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Kinemaitcs in spatially flat FLRW spacetimes

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Abstract: The kinematics on spatially flat FLRW spacetimes is presented for the first time in local charts with physical coordinates, i.e., the cosmic time and proper Cartesian space coordinates of Painlevé-type. It is shown that there exists a conserved momentum that determines the form of the covariant four-momentum on geodesics in terms of physical coordinates. Moreover, with the help of this conserved momentum, the peculiar momentum can be defined, thus separating the peculiar and recessional motions without ambiguity. It is shown that the energy and peculiar momentum satisfy the mass-shell condition of special relativity while the recessional momentum does not produce energy. In this framework, the measurements of the kinetic quantities along geodesics performed by different observers are analyzed, pointing out an energy loss of the massive particles similar to that producing the photon redshift. The examples of the kinematics on the de Sitter expanding universe and a new Milne-type spacetime are extensively analyzed.

Keywords: FLRW specetime, peculiar momentum, recessional momentum, conserved quantities, energy loss, redshift, de Sitter expanding universe, Milne-type universe

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I. INTRODUCTION

The geodesic motion in general relativity can be described in various local charts (called here frames) as each observer may choose its proper frame with preferred coordinates. According to general relativity, these frames are equivalent as their coordinates are related through diffeomorphisms under which the mathematical objects transform covariantly [1, 2]. However, the general diffeomorphisms are not able to produce conserved quantities such that we must focus mainly on the isometries, which may give rise to conserved quantities via Noether's theorem [3-5]. Therefore, it is convenient to restrict ourselves to a class of observer's frames related through isometries where we have to apply the methods of special relativity in studying the relative motion; however, we use the specific isometry group instead of the Poincaré one, as in our recent de Sitter relativity [6, 7].

Under such circumstances, it is crucial to understand the significant physical quantities and how they may relate to the conserved quantities generated by isometries. Another challenge is to determine how the measurement depends on the choice of observer's frame, considering that the isometries transform the conserved quantities among themselves such that different observers measure different values of these quantities. In this paper, we analyze these problems in the simple case of the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes. We focus on the kinetic quantities on time-like or null geodesics in frames with proper coordinates called here the physical coordinates.

Apparently, these are trivial problems and have long been solved. However, they are still relevant because the physical space coordinates of the Painlevé type were used in static problems but were never used in proper frames of the spatially flat FLRW spacetimes. Here we introduce these coordinates obtaining the physical frames with a time-dependent metric but with spatially flat sections whose Cartesian coordinates give the physical distances as in Minkowski's specetime. The measured quantities are the components p^μ of the energy-momentum four-vector in physical frames, formed by *energy*, p^0, and *covariant* momentum, $\vec{p}$, which, in general, are functions of the cosmic time t.

Moreover, the spatially flat FLRW spacetimes have the Euclidean isometry group $E(3)$ formed by space rotations and space translations, giving the conserved angular momentum and conserved momentum, $\vec{P}$. The angular momentum is related to the symmetry under rotations which is global when we use Cartesian coordinates. The conserved momentum that does not coincide with the covariant momentum is important because it generates three prime integrals helping us to derive the energy and covariant momentum we need to integrate the geodesic equa-

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tion as determined by the initial condition and the conserved momentum. Moreover, the conserved momentum helps us separate the peculiar momentum, proportional with $\vec{P}$, from the recessional momentum, finding that the energy and peculiar momentum in the physical frame satisfy the mass-shell condition of special relativity. All these results concerning the kinematics in physical frames are presented in the first part of the next section.

It remains to be explored how different observers measure the energy and covariant momentum provided that in the FLRW geometries under consideration the translations are isometries transforming the covariant four-vectors and the conserved quantities. In the last part of the next section, we investigate how these quantities are measured by two observers staying in two different points of the same geodesic showing that, in contrast with the Minkowski spacetime, the observer position determines the form and meaning of the measured quantities. Thus, we deduce that the massive or massless particles lose some energy during propagation, which is related to the redshift in the massless case. The energy loss of the massive particles is derived here for the first time.

The third section is devoted to a well-studied example, namely the de Sitter expanding universe whose geodesics we studied in different frames, including the physical one, without paying attention to the energy and covariant momentum [8]. The de Sitter manifold has ten independent conserved Killing vectors, which generate conserved quantities. Among them, we extract the conserved momentum relating the conserved quantities to the measured ones for understanding the role of the conserved quantities in the de Sitter kinematics. The conclusion is that the conserved energy coincides with the measured energy in some points of geodesics while the other conserved quantities, including the conserved momentum, work together for closing the mass-shell condition. The mentioned problem of two observers is also discussed for time-like and null geodesics pointing out the energy loss and redshift.

