^{-1}$ so that $a_1 = a_3 = 1$, $a_2 = 3a$, $b_2 = 0$, $b_4 = -\log[g_s]$ as the most simple solution. Moreover, we require that $g_{00} \sim \tilde{\lambda}$, $g_{ij} \sim \lambda$, $g_{\Omega\Omega} \sim \nu$ has to behave the same as when in the zero-th order solution of Eq. (17) in the IR region (i.e. $U \to U_T$, $\rho \to \infty$), such that the holographic duality constructed on the N_c D4-branes basically remains in the low-energy theory. Therefore, we have the following relations

$$b_1 = \frac{1}{2}(a_2 - a_6), \quad a_4 = \frac{a_2^2 + a_6^2}{a_2 + a_6}. \tag{20}$$

In contrast, the zero-energy constraint of Eq. (9) reduces to an extra relation to determine b_3, which is

$$\begin{aligned}
b_3 =& -\frac{1}{2}b_1 - \frac{\sqrt{2}}{4}\sqrt{-3a_2^2 + 8a_4^2 - 3a_6^2 - 18b_1^2}, \\
& -3a_2^2 + 8a_4^2 - 3a_6^2 - 18b_1^2 \geq 0.
\end{aligned} \tag{21}$$

The above constraints imply that our solution would be valid only if $|q_\theta| \leqslant \frac{3}{2}ag_s$, which is consistent with our assumption that the backreaction of D-instantons is sufficiently small. The constant b_θ could be determined by additionally requiring that $g_{UU} \sim \psi$ behaves as same as in Eq. (17) at $U = U_T$, and this yields

$$b_\theta = \frac{9a^2g_s^2 - 6ag_sq_\theta}{9a^2g_s^2 + 4q_\theta^2}. \tag{22}$$

For the reader's convenience, we have summarized the current solution in terms of the U coordinate in the Appendix, which can be directly compared with the zero-th order solution of Eq. (17). Notice that our solution also has the same behaviour as Eq. (17) in the UV region (i.e. $U \to \infty$, $\rho \to 0$).

3 Dual field theory

3.1 Running coupling

To start this section, let us examine the dual field theory interpretation of the above supergravity solution in Section 2.2 by taking into account a probe D4-brane moving in our D0-D4 background. The action for a non-supersymmetric D4-brane is given as,

$$\begin{aligned}
S_{\mathrm{D}_4} =& -\mu_4\int \mathrm{d}^5x\mathrm{e}^{-\phi}\mathrm{STr}\sqrt{-\det\left(g_{(5)} + \mathcal{F}\right)} \\
&+ \frac{1}{2}\mu_4\mathrm{Tr}\int C_1 \wedge \mathcal{F} \wedge \mathcal{F},
\end{aligned} \tag{23}$$

where respectively $\mu_4 = \frac{1}{(2\pi)^4 l_s^5}$, $g_{(5)}$, $\mathcal{F} = 2\pi\alpha' F$ are the charge of the D4-brane, induced 5d metric, and gauge field strength exited on the D4-brane, respectively. We assume that the non-vanished components of F are $F_{\mu\nu}(x)\delta^{1/2}\left(x^4 - \bar{x}\right)$. Then, considering that the x^4 direction is compacted on a circle S^1 with the period β_4, the action Eq. (23) can be expanded in powers of $\mathcal{F}$ as a 4d Yang-Mills theory with a θ term,

$$S_{\mathrm{D}_4} \simeq -\frac{1}{2g_{\mathrm{YM}}^2}\mathrm{Tr}\int F \wedge {}^*F + \mathrm{i}\frac{\theta}{8\pi^2}\mathrm{Tr}\int F \wedge F + O\left(F^3\right), \tag{24}$$

where the delta function is normalized as $\beta_4 = \int \mathrm{d}x^4\delta\left(x^4 - \bar{x}\right)$,

1) Strictly speaking, the black D4-brane solution corresponds to the situation that C_1 is a constant because the IIA action (3) is invariant under the gauge transformation $C_1 \to C_1 + d\Lambda$ where Λ is an arbitrary function. Thus we can choose a particular gauge condition so that $C_1 = 0$ corresponds to the situation of the black D4-brane solution.

