Analysis of the strong decays of P_c(4312) as a pentaquark molecular state with QCD sum rules*

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Abstract: In this article, we tentatively assign P_c(4312) to be the $\bar{D}\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$, and discuss the factorizable and non-factorizable contributions in the two-point QCD sum rules for the $\bar{D}\Sigma_c$ molecular state in detail to prove the reliability of the single pole approximation in the hadronic spectral density. We study its two-body strong decays with the QCD sum rules, and special attention is paid to match the hadron side with the QCD side of the correlation functions to obtain solid duality. We obtain the partial decay widths $\Gamma(P_c(4312) \to \eta_c p) = 0.255$ MeV and $\Gamma(P_c(4312) \to J/\psi p) = 9.296^{+19.542}_{-9.296}$ MeV, which are compatible with the experimental value of the total width, and support assigning P_c(4312) to be the $\bar{D}\Sigma_c$ pentaquark molecular state.

Keywords: pentaquark molecular states, QCD sum rules, *XYZ* particles

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1   Introduction

In 2015, the LHCb collaboration observed two pentaquark candidates P_c(4380) and P_c(4450) in the $J/\psi p$ mass spectrum in the $\Lambda_b^0 \to J/\psi K^- p$ decays [1]. Recently, the LHCb collaboration observed a new narrow pentaquark candidate, P_c(4312), in the $J/\psi p$ mass spectrum with the statistical significance of 7.3σ; furthermore, they confirmed the old P_c(4450) pentaquark structure, which consists of two narrow overlapping peaks P_c(4440) and P_c(4457), with the statistical significance of 5.4σ [2]. The masses and widths are

$$P_c(4312) : M = 4311.9 \pm 0.7^{+6.8}_{-0.6} \text{ MeV},$$
$$\Gamma = 9.8 \pm 2.7^{+3.7}_{-4.5} \text{ MeV},$$
$$P_c(4440) : M = 4440.3 \pm 1.3^{+4.1}_{-4.7} \text{ MeV},$$
$$\Gamma = 20.6 \pm 4.9^{+8.7}_{-10.1} \text{ MeV},$$
$$P_c(4457) : M = 4457.3 \pm 0.6^{+4.1}_{-1.7} \text{ MeV},$$
$$\Gamma = 6.4 \pm 2.0^{+5.7}_{-1.9} \text{ MeV}. \tag{1}$$

The P_c(4312) can be assigned to be a $\bar{D}\Sigma_c$ pentaquark molecular state [3-18], a pentaquark state [19-23], and a hadrocharmonium pentaquark state [24].

The P_c(4312) lies near the $\bar{D}\Sigma_c$ threshold, which leads to the molecule assignment naturally. In Ref. [18], we performed detailed studies of the $\bar{D}\Sigma_c$, $\bar{D}\Sigma_c^*$, $\bar{D}^*\Sigma_c$ and $\bar{D}^*\Sigma_c^*$ pentaquark molecular states with the QCD sum rules by carrying out the operator product expansion up to the vacuum condensates of dimension 13 in a consistent way. The prediction $M_P = 4.32 \pm 0.11$ GeV for the $\bar{D}\Sigma_c$ molecular state supports assigning P_c(4312) to be the $\bar{D}\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$. On the other hand, our studies based on the QCD sum rules indicate that the scalar-diquark-scalar-diquark-antiquark type pentaquark state with the spin-parity $J^P = \frac{1}{2}^-$ has a mass of 4.31 ± 0.11 GeV, whereas the axialvector-diquark-axialvector-diquark-antiquark type pentaquark state with the spin-parity $J^P = \frac{1}{2}^-$ has a mass of 4.34 ± 0.14 GeV, which support assigning P_c(4312) to be a diquark-diquark-antiquark type pentaquark state [23, 25, 26]. P_c(4312) may be a diquark-diquark-antiquark type pentaquark state, which has a strong coupling to the $\bar{D}\Sigma_c$ scattering states. The strong coupling induces some $\bar{D}\Sigma_c$ components [27]. Thus, we can reproduce the experimental value of the mass of P_c(4312) in both the scenarios of the pentaquark state and pentaquark molecular state. In Ref. [28], we choose the $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ type tetraquark current to study the strong decays of Y(4660)

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with the QCD sum rules based on solid quark-hadron duality. Our calculations showed that the hadronic coupling constant $|G_{Y\psi'f_0}| \gg |G_{YJ/\psi f_0}|$ is consistent with the observation of $Y(4660)$ in the $\psi'\pi^+\pi^-$ mass spectrum, and that it favors the $\psi'f_0(980)$ molecule assignment [29, 30]. A similar mechanism may exist for $P_c(4312)$.

In this article, we tentatively assign $P_c(4312)$ to be the $\bar{D}\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$, and study its two-body strong decays with the QCD sum rules. In Ref. [31], we assigned $Z_c(3900)$ to be the diquark-antidiquark type axialvector tetraquark state, and studied the hadronic coupling constants in the strong decays $Z_c(3900) \to J/\psi\pi$, $\eta_c\rho$, $D\bar{D}^*$ with the QCD sum rules based on the solid quark-hadron duality by taking into account both the connected and disconnected Feynman diagrams in the operator product expansion. The method works well for studying the two-body strong decays of $Z_c(3900)$, $X(4140)$, $X(4274)$ and $Z_c(4600)$ [31-34]. Now, we extend the method to study the two-body strong decays of the pentaquark molecular state by carrying out the operator product expansion up to the vacuum condensates of dimension 10.

The article is arranged as follows: in Sec. 2, we present the comments on the QCD sum rules for the $\bar{D}\Sigma_c$ pentaquark molecular state; in Sec. 3, we derive the QCD sum rules for the hadronic coupling constants in the

strong decays $P_c(4312) \to \eta_c p$, $J/\psi p$; in Sec. 4, we present the numerical results and discussions; and we conclude our report in Sec. 5.

2   Comments on the QCD sum rules for the $\bar{D}\Sigma_c$ pentaquark molecular state

In this section, we present the two-point correlation function $\Pi(p)$ to study the mass and pole residue of the $\bar{D}\Sigma_c$ pentaquark molecular state with the QCD sum rules,

$$\Pi(p) = i \int d^4x e^{ip\cdot x} \langle 0|T\{J(x)\bar{J}(0)\}|0\rangle, \tag{2}$$

where the current $J(x) = J_{\bar{D}\Sigma_c}(x)$,

$$J_{\bar{D}\Sigma_c}(x) = \bar{c}(x)i\gamma_5 u(x)\varepsilon^{ijk}u_i^T(x)C\gamma_\alpha d_j(x)\gamma^\alpha\gamma_5 c_k(x), \tag{3}$$

and i, j, k are color indices. We choose the color-singlet-color-singlet type (or meson-baryon type) current $J_{\bar{D}\Sigma_c}(x)$ to interpolate the $\bar{D}\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$ [18]. For the technical details and numerical results, one can consult Ref. [18]. In the present work, we will focus on the reliability of the single pole approximation in the hadronic spectral density.

At the QCD side, the correlation function $\Pi(p)$ can be written as

$$\Pi(p) = -i\varepsilon^{ijk}\varepsilon^{i'j'k'}\int d^4x e^{ip\cdot x}\left\{-\text{Tr}[i\gamma_5 C_{m'm}(-x)i\gamma_5 U_{mm'}(x)]\,\text{Tr}\left[\gamma_\alpha D_{jj'}(x)\gamma_\beta C U_{ii'}^T(x)C\right]\gamma^\alpha\gamma_5 C_{kk'}(x)\gamma_5\gamma^\beta\right.$$
$$\left.+\text{Tr}\left[i\gamma_5 C_{m'm}(-x)i\gamma_5 U_{mi'}(x)\gamma_\beta C D_{jj'}^T(x)C\gamma_\alpha U_{im'}(x)\right]\gamma^\alpha\gamma_5 C_{kk'}(x)\gamma_5\gamma^\beta\right\}, \tag{4}$$

where $U_{ij}(x)$, $D_{ij}(x)$, and $C_{ij}(x)$ are the full u, d, and c quark propagators, respectively ($S_{ij}(x) = U_{ij}(x), D_{ij}(x)$),

$$S_{ij}(x) = \frac{i\delta_{ij}\not{x}}{2\pi^2 x^4} - \frac{\delta_{ij}\langle\bar{q}q\rangle}{12} - \frac{\delta_{ij}x^2\langle\bar{q}g_s\sigma Gq\rangle}{192} - \frac{ig_s G_{\alpha\beta}^a t_{ij}^a(\not{x}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\not{x})}{32\pi^2 x^2} - \frac{1}{8}\langle\bar{q}_j\sigma^{\mu\nu}q_i\rangle\sigma_{\mu\nu} + \cdots, \tag{5}$$

$$C_{ij}(x) = \frac{i}{(2\pi)^4}\int d^4k e^{-ik\cdot x}\left\{\frac{\delta_{ij}}{\not{k}-m_c} - \frac{g_s G_{\alpha\beta}^n t_{ij}^n}{4}\frac{\sigma^{\alpha\beta}(\not{k}+m_c) + (\not{k}+m_c)\sigma^{\alpha\beta}}{(k^2-m_c^2)^2}\right.$$
$$\left.-\frac{g_s^2(t^a t^b)_{ij}G_{\alpha\beta}^a G_{\mu\nu}^b(f^{\alpha\beta\mu\nu} + f^{\alpha\mu\beta\nu} + f^{\alpha\mu\nu\beta})}{4(k^2-m_c^2)^5} + \cdots\right\},$$

$$f^{\alpha\beta\mu\nu} = (\not{k}+m_c)\gamma^\alpha(\not{k}+m_c)\gamma^\beta(\not{k}+m_c)\gamma^\mu(\not{k}+m_c)\gamma^\nu(\not{k}+m_c), \tag{6}$$

$t^n = \frac{\lambda^n}{2}$, λ^n is the Gell-Mann matrix [35-37].