A new example whose kinematics were never studied is presented in Sec. IV. This is a spatially flat FLRW spacetime with a Milne-type scale factor which, in contrast to the genuine Milne universe, has gravitational sources determining its expansion. We inspect the kinematics briefly on this manifold, observing that this behaves somewhat complementary with respect to the de Sitter one. Finally, we present some concluding remarks.

In what follows, we use the Planck natural units and denote the conserved quantities with capital letters.

II. SPATIALLY FLAT FLRW SPACETIMES

The FLRW spacetimes are the most plausible models of our universe in various periods of evolution. The actual universe is observed as being spatially flat with a reasonable accuracy. For this reason, we focus here on such manifolds for which we consider many types of frames with Cartesian or spherical coordinates searching for measurable quantities expressed in terms of physical coordinates of Painlevé type.

A. Physical frames

The Painlevé - Gullstrand coordinates [9, 10] were proposed for studying the Schwarzschild black holes. Similar coordinates can be introduced in any isotropic manifold (M,g) having frames $\{x\}=\{t,\vec{x}\}$ with flat space sections. In these frames, the coordinates, x^μ $(\alpha,\mu,\nu,\dots=0,1,2,3)$, may be formed by the cosmic time t and either Cartesian space coordinates $\vec{x}=(x^1,x^2,x^3)$ or associated spherical ones (r,θ,ϕ) with Euclidean metric $ds_E^2=d\vec{x}\cdot d\vec{x}=dr^2+r^2d\Omega^2$, where $d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2$. For example, the line element

$$ds^2=f(r)\,dt_s^2-\frac{dr^2}{f(r)}-r^2d\Omega^2\,, \tag{1}$$

of any static frame, $\{t_s,r,\theta,\phi\}$, with static time t_s, can be put in Painlevé-Gullstrand form,

$$ds^2=f(r)dt^2+2\sqrt{1-f(r)}\,dt\,dr-dr^2-r^2d\Omega^2\,; \tag{2}$$

substituting this in Eq. (1),

$$t_s=t+\int dr\frac{\sqrt{1-f(r)}}{f(r)}\,, \tag{3}$$

where t is the cosmic time.

Similar space coordinates, known as proper coordinates, called here more intuitively the *physical* Cartesian space coordinates and denoted by $\vec{x}$, are associated with the cosmic time, t, defining the *physical* frame $\{t,\vec{x}\}$. These coordinates can be introduced in any spatially flat FLRW spacetime through those of the conformal Euclidean frame $\{t_c,\vec{x}_c\}$, i.e., the conformal time, t_c, and the *comoving* Cartesian coordinates, $\vec{x}_c$, giving the line element

$$ds^2=a(t_c)^2\left(dt_c^2-d\vec{x}_c\cdot d\vec{x}_c\right). \tag{4}$$

Substituting the physical coordinates $\{t,\vec{x}\}$ as

$$t_c=\int\frac{dt}{a(t)}\,,\quad \vec{x}_c=\frac{\vec{x}}{a(t)}\,. \tag{5}$$

we obtain the line element of the physical frame,

$$ds^2=\left(1-\frac{\dot{a}(t)^2}{a(t)^2}\vec{x}^2\right)dt^2+2\frac{\dot{a}(t)}{a(t)}\vec{x}\cdot d\vec{x}\,dt-d\vec{x}\cdot d\vec{x}\,, \tag{6}$$

where $a(t) = a[t_c(t)]$ is the usual FLRW scale factor while

$$\frac{\dot{a}(t)}{a(t)} = \frac{1}{a(t)}\frac{\mathrm{d}a(t)}{\mathrm{d}t} = \frac{1}{a(t_c)^2}\frac{\mathrm{d}a(t_c)}{\mathrm{d}t_c}, \tag{7}$$

is the Hubble function for which we do not use a special notation. The inverse transformation $\{t,\vec{x}\} \to \{t_c,\vec{x}_c\}$ is obvious,

$$t = \int a(t_c)\mathrm{d}t_c, \quad \vec{x} = a(t_c)\vec{x}_c. \tag{8}$$

We suppose that the function $a(t)$ is smooth such that the transformations (8) and (5) are diffeomorphisms.