and the coupling constant $g_{\rm YM}$, θ are defined as,

$$\begin{aligned} g_{\rm YM}^2(U) =& \left[\mu_4(2\pi\alpha')^2\beta_4 e^{-\phi}\sqrt{g_{44}}\right]^{-1} \\ =& \frac{8\pi^2 g_s l_s}{\beta_4}\cosh\left[\frac{q_\theta}{2g_s}\rho(U)\right], \\ \theta(U) =& -\frac{i}{l_s}\int_{\partial D=S^1_{x^4}} C_1 = -\frac{i}{l_s}\int_D F_2 \\ =& \theta - \frac{\beta_4}{g_s l_s}\tanh\left[\frac{q_\theta}{2g_s}\rho(U)\right], \end{aligned} \tag{25}$$

which are the running couplings. Since the asymptotic region of the bulk supergravity corresponds to the dual field theory, at the boundary $\rho\to 0$, $U\to\infty$, Eq. (25) defines the value of the Yang-Mills coupling constant and the θ angle in dual theory. In the large N_c limit, we should define the limitation $\bar{\theta}=\theta/N_c$ [1, 2] and the t'Hooft coupling,

$$\lambda_4(U) = \frac{8\pi^2 g_s l_s N_c}{\beta_4}\cosh\left[\frac{q_\theta}{2g_s}\rho(U)\right]. \tag{26}$$

According to the AdS/CFT dictionary, we remarkably find the Yang-Mills and t'Hooft coupling constant $g_{\rm YM}$, λ_4 increase in the IR region ($\rho\to\infty$, $U\to U_T$), while they become small in the UV region ($\rho\to 0$, $U\to\infty$) with our D0-D4 solution. This behavior is in qualitative agreement with the property of asymptotic freedom in QCD or Yang-Mills theory.

To summarize this subsection, we evaluate the relation between q_θ and θ. In the Dp-brane supergravity solution, the normalization of the Ramond-Ramond field F_{p+2} is given as $2k_0^2\mu_p N_p = \int_{S^{8-p}} {}^*F_{p+2}$, and this normalization with Eq. (8) would tell us that q_θ is proportional to the number of D0-branes. Hence, we have $q_\theta \sim g_s N_0$, $N_0 = g_s d_{\rm D_0} V_4$, where $d_{\rm D_0}$ is the number density of D0-branes, and $V_4 \simeq (2\pi R)^3\beta_T$ is the worldvolume of the D4-branes. To include the influence of the D-instantons, we further assume that $d_{\rm D_0}$ depends on x^4, because $x^4=\theta R_4$ is periodic. This viewpoint implies that each slice in the D4-brane with a fixed x^4 corresponds to a theta vacuum in the dual field theory if we identify the coordinate θ to the theta angle in Eq. (24). Thus, we could interpret that the 4d Yang-Mills action Eq. (24) is defined on a slice of the D4-brane with $x^4=\bar{x}$, or namely with a theta angle $\theta=\bar{x}/R_4$, and it might offer a geometric interpretation of the theta-dependence in the dual field theory. Finally, we can define the dimensionless density using β_4 as $I(\theta)=d_{\rm D_0}\beta_4^{-4}$, which leads to $|q_\theta|\simeq 2g_s V_4 I(\theta)/\beta_4^4$. Note that in the large N_c limit $I(\theta)$ may be expected to be a function of θ/N_c.

3.2 Thermodynamics

The thermodynamics in holography is based on the relation between the partition function of the bulk supergravity $Z_{\rm SUGRA}$ and the dual field theory (DFT) $Z_{\rm DFT}$ as $Z_{\rm SUGRA}=Z_{\rm DFT}$ in the large N_c limit [17-19]. Hence, the free energy density of the 4d theta-dependent Yang-Mills theory $f(\theta)$ is obtained by

$$Z = e^{-V_4 f(\theta)} = e^{-S_{\rm SUGRA}^{\rm ren\ onshell}}, \tag{27}$$

where V_4 and $S_{\rm SUGRA}^{\rm ren\ onshell}$ represent the 4d spacetime volume and the renormalized onshell action of the bulk supergravity, respectively. For the duality to the thermal field theory, $V_4=V_3\beta_T$ and $S_{\rm SUGRA}^{\rm ren\ onshell}$ refer to the Euclidean version. The temperature in the dual field theory is defined by $T=1/\beta_T$. To avoid conical singularities in the dual field theory, the relation with our D0-D4 solution is provided[1)],

$$2\pi T \simeq \left(\frac{3}{2}+\frac{q_\theta}{3ag_s}\right)\frac{U_T^{1/2}}{R^{3/2}} + O\left(q_\theta^2\right). \tag{28}$$