In Fig. 1, we plot two Feynman diagrams for the lowest order contributions, where the first diagram corresponds to the term with two Tr's and the second diagram corresponds to the term with one Tr in Eq. (4). The first Feynman diagram is factorizable and has the color factor 18, while the second Feynman diagram is non-factorizable and has the color factor 6. In the large N_c limit

$N_c \to \infty$, the contribution of the second Feynman diagram is greatly suppressed. In reality, the color number $N_c = 3$, the second Feynman diagram plays an important role.

In the second Feynman diagram, we can replace the lowest order heavy quark lines and (or) light quark lines with other terms in the full propagators in Eqs. (5), (6), and obtain other non-factorizable Feynman diagrams.

In the first Feynman diagram, we can also replace the lowest order heavy quark lines and (or) light quark lines

with other terms in the full propagators in Eqs. (5), (6), and obtain other factorizable Feynman diagrams. There are non-factorizable Feynman diagrams besides the factorizable Feynman diagrams, see Fig. 2. In Fig. 2, we plot the Feynman diagrams contributing to the vacuum condensates $\langle \bar{q} g_s \sigma G q \rangle^2$, which are the vacuum expectations of the quark-gluon operators of the order $O(\alpha_s)$ and not of the order $O(\alpha_s^2)$. In Fig. 3, we plot the non-factorizable Feynman diagrams of the order $O(\alpha_s^2)$ from the terms with two Tr's in Eq. (4), where the first, second, third, and fourth diagrams are non-planar Feynman diagrams, while the fifth, and sixth diagrams are planar Feynman diagrams. The first and the second Feynman diagrams are suppressed by a factor $\dfrac{1}{\sqrt{N_c}^4} \dfrac{1}{N_c} = \dfrac{1}{N_c^3}$ in the large N_c limit compared to the first Feynman diagram in Fig. 1, while the third, fourth, fifth, and sixth diagrams are suppressed by a factor $\dfrac{1}{\sqrt{N_c}^4} = \dfrac{1}{N_c^2}$. In reality, for the color number $N_c = 3$, the Feynman diagrams in Fig. 3 are suppressed by a factor $\left(\dfrac{4}{3} \dfrac{\alpha_s}{4\pi} \right)^2 \sim 0.0009$, and play a minor role.

In Fig. 4, we plot the non-factorizable Feynman diagrams contributing to the vacuum condensates $\langle \bar{q} g_s \sigma G q \rangle^2$
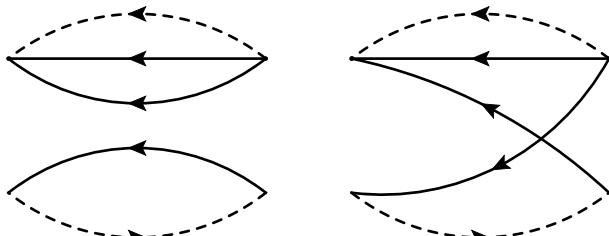

Fig. 1.   Feynman diagrams for the lowest order contributions for the baryon-meson type current $J(x)$, where the solid lines and the dashed lines denote the light quarks and heavy quarks, respectively. We have taken into account the finite spatial separation between the $\bar{c}(x) i \gamma_5 u(x)$ and $\varepsilon^{ijk} u_i^T(x) C \gamma_\alpha d_j(x) \gamma^\alpha \gamma_5 c_k(x)$ clusters in the current operator $J(x)$.
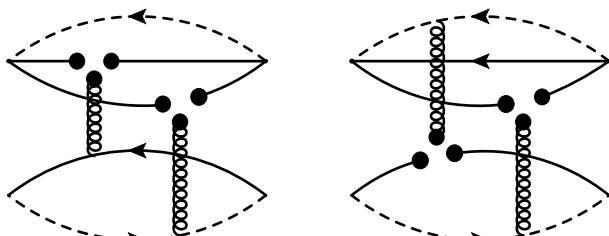

Fig. 2.   Non-factorizable Feynman diagrams contributing to the vacuum condensates $\langle \bar{q} g_s \sigma G q \rangle^2$ from the terms with two Tr's in Eq. (4), where the solid lines and dashed lines denote the light quarks and heavy quarks, respectively. Other diagrams obtained by interchanging of the light quark lines are implied.

for the meson-meson type currents. From the figure, we can see that the non-factorizable contributions begin at the order $O(\alpha_s^0)$ rather than at the order $O(\alpha_s^2)$, as argued in Ref. [38]. For the nonperturbative contributions, we absorb the strong coupling constant $g_s^2 = 4\pi\alpha_s$ into the vacuum condensates and count them as of the order $O(\alpha_s^0)$.

We insist on the viewpoint that the factorizable Feyn-
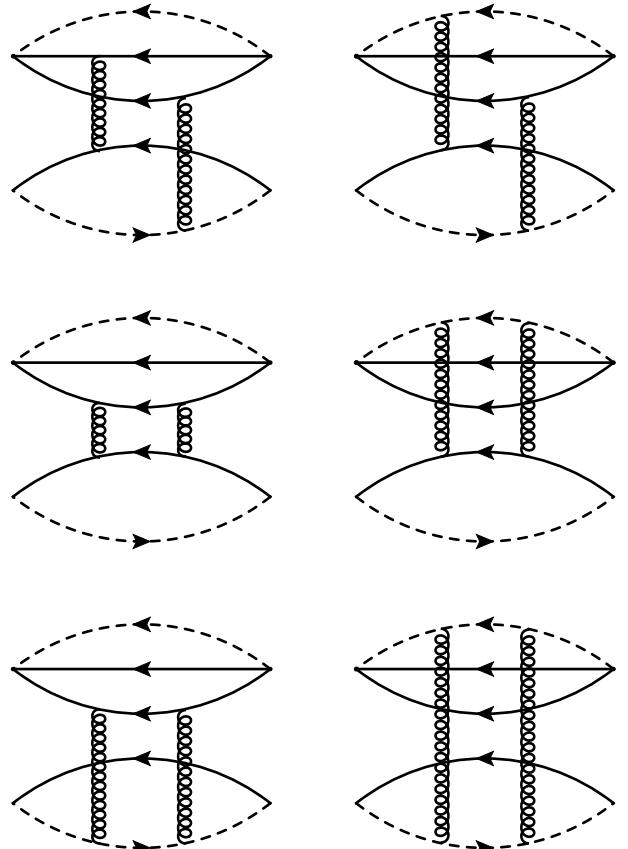
Fig. 3.   Non-factorizable Feynman diagrams of the order $O(\alpha_s^2)$ from the terms with two Tr's in Eq. (4), where the solid lines and dashed lines denote the light quarks and heavy quarks, respectively. Other diagrams obtained by interchanging of the light and heavy quark lines are implied.
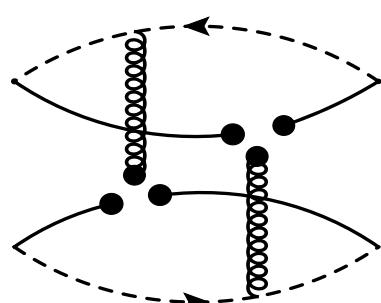
Fig. 4.   Non-factorizable Feynman diagrams contributing to the vacuum condensates $\langle \bar{q} g_s \sigma G q \rangle^2$ for the meson-meson type currents, where the solid lines and dashed lines denote the light quarks and heavy quarks, respectively.

man diagrams correspond to the two-particle reducible contributions, irrespective of the baryon-meson pair or the meson-meson pair, and give the masses of the two constituent particles. Then, the attractive interactions which originate from (or are embodied in) the non-factorizable Feynman diagrams attract the two constituent particles to form the molecular states. The non-factorizable Feynman diagrams are suppressed in the large N_c limit, which is consistent with the small bound energies of the pentaquark molecular states. The baryon-meson type or the color-singlet-color-singlet type currents couple potentially to the pentaquark molecular states.