The metric (6) is of an observer at rest in origin ($\vec{x}=0$) and having a dynamic apparent horizon,

$$\left|\vec{x}_h(t)\right| = \frac{a(t)}{\dot{a}(t)}. \tag{9}$$

This particularity makes this metric less popular in cosmology; however, it is useful in astrophysics for studying how different observers record the relative geodesic motion measuring directly the physical coordinates. Another advantage of this metric is that this is approaching to the Minkowski one in a neighbourhood of $\vec{x}=0$. However, in cosmology one prefers the FLRW coordinates $\{t,\vec{x}_c\}$ with the well-known line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2\mathrm{d}\vec{x}_c\cdot\mathrm{d}\vec{x}_c, \tag{10}$$

because the comoving space coordinates comply with the homogeneity of the universes with flat space sections. In what follows we use mainly physical frames resorting to the conformal ones as an auxiliary tool when these offer technical advantages.

B. Kinematics

Our principal objective is to derive the equations of the time-like and null geodesics and the associated kinetic quantities in the physical frame $\{t,\vec{x}\}_O$ of an observer staying at rest in the origin O. We search for the components $p^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ of the four-momentum $(p^0,\vec{p})$ formed by the measured energy p^0 and covariant momentum $\vec{p}$. Here λ is an affine parameter related to the mass m of a particle moving freely along a geodesic as $\mathrm{d}s = m\,\mathrm{d}\lambda$.

We start with an intermediate step, focusing first on the components $p_c^\mu = \frac{\mathrm{d}x_c^\mu}{\mathrm{d}\lambda}$ of the four-momentum in the conformal frame $\{t_c,\vec{x}_c\}_O$ of our observer where we meet the simple prime integral,

$$a(t_c)^2\left[p_c^0(t_c)^2 - \vec{p}_c(t_c)^2\right] = m^2, \tag{11}$$

resulted from the line element (4). In other respects, we may exploit the fact that the spatially flat FLRW spacetimes have $E(3)$ isometries formed by global rotations, $x^i \to R^I_{\ j}x^j$ $(i,j,k,\cdots = 1,2,3)$, and space translations

$$\begin{array}{l} t_c = t_{c'}, \\ x_c^i = x_c'^{\,i} + \xi^i, \end{array} \quad \to \quad \begin{array}{l} t = t', \\ x^i = x'^{\,i} + \xi^i a(t). \end{array} \tag{12}$$

The associated Killing vectors $k_{(i)}$ have the components $k^0_{(i)} = 0$ and $k^j_{(i)} = \delta_{ij}$ in the frame $\{t_c,\vec{x}_c\}_O$ giving rise to the conserved quantities

$$P^i = -k_{(i)\,j}\frac{\mathrm{d}x_c^j}{\mathrm{d}\lambda} = a(t_c)^2\frac{\mathrm{d}x_c^i}{\mathrm{d}\lambda}, \tag{13}$$

representing the components of the *conserved* momentum $\vec{P}$ which is different from the covariant momentum $\vec{p}(t)$. Then, by using the prime integrals (11) and (13) we derive the energy and covariant momentum in this frame as

$$p_c^0(t_c) = \frac{\mathrm{d}t_c}{\mathrm{d}\lambda} = \frac{1}{a(t_c)}\sqrt{m^2 + \frac{P^2}{a(t_c)^2}}, \tag{14}$$

$$p_c^i(t_c) = \frac{\mathrm{d}x_c^i}{\mathrm{d}\lambda} = \frac{P^i}{a(t_c)^2}, \tag{15}$$

where we denote $P = |\vec{P}|$. The geodesic results from integrating the equation

$$\frac{\mathrm{d}x_c^i}{\mathrm{d}t_c} = \frac{p_c^i(t_c)}{p_c^0(t_c)}, \tag{16}$$

that yields

$$x_c^i(t_c) = x_{c0}^i + \frac{P^i}{m}\int_{t_{c0}}^{t_c}\frac{\mathrm{d}t_c}{\sqrt{a(t_c)^2 + \frac{P^2}{m^2}}}. \tag{17}$$

We may conclude that any time-like geodesic is determined completely by its conserved momentum $\vec{P} = \vec{n}_P P$ and the initial condition $\vec{x}_c(t_{c0}) = \vec{x}_{c0}$. However, this equation must be integrated in each particular case, but for the massless particles (with $m=0$), we have the universal solution

$$\vec{x}_c(t_c) = \vec{x}_{c0} + \vec{n}_P(t_c - t_{c0}), \tag{18}$$

giving the null geodesics on any FLRW spacetime.