Subsequently, the renormalized Euclidean onshell action of the supergravity is given as,

$$S_{\rm SUGRA}^{\rm ren\ onshell} = S_{\rm IIA}^{\rm E} + S_{\rm GH} + S_{\rm CT}, \tag{29}$$

where $S_{\rm IIA}^{\rm E}$ refers to the Euclidean version of IIA supergravity action Eq. (3) and $S_{\rm GH}$, $S_{\rm CT}$ refers to the associated Gibbons-Hawking and the bulk counter-term, which are respectively given as [29, 53],

$$\begin{aligned} S_{\rm IIA}^{\rm E} =& -\frac{1}{2k_0^2}\int d^{10}x\sqrt{g}\left[e^{-2\phi}\left(\mathcal{R}+4(\partial\phi)^2\right)-\frac{1}{2}|F_2|^2-\frac{1}{2}|F_4|^2\right], \\ S_{\rm GH} =& -\frac{1}{k_0^2}\int d^9x\sqrt{h}e^{-2\phi}K, \\ S_{\rm CT} =& \frac{1}{k_0^2}\left(\frac{g_s^{1/3}}{R}\right)\int d^9x\sqrt{h}\frac{5}{2}e^{-7\phi/3}, \end{aligned} \tag{30}$$

here h is the determinant of the boundary metric, i.e., the slice of the bulk metric Eq. (4) at fixed $\rho=\varepsilon$ with $\varepsilon\to 0$. K is the trace of the extrinsic curvature at the boundary, which is defined as

$$K = \frac{1}{\sqrt{g}}\partial_\rho\left(\frac{\sqrt{g}}{\sqrt{g_{\rho\rho}}}\right)\Big|_{\rho=\varepsilon}. \tag{31}$$

Then, the actions in Eq. (30) can be evaluated using the D0-D4 solution discussed in Section 2.2. After some straightforward albeit complex calculations, we finally obtain

$$\begin{aligned} S_{\rm IIA}^{\rm E} =& \mathcal{V}\left[\frac{3}{2\varepsilon}-\frac{9}{4}a+\frac{7q_\theta}{2g_s}\right], \\ S_{\rm GH} =& \mathcal{V}\left[-\frac{19}{6\varepsilon}+\frac{7}{6}\frac{9a^2g_s^2-36ag_sq_\theta+4q_\theta^2}{6ag_s^2+4g_sq}\right], \\ S_{\rm CT} =& \mathcal{V}\frac{5}{3\varepsilon}, \end{aligned} \tag{32}$$

1) It is not very obvious to find a relation as Eq. (28) just by requiring no singularities outside the horizon with our gravitational solution in Section 2.2. So we assume that our solution could return to Eq. (17) continuously if $q_\theta\to 0$ then we find the relation Eq. (28) is at least valid at order $O\left(q_\theta^1\right)$.

and the free energy density $f(\theta)$ is therefore obtained using Eqs. (27), (32) with the relation of q_θ and θ, which is calculated as,

$$f(\theta,T) = -\frac{128N_c^2\pi^4T^6\lambda_4}{2187M_{KK}^2} + \frac{2M_{KK}^5\lambda_4}{3\pi^2T}I(\theta), \tag{33}$$

where we have defined the Kaluza-Klein (KK) mass $M_{KK} = 2\pi/\beta_4$ and rescaled $I(\theta) \to (2\pi l_s)^3 M_{KK}^3 I(\theta)$. The function $I(\theta)$ is found to be a periodic and even function of θ i.e., $I(\theta) = I(-\theta)$, $I(\theta) = I(\theta+2k\pi)$, $k \in \mathbb{Z}$, and the energy of the true vacuum $F(\theta)$ is obtained by minimizing the expression in Eq. (33) over k,

$$F(\theta,T) = \min_k f(\theta,T). \tag{34}$$

While at finite temperature, the exact theta-dependence of the ground-state free energy in Yang-Mills theory is less clear, especially in the large N_c limit, the computation for one-loop contribution of instantons to the functional integral at sufficiently high temperature suggests that $f(\theta) - f(0) \propto 1 - \cos\theta$ [1]. Although this theta-dependence is consistent with the gravitational constraints discussed in Section 2, i.e., $q_\theta \to 0$ if $\theta \to 0$, this does not have a definite limitation at $N_c \to \infty$. Nonetheless, if we assume the function $I(\theta)$ has a limit at $N_c \to \infty$, the topological susceptibility can be computed by expanding Eq. (33) in powers of $\bar\theta$ as,

$$f(\bar\theta) - f(0) = \frac{2M_{KK}^5\lambda_4}{3\pi^2T}\sum_{n=1}^{\infty}\frac{b_n}{2n!}\bar\theta^{2n}, \quad b_n = \left.\frac{\partial^n f(\bar\theta,T)}{\partial\bar\theta^n}\right|_{\bar\theta=0}. \tag{35}$$