On the other hand, the baryon-meson type currents also couple to the baryon-meson pairs besides the molecular states as there exist two-particle reducible contributions. The intermediate baryon-meson loops contribute a finite imaginary part to modify the dispersion relation at the hadron side [18]. Through calculations, we observe that the zero width approximation works well, and the couplings to the baryon-meson pairs can be neglected safely.

If we only take into account the non-factorizable Feynman diagrams shown in Figs. 1 and 2, even if we obtain the stable QCD sum rules, we cannot distinguish the diquark-diquark-antiquark type substructure or the baryon-meson type substructure, and cannot select the color-singlet-color-singlet type substructure and refer to it as the molecular state. We simply obtain a hidden-charm five-quark state with the spin-parity $J^P = \frac{1}{2}^-$. If we insist that it is a molecular state, then which diagram contributes to the masses of the baryon and meson constituents? In Ref. [39], the factorizable Feynman diagrams corresponding to the two-particle reducible contributions were subtracted, and only the non-factorizable Feynman diagrams were taken into account to study the pentaquark

states. We do not agree with that approach.

3   QCD sum rules for the $P_c(4312)$ decays as a pentaquark molecular state

Here, we write down the three-point correlation functions $\Pi_5(p,q)$ and $\Pi_\mu(p,q)$ in the QCD sum rules,

$$\Pi_5(p,q) = i^2 \int d^4x d^4y e^{ip\cdot x} e^{iq\cdot y} \langle 0|T\left\{J_5(x)J_N(y)\bar{J}(0)\right\}|0\rangle,$$
(7)

$$\Pi_\mu(p,q) = i^2 \int d^4x d^4y e^{ip\cdot x} e^{iq\cdot y} \langle 0|T\left\{J_\mu(x)J_N(y)\bar{J}(0)\right\}|0\rangle,$$
(8)

where

$$J_5(x) = \bar{c}(x)i\gamma_5 c(x),$$
$$J_\mu(x) = \bar{c}(x)\gamma_\mu c(x),$$
$$J_N(y) = \varepsilon^{ijk}u_i^T(y)C\gamma_\alpha u_j(y)\gamma^\alpha\gamma_5 d_k(y),$$
$$J(0) = \bar{c}(0)i\gamma_5 u(0)\varepsilon^{ijk}u_i^T(0)C\gamma_\alpha d_j(0)\gamma^\alpha\gamma_5 c_k(0),$$
(9)

and i, j, k are color indices. We choose the currents $J_5(x)$, $J_\mu(x)$, $J_N(y)$, and $J(0)$ to interpolate η_c, J/ψ, p, and $P_c(4312)$, respectively. Thereafter we will denote the proton p as N to avoid confusion due to the four-momentum p_μ.

At the hadron side, we insert a complete set of intermediate hadron states with the same quantum numbers as the current operators $J_5(x)$, $J_\mu(x)$, $J_N(y)$, and $J(0)$ into the correlation functions $\Pi_5(p,q)$ and $\Pi_\mu(p,q)$ to obtain the hadronic representation [35, 40, 41]. After isolating the pole terms of the ground states, we obtain the following results:

$$\Pi_5(p,q) = \frac{f_{\eta_c}m_{\eta_c}^2\lambda_P\lambda_N}{2m_c}\frac{-iu(q)\langle\eta_c(p)N(q)|P(p')\rangle\bar{u}(p')}{\left(m_P^2-p'^2\right)\left(m_{\eta_c}^2-p^2\right)\left(m_N^2-q^2\right)} + \cdots$$

$$= \frac{f_{\eta_c}m_{\eta_c}^2\lambda_P\lambda_N}{2m_c}\frac{-i(\slashed{q}+m_N)(\slashed{p}'+m_P)}{\left(m_P^2-p'^2\right)\left(m_{\eta_c}^2-p^2\right)\left(m_N^2-q^2\right)}ig_5+\cdots,$$
(10)

$$\Pi_\mu(p,q) = f_{J/\psi}m_{J/\psi}\lambda_P\lambda_N\frac{-i\varepsilon_\mu u(q)\langle J/\psi(p)N(q)|P(p')\rangle\bar{u}(p')}{\left(m_P^2-p'^2\right)\left(m_{J/\psi}^2-p^2\right)\left(m_N^2-q^2\right)} + \cdots$$

$$= f_{J/\psi}m_{J/\psi}\lambda_P\lambda_N\frac{-i(\slashed{q}+m_N)\left(g_V\gamma^\alpha - i\frac{g_T}{m_P+m_N}\sigma^{\alpha\beta}p_\beta\right)\gamma_5(\slashed{p}'+m_P)}{\left(m_P^2-p'^2\right)\left(m_{J/\psi}^2-p^2\right)\left(m_N^2-q^2\right)}\left(-g_{\mu\alpha}+\frac{p_\mu p_\alpha}{p^2}\right)+\cdots,$$
(11)

where we have used the definitions

$$\langle 0|J(0)|P_c(p')\rangle = \lambda_P U(p',s),$$

$$\langle 0|J_N(0)|N(q)\rangle = \lambda_N U(q,s),$$

$$\langle 0|J_\mu(0)|J/\psi(p)\rangle = f_{J/\psi} m_{J/\psi} \varepsilon_\mu(p,s),$$

$$\langle 0|J_5(0)|\eta_c(p)\rangle = \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c}, \tag{12}$$

$$\langle \eta_c(p)N(q)|P(p')\rangle = i g_5 \bar{u}(q) u(p'),$$

$$\langle J/\psi(p)N(q)|P(p')\rangle = \bar{u}(q)\varepsilon_\alpha^* \left(g_V \gamma^\alpha - i\frac{g_T}{m_P + m_N}\sigma^{\alpha\beta} p_\beta \right)\gamma_5 u(p'), \tag{13}$$

Here, the g_5, g_V, and g_T are the hadronic coupling constants, $U(p,s)$ and $U(q,s)$ are the Dirac spinors, λ_P and λ_N are the pole residues, $f_{J/\psi}$ and f_{η_c} are the decay constants, and ε_μ is the polarization vector of J/ψ.

It is important to choose the pertinent structures to study the hadronic coupling constants. If $\Pi_{5,H}(p,q) = \Pi_{5,\text{QCD}}(p,q)$ and $\Pi_{\mu,H}(p,q) = \Pi_{\mu,\text{QCD}}(p,q)$, we expect that the two relations $\text{Tr}[\Pi_{5,H}(p,q)\Gamma] = \text{Tr}[\Pi_{5,\text{QCD}}(p,q)\Gamma]$ and

$\text{Tr}[\Pi_{\mu,H}(p,q)\Gamma'] = \text{Tr}[\Pi_{\mu,\text{QCD}}(p,q)\Gamma']$ also exist, where the subscripts H and QCD denote the hadron side and QCD side of the correlation functions, respectively. Γ and Γ' are some Dirac γ-matrixes.

In this article, we choose $\Gamma = \sigma_{\mu\nu}$, $i\gamma_\mu$, $\Gamma' = \gamma_5 \not{z}$, γ_5,

$$\frac{1}{4}\text{Tr}[\Pi_5(p,q)\sigma_{\mu\nu}] = \Pi_5(p'^2,p^2,q^2)i(p_\mu q_\nu - q_\mu p_\nu) + \cdots,$$

$$\frac{1}{4}\text{Tr}[\Pi_5(p,q)i\gamma_\mu] = \overline{\Pi}_5(p'^2,p^2,q^2)iq_\mu + \cdots,$$

$$\frac{1}{4}\text{Tr}[\Pi_\mu(p,q)\gamma_5 \not{z}] = \Pi_A(p'^2,p^2,q^2)iq_\mu p\cdot z + \cdots,$$

$$\frac{1}{4}\text{Tr}[\Pi_\mu(p,q)\gamma_5] = \Pi_B(p'^2,p^2,q^2)iq_\mu + \cdots, \tag{14}$$

and choose the tensor structures $p_\mu q_\nu - q_\mu p_\nu$, q_μ, $q_\mu p\cdot z$ and q_μ to study the hadronic coupling constants g_5, g_V and g_T, respectively, where z_μ is a four-vector.