The corresponding physical quantities measured by the observer O in his proper frame, $\{t,\vec{x}\}_O$, may be obtained by substituting the physical coordinates according to Eq. (5). Thus we obtain the covariant components,

$$p^0(t) = \frac{dt}{d\lambda} = \sqrt{m^2 + \frac{P^2}{a(t)^2}}, \tag{19}$$

$$p^i(t) = \frac{dx^i}{d\lambda} = \frac{P^i}{a(t)} + x^i(t)\frac{\dot{a}(t)}{a(t)}\sqrt{m^2 + \frac{P^2}{a(t)^2}}, \tag{20}$$

which represent the measured energy and covariant momentum in the point $[t, \vec{x}(t)]$ of the time-like geodesic,

$$\vec{x}(t) = \vec{x}_0\frac{a(t)}{a(t_0)} + \frac{\vec{P}}{m}a(t)\int_{t_0}^{t}\frac{dt}{a(t)\sqrt{a(t)^2 + \frac{P^2}{m^2}}}, \tag{21}$$

which is passing through the space point $\vec{x}(t_0) = \vec{x}_0$ at the initial time t_0. In the physical frame $\{t, \vec{x}\}_O$ the equation of the null geodesics,

$$\vec{x}(t) = \vec{x}_0\frac{a(t)}{a(t_0)} + \vec{n}_P\, a(t)\,[t_c(t) - t_c(t_0)], \tag{22}$$

results from Eq. (18).

A special and delicate problem is that of the tachyons whose kinetic quantities on space-like geodesics can be obtained by substituting $m^2 \to -m^2$ in the above equations. Then the energy

$$p^0_{\rm tach}(t) = \sqrt{-m^2 + \frac{P^2}{a(t)^2}}, \tag{23}$$

is real valued only when $a(t) < \frac{P}{m}$. This means that, in expanding universes, a tachyon with conserved momentum $\vec{P}$ disappears when $a(t)$ reaches the value $\frac{P}{m}$. This may survive only in collapsing universes for smaller values of the scale factor. As here we focus only on expanding geometries, we ignore the space-like geodesics remaining to study the time-like and null ones.

The covariant momentum defined by Eq. (20) can be split as $\vec{p}(t) = \vec{\hat{p}}(t) + \vec{\tilde{p}}(t)$ where

$$\vec{\hat{p}} = \frac{\vec{P}}{a(t)}, \quad \vec{\tilde{p}} = \vec{x}(t)\frac{\dot{a}(t)}{a(t)}p^0(t), \tag{24}$$

are the *peculiar* and respectively *recessional* momenta. We must stress that this splitting can be performed only in the physical frames where the prime integral derived from the line element (6) gives the familiar identity

$$p^0(t)^2 - \vec{\hat{p}}(t)^2 = m^2, \tag{25}$$

which is just the mass-shell condition of special relativity satisfied by the energy and peculiar momentum. Therefore, we may conclude that the peculiar momentum produces the entire energy of the geodesic motion as in special relativity. Thus for $\vec{P} = 0$ and $p^0(t) = m$, the particle remains at rest in the point $\vec{x}(t) = \frac{a(t)}{a(t_0)}\vec{x}_0$ but moving with the recessional momentum $\vec{\tilde{p}}$ with respect the observer O.

We must emphasize that these properties hold only in physical frames because, in the conformal frames with comoving space coordinates, this separation is not possible, and the recessional momentum remains hidden, provided that the whole covariant momentum (15) is proportional to the conserved one. Moreover, in these frames, the momenta satisfy inappropriate dispersion relations, depending explicitly on time, as in Eq. (11) or the identity

$$p_c^0(t)^2 - a(t)^2\vec{p}_c(t)^2 = m^2, \tag{26}$$

that holds in FLRW comoving frames $\{t, \vec{x}_c\}$.

In physical frames, other interesting kinetic quantities can be derived as, for example, the velocity

$$\vec{v}(t) = \frac{\vec{p}(t)}{p^0(t)} = \vec{x}(t)\frac{\dot{a}(t)}{a(t)} + \frac{\vec{\hat{p}}(t)}{\sqrt{m^2 + \vec{\hat{p}}(t)^2}}, \tag{27}$$

where the first term is the recessional velocity due to the space evolution, complying with the velocity law, which is sometimes confused with the Hubble one [11, 12]; the second term is the peculiar velocity, which depends on the peculiar momentum as in special relativity.