Thus, the topological susceptibility reads[1)],

$$\chi(T) = \left.\frac{\partial^2 f(\bar\theta,T)}{\partial\theta^2}\right|_{\theta=0} = \frac{2M_{KK}^5\lambda_4}{3\pi^2N_c^2T}b_2. \tag{36}$$

where b_2 should be a positive numerical number[2)]. As expected, the topological susceptibility (36) depends on temperature and vanishes in the large N_c limit. Our holographic approach implies the behavior of the topological susceptibility in deconfined phase is different from its behavior in the confined phase, as in Ref. [45]. We notice this large N_c behavior agrees remarkably with the simulation results reviewed in Ref. [1], which indicates that the topological susceptibility has a vanishing large N_c above the deconfinement temperature.

4 Summary and discussion

In this letter, we holographically combine the IIA supergravity with the theta-dependent Yang-Mills theory at finite temperature. The bulk geometry is sourced by a stack of N_c black D4-branes and N_0 D0-branes as D-instantons. In the pure black D4-brane solution, the dual field theory indicates deconfinement at finite temperature, and adding D-instantons to the D4 background could describe the dynamics of the theta angle in the bulk. To keep this duality image and include the dynamics of the D-instantons, we therefore consider a sufficiently small backreaction from the D-instantons to the bulk geometry. Then, a particular solution is found by solving the IIA supergravity action. After using our supergravity solution, we investigate the coupling constant and the ground-state energy as two most fundamental properties in the dual field theory. The behavior of the coupling constant exhibits the asymptotic freedom as in QCD or Yang-Mills theory, and the theta contribution to the free energy density is suppressed at high temperature. The topological susceptibility is vanished in the large N_c limit. Remarkably, all these results are in qualitative agreement with various simulation results of the theta-dependent Yang-Mills theory at finite temperature discussed in Ref. [1]. Furthermore, we propose a geometric interpretation of the theta-dependence in this system.

In our D0-D4 background, the dual theory should deconfine at the temperature $T \geq T_c$, where T_c refers to the critical temperature of the deconfinement transition. Below T_c, the current supergravity solution would be invalid, and the confinement in the dual theory should be described by the bubble D0-D4 background, as discussed in Refs. [42-50]. The thermodynamical variables have different large N_c limits in these two D0-D4 backgrounds. The T_c could be obtained by comparing the free energy of our black Eq. (33) and the bubble D0-D4 system [42-45]. However, T_c remains substantially unchanged, as described in Ref. [45] in the large N_c limit[3)]. Another noteworthy point is that Eq. (36) implies that the instantons would be more unstable in the dual theory at high temperatures due to the definition of the topological susceptibil-

1) Here the reader should notice the relation of $\bar\theta$ and θ as $\bar\theta = \theta/N_c$ in the formulas.

2) If we phenomenologically choose $I(\theta) = 1 - \cos\theta$ at finite N_c and $I(\bar\theta) = 1 - \cos\bar\theta$ in the large N_c limit, the topological susceptibility would be $\chi(T) = \frac{2M_{KK}^5\lambda_4}{3\pi^2N_c^2T}b_2$ with $b_2 = 1/2$.