Now we write down the components $\Pi_5(p'^2,p^2,q^2)$, $\overline{\Pi}_5(p'^2,p^2,q^2)$, $\Pi_A(p'^2,p^2,q^2)$, and $\Pi_B(p'^2,p^2,q^2)$ explicitly,

$$\Pi_5(p'^2,p^2,q^2) = \frac{f_{\eta_c} m_{\eta_c}^2 \lambda_P \lambda_N}{2m_c} \frac{g_5}{\left(m_P^2 - p'^2\right)\left(m_{\eta_c}^2 - p^2\right)\left(m_N^2 - q^2\right)} + \frac{1}{(m_P^2 - p'^2)(m_{\eta_c}^2 - p^2)} \int_{s_N^0}^\infty dt \frac{\rho_{PN'}^5(p'^2,p^2,t)}{t - q^2}$$

$$+ \frac{1}{(m_P^2 - p'^2)(m_N^2 - q^2)} \int_{s_{\eta_c}^0}^\infty dt \frac{\rho_{P\eta_c}^5(p'^2,t,q^2)}{t - p^2} + \frac{1}{(m_{\eta_c}^2 - p^2)(m_N^2 - q^2)} \int_{s_P^0}^\infty dt \frac{\rho_{P'\eta_c}^5(t,p^2,q^2) + \rho_{P'N}^5(t,p^2,q^2)}{t - p'^2} + \cdots, \tag{15}$$

$$\overline{\Pi}_5(p'^2,p^2,q^2) = \frac{f_{\eta_c} m_{\eta_c}^2 \lambda_P \lambda_N}{2m_c} \frac{(m_P + m_N) g_5}{\left(m_P^2 - p'^2\right)\left(m_{\eta_c}^2 - p^2\right)\left(m_N^2 - q^2\right)} + \frac{1}{(m_P^2 - p'^2)(m_{\eta_c}^2 - p^2)} \int_{s_N^0}^\infty dt \frac{\overline{\rho}_{PN'}^5(p'^2,p^2,t)}{t - q^2}$$

$$+ \frac{1}{(m_P^2 - p'^2)(m_N^2 - q^2)} \int_{s_{\eta_c}^0}^\infty dt \frac{\overline{\rho}_{P\eta_c}^5(p'^2,t,q^2)}{t - p^2} + \frac{1}{(m_{\eta_c}^2 - p^2)(m_N^2 - q^2)} \int_{s_P^0}^\infty dt \frac{\overline{\rho}_{P'\eta_c}^5(t,p^2,q^2) + \overline{\rho}_{P'N}^5(t,p^2,q^2)}{t - p'^2} + \cdots, \tag{16}$$

$$\Pi_A(p'^2,p^2,q^2) = f_{J/\psi} m_{J/\psi} \lambda_P \lambda_N \frac{g_T - g_V}{\left(m_P^2 - p'^2\right)\left(m_{J/\psi}^2 - p^2\right)\left(m_N^2 - q^2\right)} + \frac{1}{(m_P^2 - p'^2)(m_{J/\psi}^2 - p^2)} \int_{s_N^0}^\infty dt \frac{\rho_{PN'}^A(p'^2,p^2,t)}{t - q^2}$$

$$+ \frac{1}{(m_P^2 - p'^2)(m_N^2 - q^2)} \int_{s_{J/\psi}^0}^\infty dt \frac{\rho_{P\psi'}^A(p'^2,t,q^2)}{t - p^2} + \frac{1}{(m_{J/\psi}^2 - p^2)(m_N^2 - q^2)} \int_{s_P^0}^\infty dt \frac{\rho_{P'J/\psi}^A(t,p^2,q^2) + \rho_{P'N}^A(t,p^2,q^2)}{t - p'^2} + \cdots, \tag{17}$$

$$\Pi_B(p'^2,p^2,q^2) = f_{J/\psi} m_{J/\psi} \lambda_P \lambda_N \frac{(m_P - m_N) g_V - g_T \dfrac{m_{J/\psi}^2}{m_P + m_N}}{\left(m_P^2 - p'^2\right)\left(m_{J/\psi}^2 - p^2\right)\left(m_N^2 - q^2\right)} + \frac{1}{(m_P^2 - p'^2)(m_{J/\psi}^2 - p^2)} \int_{s_N^0}^\infty dt \frac{\rho_{PN'}^B(p'^2,p^2,t)}{t - q^2}$$

$$+ \frac{1}{(m_P^2 - p'^2)(m_N^2 - q^2)} \int_{s_{J/\psi}^0}^\infty dt \frac{\rho_{P\psi'}^B(p'^2,t,q^2)}{t - p^2} + \frac{1}{(m_{J/\psi}^2 - p^2)(m_N^2 - q^2)} \int_{s_P^0}^\infty dt \frac{\rho_{P'J/\psi}^B(t,p^2,q^2) + \rho_{P'N}^B(t,p^2,q^2)}{t - p'^2} + \cdots, \tag{18}$$

where we introduce the formal functions $\rho_{PN'}^5(p'^2,p^2,t)$, $\rho_{P\eta_c}^5(p'^2,t,q^2)$, $\rho_{P'\eta_c}^5(t,p^2,q^2)$, $\rho_{P'N}^5(t,p^2,q^2)$, $\overline{\rho}_{PN'}^5(p'^2,p^2,t)$,

$\overline{\rho}_{P\eta_c}^5(p'^2,t,q^2)$, $\overline{\rho}_{P'\eta_c}^5(t,p^2,q^2)$, $\overline{\rho}_{P'N}^5(t,p^2,q^2)$, $\rho_{PN'}^A(p'^2,p^2,t)$, $\rho_{P\psi'}^A(p'^2,t,q^2)$, $\rho_{P'J/\psi}^A(t,p^2,q^2)$, $\rho_{P'N}^A(t,p^2,q^2)$, $\rho_{PN'}^B(p'^2,p^2,t)$,

$\rho_{P'\psi'}^B(p'^2,t,q^2)$, $\rho_{P'J/\psi}^B(t,p^2,q^2)$, and $\rho_{P'N}^B(t,p^2,q^2)$ to para-meterize the transitions between the ground states and the excited states. $s_{\eta_c}^0$, $s_{J/\psi}^0$, s_N^0, and s_P^0 are the threshold para-

meters for the radial excited states.

Now we smear the indexes 5, A, B, etc, and rewrite (components of) the correlation functions $\Pi_H(p'^2,p^2,q^2)$ at the hadron side as

$$\Pi_H(p'^2,p^2,q^2) = \int_{(m_{\eta_c}+m_N)^2}^{s_P^0} \mathrm{d}s' \int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{\rho_H(s',s,u)}{(s'-p'^2)(s-p^2)(u-q^2)}$$
$$+ \int_{s_P^0}^{\infty} \mathrm{d}s' \int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{\rho_H(s',s,u)}{(s'-p'^2)(s-p^2)(u-q^2)} + \cdots, \tag{19}$$

through dispersion relation, and take $\eta_c = \eta_c$, J/ψ for sim-plicity, where $\rho_H(s',s,u)$ represents the hadronic spectral densities.

We carry out the operator product expansion at the QCD side, and write (components of) the correlation functions $\Pi_{\mathrm{QCD}}(p'^2,p^2,q^2)$ as

$$\Pi_{\mathrm{QCD}}(p'^2,p^2,q^2) = \int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{\rho_{\mathrm{QCD}}(p'^2,s,u)}{(s-p^2)(u-q^2)} + \cdots, \tag{20}$$

through dispersion relation, where the $\rho_{\mathrm{QCD}}(p'^2,s,u)$ rep-resents the QCD spectral densities. As the QCD spectral density $\rho_{\mathrm{QCD}}(s',s,u)$ does not exist,

$$\rho_{\mathrm{QCD}}(s',s,u) = \lim_{\epsilon\to 0} \frac{\mathrm{Im}_{s'}\,\Pi_{\mathrm{QCD}}(s'+i\epsilon,s,u)}{\pi} = 0, \tag{21}$$

we can write the QCD spectral density $\rho_{\mathrm{QCD}}(p'^2,s,u)$ as $\rho_{\mathrm{QCD}}(s,u)$ for simplicity.

Now we match the hadron side with the QCD side of the correlation functions, and carry out the integration over $\mathrm{d}s'$ first, to obtain the solid duality [31],

$$\int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{\rho_{\mathrm{QCD}}(s,u)}{(s-p^2)(u-q^2)} = \int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{1}{(s-p^2)(u-q^2)} \left[\int_{(\eta_c+m_N)^2}^{\infty} \mathrm{d}s' \frac{\rho_H(s',s,u)}{s'-p'^2} \right]. \tag{22}$$

It is impossible to carry out the integration over s' ex-plicitly due to the unknown functions $\rho_{P'\eta_c}^5(t,p^2,q^2)$, $\rho_{P'N}^5(t,p^2,q^2)$, $\overline{\rho}_{P'\eta_c}^5(t,p^2,q^2)$, $\overline{\rho}_{P'N}^5(t,p^2,q^2)$, $\rho_{P'J/\psi}^A(t,p^2,q^2)$, $\rho_{P'N}^A(t,p^2,q^2)$, $\rho_{P'J/\psi}^B(t,p^2,q^2)$, and $\rho_{P'N}^B(t,p^2,q^2)$. Now we introduce the parameters C_5, $\overline{C}_5$, C_A, and C_B to paramet-erize the net effects,

$$C_5 = \int_{s_P^0}^{\infty} \mathrm{d}t \frac{\rho_{P'\eta_c}^5(t,p^2,q^2)+\rho_{P'N}^5(t,p^2,q^2)}{t-p'^2}, \qquad \overline{C}_5 = \int_{s_P^0}^{\infty} \mathrm{d}t \frac{\overline{\rho}_{P'\eta_c}^5(t,p^2,q^2)+\overline{\rho}_{P'N}^5(t,p^2,q^2)}{t-p'^2},$$

$$C_A = \int_{s_P^0}^{\infty} \mathrm{d}t \frac{\rho_{P'J/\psi}^A(t,p^2,q^2)+\rho_{P'N}^A(t,p^2,q^2)}{t-p'^2}, \qquad C_B = \int_{s_P^0}^{\infty} \mathrm{d}t \frac{\rho_{P'J/\psi}^B(t,p^2,q^2)+\rho_{P'N}^B(t,p^2,q^2)}{t-p'^2}. \tag{23}$$