The covariance under rotations, which behave here as a global symmetry, gives rise to the conserved angular momentum, which depends only on the peculiar momentum

$$\vec{L} = \vec{x}(t) \wedge \vec{p}(t) = \vec{x}(t) \wedge \vec{\hat{p}}(t) = \frac{\vec{x}(t) \wedge \vec{P}}{a(t)} = \frac{\vec{x}_0 \wedge \vec{P}}{a(t_0)}, \tag{28}$$

and it can be related to the initial condition. Obviously, this vanishes when the observer O stays at rest in a space point of the measured geodesic.

C. Two observers problem

The physical quantities $p^0(t)$, $\vec{\hat{p}}(t)$ and $\vec{\tilde{p}}(t)$ are functions of time; however, the last one depends explicitly on coordinate such that the experimental results will depend on the relative position between the detector and the source of the measured particle. However, this does not impede as the peculiar and recession contributions can be separated at any time without ambiguities. Nevertheless, to avoid confusions, we take care of this dependence, searching for suitable positions of the observer's proper frames to obtain intuitive results.

The example we would like to discuss here is of two observers measuring the motion of a massive particle on a time-like geodesic which is passing through the origins O and O' of their proper frames, $\{t,\vec{x}\}_O$ and $\{t,\vec{x}'\}_{O'}$. We assume that at the initial time t_0 the origin O' is translated with respect to O as $\vec{x}(t_0)=\vec{x}'(t_0)+\vec{d}(t_0)$ where $\vec{d}(t_0)=\vec{d}a(t_0)$ depends on the translation parameters of Eq. (12) denoted now by $\xi^i=d^i$. Then it is convenient to introduce the unit vector $\vec{n}$ of the direction OO' such that $\vec{d}=\vec{n}d$.

Our experiment starts in this layout at t_0 when the observer O' launches a particle of mass m, momentum $\vec{p}=-\vec{n}p$ and energy

$$p^0=\sqrt{m^2+p^2}\,, \tag{29}$$

on the geodesic $O'\to O$. The problem is to obtain which are the energy and momentum of this particle measured in the origin O at the final time t_f when the particle reaches this point. For solving this problem we first consider for the conserved momentum that can be derived in O' as

$$\vec{P}'=\vec{P}=\vec{n}_P P=\vec{p}\,a(t_0)\,, \tag{30}$$

such that $P=p\,a(t_0)$ and $\vec{n}_P=-\vec{n}$. Then by substituting these results in Eqs. (19) and (20) we obtain the energy and momentum in O,

$$p^0(t_f)=\sqrt{m^2+p^2\frac{a(t_0)^2}{a(t_f)^2}}\,, \tag{31}$$

$$\vec{p}(t_f)=\vec{\hat{p}}(t_f)=-\vec{n}\,p\,\frac{a(t_0)}{a(t_f)}\,, \tag{32}$$

where t_f solves the equation $\vec{n}\cdot\vec{x}(t_f)=0$ with $\vec{x}(t)$ given by Eq. (21) that can be written as

$$\frac{P}{m}\int_{t_0}^{tf}\frac{\mathrm{d}t}{a(t)\sqrt{a(t)^2+\dfrac{P^2}{m^2}}}=d\,, \tag{33}$$

where d is the time-independent translation parameter defined above. Solving the integral (33) we obtain an algebraic equation that can be solved in term of the final time obtaining the function $t_f(P,t_0)$. We observe that this function must be singular in $P=0$ for preventing the left handed term of this equation on vanishing in this limit. Once we have the value of t_f, we can calculate the propagation time t_f-t_0, the distance $d(t_f)$ between O and O' at t_f, and the final peculiar velocity $\hat{v}(t_f)$ of the particle arriving in O. According to Eqs. (12) and (31), we obtain

$$d(t_f)=d\,a(t_f)=d(t_0)\frac{a(t_f)}{a(t_0)}\,, \tag{34}$$

$$\hat{v}(t_f)=\left(1+\frac{m^2}{p^2}\frac{a(t_f)^2}{a(t_0)^2}\right)^{-\frac{1}{2}}\,, \tag{35}$$

completing thus the collection of the kinetic quantities related to this problem.