3) T_c can be obtained by solving $f(\theta,T)\big|_{T=T_c} = f_c(\theta)$ where $f(\theta,T)$ refers to the deconfined free energy as given in Eq. (33). And $f_c(\theta)$ refers to the confined free energy of this system as $f_c(\theta) = -\frac{2N_c^2\lambda}{3^7\pi^2}\frac{M_{KK}^4}{\left(1+\frac{\lambda^2}{16\pi^4}\frac{\theta^2}{N_c^2}\right)^3}$. In this sense, T_c remains to be $T_c = \frac{M_{KK}}{2\pi}\frac{1}{\sqrt{1+\frac{\lambda^2}{16\pi^4}\bar\theta}}$ in the large N_c limit while the dynamics of the D0-branes are not considered in the deconfined phase.

ity in QFT $\chi=-\mathrm{i}\int \mathrm{d}^4x\langle O(x)O(0)\rangle$, where $O(x)=\mathrm{Tr}F\wedge F$ is the glueball condensate operator. Hence, at extremely high temperatures $T\gg T_{\mathrm{c}}$ the quantum fluctuations would destroy the glueball condensate in the dual theory in a very short time, and the theta vacuum in the dual field theory decays quickly to the true vacuum. This conclusion is basically consistent with e.g., Refs. [7-9] and the D3-D(-1) approach in Ref. [52].

To summarize this study, we provide the final comments. Despite our holographic interpretation of the theta-dependence, the exact thermodynamics involving the theta angle is still challenging both in gauge-gravity duality and QFT, especially at finite temperature. In our theory, this is reflected in the fact that the specific relation of q_θ and θ could not be determined naturally through holographic duality. Thus, we have to further require that the density of the D0-branes exactly controls the ground-state energy, through the role of the theta parameter in dual field theory. While this could consistently resolve the problem as done in this study, the physical understanding of this constraint is not clear. Furthermore, unfortunately, the analysis in QFT has not implied any constructive results to date, hence we have to treat it as a particular constraint in this system and leave it to a future study.

I would like to thank Wenhe Cai and Chao Wu for valuable comments and discussions.

Appendix: D0-D4 solution in the *U* coordinate

We summarize the D0-D4 solution discussed in Section 2.2 in terms of the U coordinate. The components of the metric are written as,

$$\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu+g_{44}\left(\mathrm{d}x^4\right)^2+g_{UU}\mathrm{d}U^2+g_{\Omega\Omega}\mathrm{d}\Omega_4^2, \tag{A1}$$

where

$$\begin{aligned}
g_{00}&=-\left(\frac{U_T}{R}\right)^{3/2}\frac{f_T(U)^{\frac{9-4Q^2}{9+4Q^2}}}{\sqrt{2}}g_1(U)^{1/2}g_2(U)^{-1/2},\\
g_{ij}&=\frac{1}{\sqrt{2}}\left(\frac{U_T}{R}\right)^{3/2}g_1(U)^{1/2}g_2(U)^{-1/2}\delta_{ij},\\
g_{44}&=\left(\frac{U_T}{R}\right)^{3/2}\sqrt{2}f_T(U)^{\frac{12Q}{9+4Q^2}}\left[g_1(U)g_2(U)\right]^{-1/2},\\
g_{UU}&=\left(\frac{9+4Q^2}{9+6Q}\right)^{2/3}\left(\frac{R}{U_T}\right)^{3/2}\frac{\left[g_1(U)g_2(U)\right]^{1/2}}{\sqrt{2}f_T(U)},\\
g_{\Omega\Omega}&=\left(\frac{9+4Q^2}{9+6Q}\right)^{2/3}\left(\frac{R}{U_T}\right)^{3/2}\frac{U^2}{\sqrt{2}}\left[g_1(U)g_2(U)\right]^{1/2},
\end{aligned} \tag{A2}$$

and the dilaton is

$$e^\phi=g_s\left(\frac{U_T}{R}\right)^{3/4}f_T(U)^{\frac{Q(3-2Q)}{9+4Q^2}}\frac{g_1(U)^{3/4}g_2(U)^{-1/4}}{2^{3/4}}. \tag{A3}$$

The parameter Q and functions $g_{1,2}$ are defined as

$$\begin{aligned}
g_1(U)&=1+f_T(U)^{\frac{2Q(3+2Q)}{9+4Q^2}},\\
g_2(U)&=1-f_T(U)^{\frac{9+6Q}{9+4Q^2}},\\
Q&=\frac{|q_\theta|}{a g_s}.
\end{aligned} \tag{A4}$$

Note that Q is a positive number, and if sufficiently small, we obtain $f_T(U)^{\frac{12Q}{9+4Q^2}}\simeq 1$, $g_1(U)\simeq 2$ in the region $U\in(U_T+\varepsilon,\infty)$, where $\varepsilon\to 0$. The metric Eq. (A2) and the dilaton Eq. (A3) consistently return to the zero-th order solution in Eq. (17) if we set q_θ, $Q=0$.

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