Here, we write down the quark-hadron duality explicitly,

$$\int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{\rho_{\mathrm{QCD}}^5(s,u)}{(s-p^2)(u-q^2)} = \frac{f_{\eta_c} m_{\eta_c}^2 \lambda_P \lambda_N}{2m_c} \frac{g_5}{\left(m_P^2-p'^2\right)\left(m_{\eta_c}^2-p^2\right)\left(m_N^2-q^2\right)} + \frac{C_5}{(m_{\eta_c}^2-p^2)(m_N^2-q^2)}, \tag{24}$$

$$\int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{\overline{\rho}_{\mathrm{QCD}}^5(s,u)}{(s-p^2)(u-q^2)} = \frac{f_{\eta_c} m_{\eta_c}^2 \lambda_P \lambda_N}{2m_c} \frac{(m_P+m_N)g_5}{\left(m_P^2-p'^2\right)\left(m_{\eta_c}^2-p^2\right)\left(m_N^2-q^2\right)} + \frac{\overline{C}_5}{(m_{\eta_c}^2-p^2)(m_N^2-q^2)}, \tag{25}$$

$$\int_{4m_c^2}^{s_{J/\psi}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{\rho_{\mathrm{QCD}}^A(s,u)}{(s-p^2)(u-q^2)} = f_{J/\psi} m_{J/\psi} \lambda_P \lambda_N \frac{g_T-g_V}{\left(m_P^2-p'^2\right)\left(m_{J/\psi}^2-p^2\right)\left(m_N^2-q^2\right)} + \frac{C_A}{(m_{J/\psi}^2-p^2)(m_N^2-q^2)}, \tag{26}$$

$$\int_{4m_c^2}^{s_{J/\psi}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \frac{\rho_{\mathrm{QCD}}^B(s,u)}{(s-p^2)(u-q^2)} = f_{J/\psi} m_{J/\psi} \lambda_P \lambda_N \frac{(m_P-m_N)g_V-g_T\dfrac{m_{J/\psi}^2}{m_P+m_N}}{\left(m_P^2-p'^2\right)\left(m_{J/\psi}^2-p^2\right)\left(m_N^2-q^2\right)} + \frac{C_B}{(m_{J/\psi}^2-p^2)(m_N^2-q^2)}. \tag{27}$$

We set $p'^2 = p^2$ and perform a double Borel transform with respect to the variables $P^2 = -p^2$ and $Q^2 = -q^2$, respectively, to obtain the QCD sum rules,

$$
\frac{f_{\eta_c} m_{\eta_c}^2 \lambda_P \lambda_N}{2m_c} \frac{g_5}{m_P^2 - m_{\eta_c}^2} \left[\exp\left(-\frac{m_{\eta_c}^2}{T_1^2}\right) - \exp\left(-\frac{m_P^2}{T_1^2}\right) \right] \exp\left(-\frac{m_N^2}{T_2^2}\right) + C_5 \exp\left(-\frac{m_{\eta_c}^2}{T_1^2} - \frac{m_N^2}{T_2^2}\right)
$$
$$
= \int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \, \rho_{\mathrm{QCD}}^5(s,u) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right), \tag{28}
$$

$$
\frac{f_{\eta_c} m_{\eta_c}^2 \lambda_P \lambda_N}{2m_c} \frac{(m_P + m_N) g_5}{m_P^2 - m_{\eta_c}^2} \left[\exp\left(-\frac{m_{\eta_c}^2}{T_1^2}\right) - \exp\left(-\frac{m_P^2}{T_1^2}\right) \right] \exp\left(-\frac{m_N^2}{T_2^2}\right) + \overline{C}_5 \exp\left(-\frac{m_{\eta_c}^2}{T_1^2} - \frac{m_N^2}{T_2^2}\right)
$$
$$
= \int_{4m_c^2}^{s_{\eta_c}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \, \overline{\rho}_{\mathrm{QCD}}^5(s,u) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right), \tag{29}
$$

$$
f_{J/\psi} m_{J/\psi} \lambda_P \lambda_N \frac{g_{V/T}}{m_P^2 - m_{J/\psi}^2} \left[\exp\left(-\frac{m_{J/\psi}^2}{T_1^2}\right) - \exp\left(-\frac{m_P^2}{T_1^2}\right) \right] \exp\left(-\frac{m_N^2}{T_2^2}\right) + C_{V/T} \exp\left(-\frac{m_{J/\psi}^2}{T_1^2} - \frac{m_N^2}{T_2^2}\right)
$$
$$
= \int_{4m_c^2}^{s_{J/\psi}^0} \mathrm{d}s \int_0^{s_N^0} \mathrm{d}u \, \rho_{\mathrm{QCD}}^{V/T}(s,u) \exp\left(-\frac{s}{T_1^2} - \frac{u}{T_2^2}\right), \tag{30}
$$

$$
C_V = \left[\frac{m_{J/\psi}^2}{m_P + m_N} C_A + C_B \right] \frac{m_P + m_N}{m_P^2 - m_{J/\psi}^2 - m_N^2}, \qquad C_T = \left[(m_P - m_N) C_A + C_B \right] \frac{m_P + m_N}{m_P^2 - m_{J/\psi}^2 - m_N^2},
$$
$$
\rho_{\mathrm{QCD}}^V(s,u) = \left[\frac{m_{J/\psi}^2}{m_P + m_N} \rho_{\mathrm{QCD}}^A(s,u) + \rho_{\mathrm{QCD}}^B(s,u) \right] \frac{m_P + m_N}{m_P^2 - m_{J/\psi}^2 - m_N^2},
$$
$$
\rho_{\mathrm{QCD}}^T(s,u) = \left[(m_P - m_N) \rho_{\mathrm{QCD}}^A(s,u) + \rho_{\mathrm{QCD}}^B(s,u) \right] \frac{m_P + m_N}{m_P^2 - m_{J/\psi}^2 - m_N^2}, \tag{31}
$$

$$
\rho_{\mathrm{QCD}}^5(s,u) = \frac{m_c}{4096\pi^6} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, u^2 - \frac{m_c^3}{36864\pi^4 T_1^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{1}{x^3} u^2 \delta\left(s - \widetilde{m}_c^2\right)
$$
$$
+ \frac{m_c}{12288\pi^4 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{1-x}{x^2} u^2 \delta\left(s - \widetilde{m}_c^2\right) + \frac{m_c}{24576\pi^4 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{1}{x(1-x)} u^2 \delta\left(s - \widetilde{m}_c^2\right)
$$
$$
+ \frac{m_c}{2048\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x - \frac{m_c}{2304\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{1}{x} u \delta\left(s - \widetilde{m}_c^2\right) + \frac{m_c \langle \bar{q}q \rangle \langle \bar{q} g_s \sigma G q \rangle}{576\pi^2} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{1}{x(1-x)} \delta\left(s - \widetilde{m}_c^2\right) \delta(u)
$$
$$
+ \frac{m_c \langle \bar{q}q \rangle \langle \bar{q} g_s \sigma G q \rangle}{576\pi^2 T_2^2} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \delta(u) + \frac{m_c \langle \bar{q} g_s \sigma G q \rangle^2}{2304\pi^2 T_1^2} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{1}{x(1-x)} \delta\left(s - \widetilde{m}_c^2\right) \delta(u), \tag{32}
$$

$$
\overline{\rho}_{\mathrm{QCD}}^5(s,u) = \frac{1}{2048\pi^6} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, x s u^2 + \frac{m_c \langle \bar{q}q \rangle}{384\pi^4} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, u + \frac{m_c \langle \bar{q} g_s \sigma G q \rangle}{768\pi^4} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{u}{x} \delta\left(s - \widetilde{m}_c^2\right)
$$
$$
+ \frac{\langle \bar{q}q \rangle^2}{48\pi^2} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, x s \, \delta(u) + \frac{m_c^2}{18432\pi^4 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{u^2}{x^2} \left(2 - \frac{s}{T_1^2}\right) \delta\left(s - \widetilde{m}_c^2\right)
$$
$$
+ \frac{1}{768\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, u s \, \delta\left(s - \widetilde{m}_c^2\right) + \frac{1}{3072\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, (u + 3xs)
$$
$$
+ \frac{1}{12288\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, (1+x)\left(2us + u^2\right) \delta\left(s - \widetilde{m}_c^2\right) + \frac{1}{12288\pi^4 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{2-x}{1-x} s u^2 \delta\left(s - \widetilde{m}_c^2\right)
$$
$$
- \frac{m_c^3}{3456\pi^2 T_1^4} \langle \bar{q}q \rangle \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{u}{x^3} \delta\left(s - \widetilde{m}_c^2\right) + \frac{m_c \langle \bar{q}q \rangle}{1152\pi^2 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} \mathrm{d}x \, \frac{1-x}{x^2} u \delta\left(s - \widetilde{m}_c^2\right)
$$