Eq. (31) shows that, in expanding universes, a part of energy is lost during the propagation. This can be measured by the relative energy loss we define here for the first time as

$$e=1-\frac{p^0(t_f)}{p^0(t_0)}=1-\frac{1}{p^0}\sqrt{m^2+p^2\frac{a(t_0)^2}{a(t_f)^2}}\,, \tag{36}$$

because $p^0(t_0)=p^0$ as it results from Eqs. (19) and (30). This phenomenon is similar to the redshift of the photons with $m=0$ for which we recover the Lemaître equation [13, 14] of Hubble's law [15] as

$$\frac{1}{1-e}=1+z=\frac{p^0(t_0)}{p^0(t_f)}=\frac{a(t_f)}{a(t_0)}\,, \tag{37}$$

where z is the usual redshift defined as the relative dilation of the wave length. As expected, for $m=0$, the final velocity $\hat{v}(t_f)=1$ is the speed of light.

All the results presented here can be exploited effectively only in concrete geometries where the geodesic, Eq. (21), can be integrated. In what follows, we discuss two such examples starting with one of the most studied geometries.

III. DE SITTER EXPANDING UNIVERSE

The first example is the expanding portion of the de Sitter spacetime defined as the hyperboloid of radius $1/\omega_{\rm H}$ in the five-dimensional flat spacetime (M^5,η^5) of coordinates z^A (labelled by the indices $A,B,\cdots=0,1,2,3,4$) having the metric $\eta^5=\mathrm{diag}(1,-1,-1,-1,-1)$. The coordinates $\{x\}$ can be introduced giving the set of functions $z^A(x)$, which solve the hyperboloid equation,

$$\eta^5_{AB}z^A(x)z^B(x)=-\frac{1}{\omega_{\rm H}^2}\,. \tag{38}$$

where $\omega_{\rm H}$ is the Hubble de Sitter constant in our notations.

There are conformal frames, $\{t_c,\vec{x}_c\}$ [16], associated to physical ones, $\{t,\vec{x}\}$, having the conformal and cosmic times related as

$$t_c = -\frac{1}{\omega_{\rm H}}e^{-\omega_{\rm H}t} \tag{39}$$

and the scale factors

$$a(t) = e^{\omega_{\rm H}t} \ \to\ a(t_c) = -\frac{1}{\omega t_c}\,, \tag{40}$$

defined for $t\in\mathbb{R}$ and $t_c<0$ corresponding to the expanding portion. In this case, the Hubble function becomes the constant $\omega_{\rm H}$. In addition, this manifold allows even a static frame $\{t_s,\vec{x}\}$ with the line element (1) where $f(r)=1-\omega_{\rm H}^2r^2$ and $t_s=t-\ln f(r)$.

A. Conserved quantities

The de Sitter manifold has a rich isometry group which is the gauge group $SO(1,4)$ of the embedding manifold (M^5,η^5) that leave invariant its metric and implicitly Eq. (38). Therefore, given a system of coordinates defined by the functions $z=z(x)$, each transformation $\mathfrak{g}\in SO(1,4)$ defines the isometry $x\to x'=\phi_{\mathfrak{g}}(x)$ derived from the system of equations

$$z[\phi_{\mathfrak{g}}(x)] = \mathfrak{g}z(x) \tag{41}$$

that holds for any type of coordinates because these isometries are defined globally. The set of frames related through these isometries play the role of the inertial frames similar to those of special relativity.

Given an isometry $x\to x'=\phi_{\mathfrak{g}(\xi)}(x)$ depending on the group parameter ξ there exists an associated Killing vector, $k=\partial_\xi\phi_\xi|_{\xi=0}$ (which satisfy the Killing equation $k_{\mu;\nu}+k_{\nu;\mu}=0$). Thus, in a canonical parametrization of the $SO(1,4)$ group, with real skew-symmetric parameters $\xi^{AB}=-\xi^{BA}$, any infinitesimal isometry can be written as $\phi^\mu_{\mathfrak{g}(\xi)}(x)=x^\mu+\xi^{AB}k^\mu_{(AB)}(x)+\cdots$. Starting with the general definition of the Killing vectors in the pseudo-Euclidean spacetime (M^5,η^5), we may consider the following identity

$$K_C^{(AB)}dz^C = z^Adz^B - z^Bdz^A = k_\mu^{(AB)}dx^\mu\,, \tag{42}$$

giving the covariant components of the Killing vectors in an arbitrary frame $\{x\}$ of the de Sitter manifold as

$$k_{(AB)\mu} = \eta^5_{AC}\eta^5_{BD}k_\mu^{(CD)} = z_A\partial_\mu z_B - z_B\partial_\mu z_A\,, \tag{43}$$

where $z_A=\eta_{AB}z^B$.

The classical conserved quantities along the time-like geodesics have the general form $\mathcal{K}_{(AB)}(x,\vec{P})=\omega_{\rm H}k_{(AB)\mu}p^\mu$, where p^μ are the components of the covariant four-vector defined above. The conserved quantities with physical meaning [17] are the energy, $E=\omega_{\rm H}k_{(04)\mu}p^\mu$, the angular momentum components, $L_i=\frac{1}{2}\varepsilon_{ijk}k_{(jk)\mu}p^\mu$, and the components $K_i=k_{(0i)\mu}p^\mu$ and $R_i=k_{(i4)\mu}p^\mu$ of two vectors related to the conserved momentum $\vec{P}$ and its associated dual momentum $\vec{Q}$:

$$\vec{P} = -\omega_{\rm H}(\vec{R}+\vec{K})\,,\quad \vec{Q}=\omega_{\rm H}(\vec{K}-\vec{R})\,. \tag{44}$$

satisfying the identity

$$E^2-\omega_{\rm H}^2\vec{L}^2-\vec{P}\cdot\vec{Q}=m^2\,, \tag{45}$$

corresponding to the first Casimir invariant of the $so(1,4)$ algebra [17]. In the flat limit, $\omega_{\rm H}\to 0$ and $-\omega_{\rm H}t_c\to 1$, we have $\vec{Q}\to\vec{P}$ such that this identity becomes just the usual null mass-shell condition $E^2-\vec{P}^2=m^2$ of special relativity.

Note that the conserved quantities transform among themselves under de Sitter isometries including the simple translations, which in this case transform the energy and dual momentum as we have shown recently [6].

B. Time-like geodesics

The coordinates of the physical frame $\{t,\vec{x}\}_O$ are introduced by the functions

$$\begin{aligned}
z^0(x) &= \frac{1}{2\omega_{\rm H}}\left[e^{\omega_{\rm H}t}-e^{-\omega_{H}t}\left(1-\omega_{\rm H}^2\vec{x}^2\right)\right],\\
z^i(x) &= x^i\,,\\
z^4(x) &= \frac{1}{2\omega_{\rm H}}\left[e^{\omega_{\rm H}t}+e^{-\omega_{\rm H}t}\left(1-\omega_{\rm H}^2\vec{x}^2\right)\right],
\end{aligned} \tag{46}$$

giving the line element

$$ds^2 = (1-\omega_H^2\vec{x}^2)dt^2+2\omega_H\vec{x}\cdot d\vec{x}dt - d\vec{x}\cdot d\vec{x}, \tag{47}$$

with observer's horizon at $|\vec{x}_h|=\omega_{\rm H}^{-1}$, such that the condition $\omega_{\rm H}|\vec{x}|<1$ is mandatory. In this frame, the equation of a time-like geodesic can be obtained by solving the integral of Eq. (21) that yields [8]

$$\begin{aligned}
\vec{x}(t) =& \vec{x}_0e^{\omega_{\rm H}(t-t_0)}+\vec{n}_P\frac{e^{\omega_{\rm H}t}}{\omega_{\rm H}P}\left(\sqrt{m^2+P^2e^{-2\omega_{\rm H}t_0}}\right.\\
&\left. -\sqrt{m^2+P^2e^{-2\omega_{\rm H}t}}\right),
\end{aligned} \tag{48}$$

which is determined by the conserved momentum $\vec{P}=\vec{n}_PP$ and the initial condition $\vec{x}(t_0)=\vec{x}_0$ fixed at time t_0.