$$
\begin{aligned}
&+ \frac{m_c\langle\bar{q}q\rangle}{2304\pi^2 T_1^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1}{x(1-x)}u\delta\left(s-\widetilde{m}_c^2\right) - \frac{m_c\langle\bar{q}q\rangle}{1152\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\delta(u)\\[4pt]
&+ \frac{m_c^2\langle\bar{q}q\rangle^2}{432 T_1^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1}{x^2}\left(2-\frac{s}{T_1^2}\right)\delta\left(s-\widetilde{m}_c^2\right)\delta(u) + \frac{\langle\bar{q}q\rangle^2}{864 T_2^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\left[4s\delta\left(s-\widetilde{m}_c^2\right)+1\right]\delta(u)\\[4pt]
&+ \frac{\langle\bar{q}q\rangle^2}{288 T_1^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{2-x}{1-x}s\delta\left(s-\widetilde{m}_c^2\right)\delta(u) - \frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{96\pi^4 T_2^2}\int_{x_i}^{x_f}dx\,xs\delta(u) + \frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{288\pi^2}\int_{x_i}^{x_f}dx\,s\delta\left(s-\widetilde{m}_c^2\right)\delta(u)\\[4pt]
&+ \frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{576\pi^4}\int_{x_i}^{x_f}dx\,\frac{1}{1-x}s\delta\left(s-\widetilde{m}_c^2\right)\delta(u) + \frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{384\pi^2 T_2^2}\int_{x_i}^{x_f}dx\,s\delta(u) + \frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{2304\pi^2 T_2^2}\int_{x_i}^{x_f}dx\,\delta(u)\\[4pt]
&- \frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{1728\pi^2 T_2^2}\int_{x_i}^{x_f}dx\,s\delta\left(s-\widetilde{m}_c^2\right)\delta(u) - \frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{13824\pi^2 T_2^2}\int_{x_i}^{x_f}dx\,\frac{1}{1-x}s\delta\left(s-\widetilde{m}_c^2\right)\delta(u)\\[4pt]
&+ \frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{13824\pi^2 T_1^2}\int_{x_i}^{x_f}dx\,s\delta\left(s-\widetilde{m}_c^2\right)\delta(u) - \frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{13824\pi^2 T_1^2}\int_{x_i}^{x_f}dx\,\frac{1}{1-x}s\delta\left(s-\widetilde{m}_c^2\right)\delta(u) - \frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{6912\pi^2}\int_{x_i}^{x_f}dx\,\delta\left(s-\widetilde{m}_c^2\right)\delta(u),
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\rho_{\text{QCD}}^A(s,u) =\ & \frac{m_c\langle\bar{q}q\rangle^2}{48\pi^2}\int_{x_i}^{x_f}dx\,\delta(u) - \frac{5m_c\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{576\pi^2 T_2^2}\int_{x_i}^{x_f}dx\,\delta(u) + \frac{m_c\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{288\pi^2}\int_{x_i}^{x_f}dx\,\frac{1}{x}\delta\left(s-\widetilde{m}_c^2\right)\delta(u)\\[4pt]
&+ \frac{m_c}{9216\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1}{x(1-x)}u\delta\left(s-\widetilde{m}_c^2\right) - \frac{m_c^3\langle\bar{q}q\rangle^2}{432 T_1^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1}{x^3}\delta\left(s-\widetilde{m}_c^2\right)\delta(u)\\[4pt]
&+ \frac{m_c\langle\bar{q}q\rangle^2}{144 T_1^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1-x}{x^2}\delta\left(s-\widetilde{m}_c^2\right)\delta(u) + \frac{m_c\langle\bar{q}q\rangle^2}{288 T_2^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\left(\frac{1}{x}-\frac{2}{3}\right)\delta\left(s-\widetilde{m}_c^2\right)\delta(u)\\[4pt]
&+ \frac{m_c\langle\bar{q}q\rangle^2}{864 T_2^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1-2x}{1-x}\delta\left(s-\widetilde{m}_c^2\right)\delta(u) - \frac{m_c\langle\bar{q}q\rangle^2}{864 T_1^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1}{x(1-x)}\delta\left(s-\widetilde{m}_c^2\right)\delta(u)\\[4pt]
&- \frac{m_c\langle\bar{q}g_s\sigma Gq\rangle^2}{4608\pi^2 T_2^2}\int_{x_i}^{x_f}dx\,\frac{1}{x(1-x)}\delta\left(s-\widetilde{m}_c^2\right)\delta(u) - \frac{m_c\langle\bar{q}g_s\sigma Gq\rangle^2}{13824\pi^2 T_1^2}\int_{x_i}^{x_f}dx\,\frac{1}{x(1-x)}\delta\left(s-\widetilde{m}_c^2\right)\delta(u),
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\rho_{\text{QCD}}^B(s,u) =\ & -\frac{1}{4096\pi^6}\int_{x_i}^{x_f}dx\left[xs+x(1-x)\left(s-\widetilde{m}_c^2\right)\right]u^2 - \frac{\langle\bar{q}q\rangle^2}{24\pi^2}\int_{x_i}^{x_f}dx\left[xs+x(1-x)\left(s-\widetilde{m}_c^2\right)\right]\delta(u)\\[4pt]
&- \frac{m_c^2}{36864\pi^4 T_1^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1}{x^2}\left(2-\frac{s}{T_1^2}\right)u^2\delta\left(s-\widetilde{m}_c^2\right) + \frac{m_c^2}{36864\pi^4 T_1^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\,\frac{1-x}{x^2}u^2\delta\left(s-\widetilde{m}_c^2\right)\\[4pt]
&- \frac{1}{2304\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\left[\frac{(3-2x)us}{2(1-x)}\delta\left(s-\widetilde{m}_c^2\right)+u\right] - \frac{1}{2048\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\left[\frac{4u}{9}+xs+x(1-x)\left(s-\widetilde{m}_c^2\right)\right]\\[4pt]
&+ \frac{1}{73728\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\left(\frac{x}{1-x}\frac{s}{T_1^2}-1\right)u^2\delta\left(s-\widetilde{m}_c^2\right) + \frac{m_c^2\langle\bar{q}q\rangle^2}{216 T_1^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\left[-\frac{1}{x^2}\left(2-\frac{s}{T_1^2}\right)+\frac{1-x}{x^2}\right]\delta\left(s-\widetilde{m}_c^2\right)\delta(u)\\[4pt]
&- \frac{\langle\bar{q}q\rangle^2}{432 T_2^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\left[\frac{(3-2x)s}{1-x}\delta\left(s-\widetilde{m}_c^2\right)+3\right]\delta(u) + \frac{\langle\bar{q}q\rangle^2}{432}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{x_i}^{x_f}dx\left(\frac{x}{1-x}\frac{s}{T_1^2}-1\right)\delta\left(s-\widetilde{m}_c^2\right)\delta(u)\\[4pt]
&- \frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{576\pi^2}\int_{x_i}^{x_f}dx\left[1+\frac{1+2x}{1-x}s\delta\left(s-\widetilde{m}_c^2\right)\right]\delta(u) + \frac{\langle\bar{q}q\rangle\langle\bar{q}g_s\sigma Gq\rangle}{64\pi^2 T_2^2}\int_{x_i}^{x_f}dx\left[xs+x(1-x)\left(s-\widetilde{m}_c^2\right)\right]\delta(u)\\[4pt]
&+ \frac{\langle\bar{q}g_s\sigma Gq\rangle^2}{2304\pi^2 T_2^2}\int_{x_i}^{x_f}dx\left[1+\frac{1+2x}{1-x}s\delta\left(s-\widetilde{m}_c^2\right)\right]\delta(u),
\end{aligned}
\tag{35}
$$

where $x_f = \dfrac{1+\sqrt{1-4m_c^2/s}}{2}$, $x_i = \dfrac{1-\sqrt{1-4m_c^2/s}}{2}$, $\widetilde{m}_c^2 = \dfrac{m_c^2}{x(1-x)}$, and $\int_{x_i}^{x_f}dx \to \int_0^1 dx$, when the δ function $\delta(s-\widetilde{m}_c^2)$ appears.

In this study, we carried out the operator product expansion to the vacuum condensates up to dimension 10,

and assumed vacuum saturation for the higher dimension vacuum condensates. As the vacuum condensates are vacuum expectations of the quark-gluon operators, we take the truncations $n \leqslant 10$ and $k \leqslant 1$ in a consistent way, where the operators of the orders $O(\alpha_s^k)$ with $k > 1$ are neglected. Furthermore, we set the two Borel parameters to be $T_1^2 = T_2^2 = T^2$ for simplicity. If we take T_1^2 and T_2^2 as two independent parameters, it is difficult to obtain stable QCD sum rules. In numerical calculations, we take C_5, $\overline{C}_5$, C_V, and C_T as free parameters and choose the suitable values to obtain the stable QCD sum rules.

In the operator product expansion for the correlation functions $\Pi_5(p, q)$ and $\Pi_\mu(p, q)$, if we take into account the finite spatial separation between the clusters $\bar{c}(0)i\gamma_5 u(0)$ and $\varepsilon^{ijk}u_i^T(0)C\gamma_\alpha d_j(0)\gamma^\alpha\gamma_5 c_k(0)$ in the current operator $J(0)$, the current $J(0)$ is modified to

$$J(0) = \bar{c}(\epsilon)i\gamma_5 u(\epsilon)\varepsilon^{ijk}u_i^T(0)C\gamma_\alpha d_j(0)\gamma^\alpha\gamma_5 c_k(0), \quad (36)$$

By adding a small four-vector ϵ, the Feynman diagrams for the decays into the charmonium states are made non-factorizable; see the first Feynman diagram in Fig. 5, where we split the point 0 into two points to site the baryon and meson clusters. In the limit $\epsilon \to 0$, the lowest order Feynman diagrams for the decays into the charmonium states are factorizable; see the second Feynman diagram in Fig. 5. We observed in our calculations that there are both connected and disconnected Feynman diagrams contributing to the decays. The non-factorizable contributions begin at the order $O(\sqrt{\alpha_s})$ due to the quark-gluon operators $\bar{q}g_s\sigma_{\alpha\beta}G^{\alpha\beta}q$, while at the order $O(\alpha_s)$ of the quark-gluon operators, the non-factorizable contributions are of the forms $\langle\frac{\alpha_s}{\pi}GG\rangle$ and $\langle\bar{q}g_s\sigma Gq\rangle^2$. We absorb the strong coupling constant $g_s^2 = 4\pi\alpha_s$ into the vacuum condensates and count them as of the order $O(\alpha_s^0)$. In Fig. 6, we draw the non-factorizable Feynman diagrams contributing to the gluon condensate as an example. Although the correlation functions $\Pi(p'^2, p^2, p^2)$ can be written as

$$\Pi(p'^2, p^2, p^2) = \int_{4m_c^2}^{\infty} ds \int_0^{\infty} du \frac{\rho_{\text{QCD}}(s, u)}{(s - p^2)(u - q^2)}$$
$$+ \text{nonsingular terms of } p'^2, \quad (37)$$
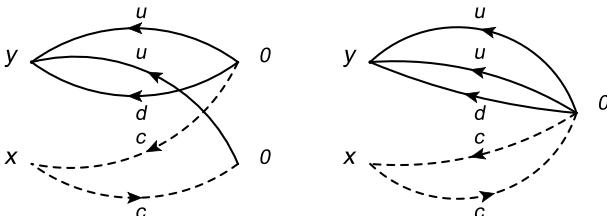
Fig. 5.   Lowest order Feynman diagrams contributing to decays of the pentaquark molecular state. In the first diagram, we take into account the finite spatial separation between the baryon and meson clusters.
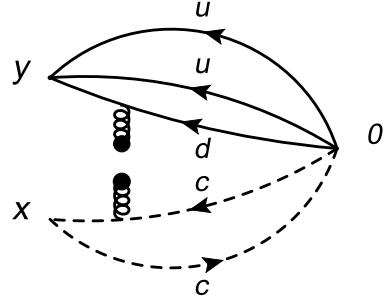

Fig. 6.   Non-factorizable Feynman diagrams contributing to the gluon condensate. Other diagrams obtained by interchanging of the light quark lines or heavy quark lines are implied.

at the QCD side; there are both factorizable and non-factorizable contributions.

In previous section, we have proved that the current operator $J(x)$ couples potentially to the $\Sigma_c\bar{D}$ molecular state, which receives both factorizable and non-factorizable contributions, while the couplings to the baryon-meson scattering states can be neglected. From Eqs. (15)-(18), we can see that there is a pole term $\frac{1}{m_P^2 - p'^2}$ at the hadron side, which should have originated at the QCD side. However, at the QCD side, there is no singular term with respect to the variable p'^2; see Eq. (21). It does not imply that there is no contribution from $P_c(4312)$ or that the current-molecule coupling λ_P is zero, it just means that $P_c(4312)$ may not be on the mass-shell. If we set $p'^2 = p^2$ to obtain the QCD sum rules, the terms $\frac{1}{m_P^2 - p^2}$ and $\frac{1}{m_{J/\psi}^2 - p^2}$ at the hadron side cannot be singular simultaneously. A reasonable explanation is that the current operator $J(0)$ in the three-point correlation functions $\Pi_5(p, q)$ and $\Pi_\mu(p, q)$ couples potentially to the $\Sigma_c\bar{D}$ molecular state or $P_c(4312)$. However, the $P_c(4312)$ may not be on the mass-shell, which facilitates setting $p'^2 = p^2$.

4   Numerical results and discussions

At the hadron side, we take the hadronic parameters as $m_{J/\psi} = 3.0969$ GeV, $m_N = 0.93827$ GeV, $m_{\eta_c} = 2.9839$ GeV, $\sqrt{s_{J/\psi}^0} = 3.6$ GeV, $\sqrt{s_{\eta_c}^0} = 3.5$ GeV, $\sqrt{s_N^0} = 1.3$ GeV [42], $m_P = 4.3119$ GeV [2], $f_{J/\psi} = 0.418$ GeV, $f_{\eta_c} = 0.387$ GeV [43], $\lambda_N = 0.032$ GeV^3 [44], and $\lambda_P = 1.95 \times 10^{-3}$ GeV^6 [18].

At the QCD side, we take the standard values of the vacuum condensates $\langle\bar{q}q\rangle = -(0.24 \pm 0.01\,\text{GeV})^3$, $\langle\bar{q}g_s\sigma Gq\rangle = m_0^2\langle\bar{q}q\rangle$, $m_0^2 = (0.8 \pm 0.1)\,\text{GeV}^2$, and $\langle\frac{\alpha_s GG}{\pi}\rangle = (0.33\,\text{GeV})^4$ at the energy scale $\mu = 1\,\text{GeV}$ [35, 40, 41, 45], and choose the $\overline{MS}$ mass $m_c(m_c) = (1.275 \pm 0.025)\,\text{GeV}$ from the Particle Data Group [42]. Moreover, we take into ac-

count the energy-scale dependence of the parameters,

$$\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(1\,\text{GeV}) \left[\frac{\alpha_s(1\,\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{12}{25}},$$

$$\langle \bar{q}g_s\sigma Gq \rangle(\mu) = \langle \bar{q}g_s\sigma Gq \rangle(1\,\text{GeV}) \left[\frac{\alpha_s(1\,\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{2}{25}},$$

$$m_c(\mu) = m_c(m_c) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{12}{25}},$$

$$\alpha_s(\mu) = \frac{1}{b_0 t} \left[1 - \frac{b_1}{b_0^2} \frac{\log t}{t} + \frac{b_1^2(\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right],$$

(38)

where $t = \log\frac{\mu^2}{\Lambda^2}$; $b_0 = \frac{33 - 2n_f}{12\pi}$; $b_1 = \frac{153 - 19n_f}{24\pi^2}$; $b_2 = \frac{2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2}{128\pi^3}$; and $\Lambda = 210,\ 292,\ 332\,\text{MeV}$ for the flavors $n_f = 5,\ 4$, and 3, respectively [42, 46], and evolve all the parameters to the ideal energy scale μ with $n_f = 4$ to extract the hadronic coupling constants g_5, g_V, and g_T.

In the QCD sum rules for the mass of the $\bar{D}\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$ or $P_c(4312)$, the ideal energy scale of the QCD spectral density is $\mu = 2.2\,\text{GeV}$ [18], which is determined by the energy scale formula $\mu = \sqrt{M_{X/Y/Z/P}^2 - (2\mathbb{M}_c)^2}$ with the effective c-quark mass $\mathbb{M}_c = 1.85\,\text{GeV}$ [47]. The energy scale $\mu = 2.2\,\text{GeV}$ is too large for N, η_c, and J/ψ. In this
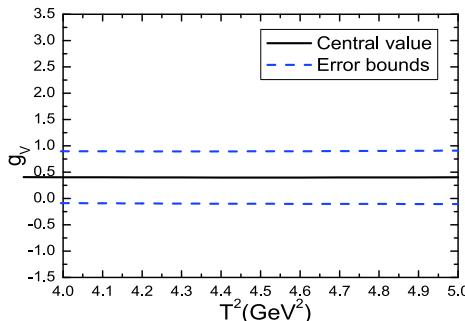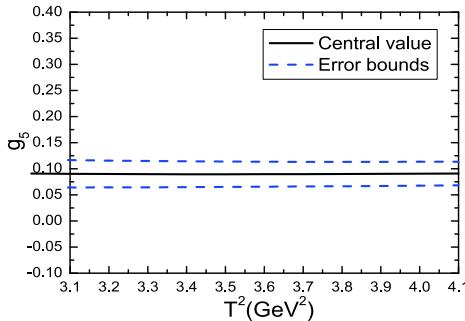
study, we take the energy scales of the QCD spectral densities to be $\mu = \frac{m_{\eta_c}}{2} = 1.5\,\text{GeV}$, which is acceptable for the charmonium states [37].

We choose the values of the free parameters as $C_5 = 1.18 \times 10^{-6}\,\text{GeV}^9$, $\overline{C}_5 = 1.94 \times 10^{-5}\,\text{GeV}^{10}$, $C_V = -1.77 \times 10^{-5}\,\text{GeV}^9$, and $C_T = -1.67 \times 10^{-5}\,\text{GeV}^9$ to obtain flat platforms in the Borel windows $T^2 = (3.1 - 4.1)\,\text{GeV}^2$, $(3.3 - 4.3)\,\text{GeV}^2$, $(4.0 - 5.0)\,\text{GeV}^2$, and $(3.9 - 4.9)\,\text{GeV}^2$ for the hadronic coupling constants g_5, g_V, and g_T, respectively. We fit the free parameters C_5, $\overline{C}_5$, C_V, and C_T to obtain the same intervals of flat platforms $T_{\max}^2 - T_{\min}^2 = 1.0\,\text{GeV}^2$, where $T_{\max}^2$ and $T_{\min}^2$ denote the maximum and minimum values of the Borel parameters, respectively.

We take into account the uncertainties of the input parameters, and obtain the values of the hadronic coupling constants g_5, g_V, and g_T, which are shown in Fig. 7,

$$g_5 = 0.09 \pm 0.03 \text{ from Eq. (28)},$$
$$g_5 = 0.09 \pm 0.07 \text{ from Eq. (29)},$$
$$g_V = 0.40 \pm 0.50,$$
$$g_T = 0.10 \pm 0.40,$$

(39)

where we have redefined the hadronic coupling constants g_V/g_T in Eq. (13) with a simple replacement, $g_V/g_T \to -g_V/g_T$, as the central values of g_V/g_T are negative from the QCD sum rules in Eq. (30).

Now it is straightforward to calculate the partial decay widths of the decays $P_c(4312) \to \eta_c N$, $J/\psi N$,
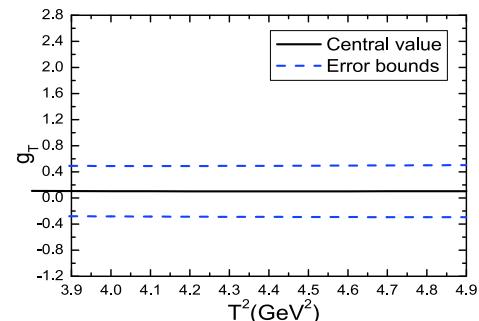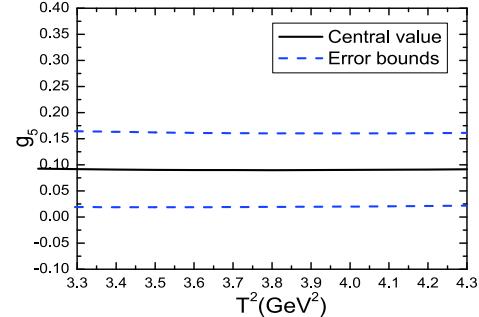

Fig. 7.   (color online) Hadronic coupling constants g_5, g_V, and g_T with variations of the Borel parameters T^2, the values of g_5 in the first diagram and second diagram come from the QCD sum rules in Eq. (28) and Eq. (29), respectively.

$$\Gamma(P_c(4312) \to \eta_c N) = \frac{p(m_P, m_{\eta_c}, m_N)}{16\pi m_P^2} |T|^2$$

$$= 31.488 g_5^2 \text{ MeV}$$

$$= 0.255^{+0.198}_{-0.142} \text{ MeV from Eq. (28),}$$

$$= 0.255^{+0.551}_{-0.242} \text{ MeV from Eq. (29),} \tag{40}$$

where

$$T = \bar{u}(q) \mathrm{i} g_5 u(p'), \tag{41}$$

and $p(a, b, c) = \dfrac{\sqrt{[a^2 - (b+c)^2][a^2 - (b-c)^2]}}{2a}$,

$$\Gamma(P_c(4312) \to J/\psi N) = \frac{p(m_P, m_{J/\psi}, m_N)}{16\pi m_P^2} |T|^2$$

$$= 29.699 g_T^2 - 97.554 g_V g_T + 80.633 g_V^2 \text{ MeV}$$

$$= 9.296^{+19.542}_{-9.296} \text{ MeV}, \tag{42}$$

where

$$T = \varepsilon_\alpha^* \bar{u}(q) \left(g_V \gamma^\alpha - \mathrm{i} \frac{g_T}{m_P + m_N} \sigma^{\alpha\beta} p_\beta \right) \gamma_5 u(p'). \tag{43}$$

The partial decay width $\Gamma(P_c(4312) \to \eta_c N) = 0.255$ MeV is extremely small, and the total width $\Gamma_{P_c(4312)}$ can be saturated with the strong decay $P_c(4312) \to J/\psi N$. The predicted width $\Gamma(P_c(4312) \to J/\psi N) = 9.296^{+19.542}_{-9.296}$ MeV is compatible with the experimental data $\Gamma_{P_c(4312)} = 9.8 \pm 2.7^{+3.7}_{-4.5}$ MeV from the LHCb collaboration [2]. The present calculations support assigning $P_c(4312)$ to be the $\bar{D}\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$. We can search for $P_c(4312)$ in the $\eta_c N$ mass spectrum, and measure the branching fraction $\mathrm{Br}(P_c(4312) \to \eta_c N)$, which may shed light on the nature of $P_c(4312)$ and test the predictions of the QCD sum rules.

The thresholds of $\bar{D}\Lambda_c$ and $\bar{D}^*\Lambda_c$ are 4.15 GeV and 4.29 GeV, respectively, and the decays into the final states $\bar{D}\Lambda_c$ and $\bar{D}^*\Lambda_c$ are kinematically allowed. At the quark level, the decays of the $\bar{D}\Sigma_c$ pentaquark molecular state to the $\bar{D}\Lambda_c$ and $\bar{D}^*\Lambda_c$ states take place through dissolving of the Σ-type diquark states to form the Λ-type diquark states by emitting an isospin $I = 1$ quark-antiquark pair. At the hadron level, the decay $P_c(4312) \to \bar{D}^*\Lambda_c$ can take place through the process $P_c(4312) \to \bar{D}\Sigma_c \to \bar{D}\Lambda_c\pi \to \bar{D}^*\Lambda_c$ with the subprocesses $\Sigma_c \to \pi\Lambda_c$ and $\bar{D}\pi \to \bar{D}^*$; the partial decay width $\Gamma(P_c(4312) \to \bar{D}^*\Lambda_c)$ may be as large as 10.7 MeV [48]. Direct calculations of these partial decay widths with the QCD sum rules are necessary to make a definite conclusion, which we aim to address in our next work.

5 Conclusion

In this article, we tentatively assign $P_c(4312)$ to be the $\bar{D}\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$, and discuss the factorizable and non-factorizable contributions in the two-point QCD sum rules for the $\bar{D}\Sigma_c$ molecular state in detail to prove the reliability of the single pole approximation in the hadronic spectral density. We study its two-body strong decays with the QCD sum rules by carrying out the operator product expansion up to the vacuum condensates of dimension 10. In calculations, special attention was paid to match the hadron side with the QCD side of the correlation functions to obtain solid duality. We obtain the partial decay widths $\Gamma(P_c(4312) \to \eta_c p) = 0.255$ MeV and $\Gamma(P_c(4312) \to J/\psi p) = 9.296^{+19.542}_{-9.296}$ MeV, which are compatible with the experimental data $\Gamma_{P_c(4312)} = 9.8 \pm 2.7^{+3.7}_{-4.5}$ MeV from the LHCb collaboration. The present calculations support assigning $P_c(4312)$ to be the $\bar{D}\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$. We can search for the decay $P_c(4312) \to \eta_c p$ to diagnose the nature of $P_c(4312)$.

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