The conserved quantities in an arbitrary point $(t,\vec{x}(t))$ of this geodesic can be expressed as [6, 8]

$$E=\omega_{\rm H}\,\vec{x}(t)\cdot\vec{P}{\rm e}^{-\omega_{\rm H}t}+\sqrt{m^2+P^2{\rm e}^{-2\omega_{\rm H}t}}\,, \tag{49}$$

$$\vec{L}=\vec{x}(t)\wedge\vec{P}{\rm e}^{-\omega_{\rm H}t}\,, \tag{50}$$

$$\vec{Q}=2\omega_{\rm H}\,\vec{x}(t)E{\rm e}^{-\omega_{\rm H}t}+\vec{P}{\rm e}^{-2\omega_{\rm H}t}\left[1-\omega_{\rm H}^2\vec{x}(t)^2\right]. \tag{51}$$

satisfying the identity (45). Moreover, Eqs. (19) and (20) give the energy and covariant momentum components,

$$p^0(t)=\frac{{\rm d}t}{{\rm d}\lambda}=\sqrt{m^2+P^2{\rm e}^{-2\omega_{\rm H}t}}\,, \tag{52}$$

$$p^i(t)=\frac{{\rm d}x^i}{{\rm d}\lambda}={\rm e}^{-\omega_{\rm H}t}P^i+\omega_{\rm H}x^i(t)\sqrt{m^2+P^2{\rm e}^{-2\omega_{\rm H}t}}\,, \tag{53}$$

that can be measured by the observer O in its proper frame $\{t,\vec{x}\}_O$. The conserved quantities are related to the measured ones as

$$E=\omega_{\rm H}\,\vec{x}(t)\cdot\vec{p}(t)+p^0(t)\,, \tag{54}$$

$$\vec{L}=\vec{x}(t)\wedge\vec{p}(t)\,, \tag{55}$$

$$\vec{P}=\vec{p}(t)\,{\rm e}^{\omega_{\rm H}t} \tag{56}$$

$$\vec{Q}={\rm e}^{-\omega_{\rm H}t}\left\{2\omega_{\rm H}\,\vec{x}(t)E+\vec{p}(t)[1-\omega_{\rm H}^2\vec{x}(t)^2]\right\}. \tag{57}$$

Hereby we conclude that the conserved quantities depend only on position and peculiar momentum. Among them, only E and $\vec{L}$ can be measured while $\vec{P}$ and $\vec{Q}$ are not accessible directly, with their role pertaining to only closing the invariant (45) as

$$E^2-\omega_{\rm H}^2\vec{L}^2-\vec{P}\cdot\vec{Q}=p^0(t)^2-\vec{p}(t)^2=m^2\,. \tag{58}$$

For example, a measurement in observer's origin O gives $E=p^0$, $\vec{L}=0$, $\vec{P}=\vec{p}{\rm e}^{\omega_{\rm H}t}$, and $\vec{Q}=\vec{p}{\rm e}^{-\omega_{\rm H}t}$ such that $\vec{P}\cdot\vec{Q}=\vec{p}^{\,2}$. Note that there is a natural choice of the initial moment, $t_0=0$, for which we have $\vec{p}(0)=\vec{P}=\vec{Q}$, and the calculations become simpler.

Now we can revisit the problem presented in Sec. II.C, searching for the value of t_f that solves Eq. (33). Taking into account that now $\vec{P}=\vec{p}{\rm e}^{\omega_{\rm H}t_0}=-\vec{n}\,p\,{\rm e}^{\omega_{\rm H}t_0}$ and $\vec{x}_0=\vec{d}(t_0)=\vec{n}d(t_0)$, we obtain the identity

$$\frac{a(t_0)^2}{a(t_f)^2}={\rm e}^{-2\omega_{\rm H}(t_f-t_0)}=\frac{1}{p^2}\left(p^0-\omega_{\rm H}d(t_0)p\right)^2-\frac{m^2}{p^2}\,, \tag{59}$$

which may be substituted in Eq. (31) leading to the final result

$$p^0(t_f)=p^0-\omega_{\rm H}d(t_0)p=p^0+\omega_{\rm H}\vec{d}(t_0)\cdot\vec{p}\,, \tag{60}$$

$$\vec{p}(t_f)=\vec{\hat{p}}(t_f)=-\vec{n}\sqrt{p^0(t_f)^2-m^2}\,, \tag{61}$$

expressed exclusively in terms of physical quantities with p^0 given by Eq. (29). Hereby we deduce the relative energy loss

$$e=\omega_{\rm H}d(t_0)\frac{p}{p^0}=\omega_{\rm H}d(t_0)v\,, \tag{62}$$

proportional to the initial velocity v of the particle lunched by O'. In the case of the massless photons, $v=1$ recovering the energy loss producing the redshift. It remains to derive the final distance and velocity, which take the form

$$d(t_f)=d(t_0)\frac{p}{|\vec{p}(t_f)|}\,, \tag{63}$$

$$\hat{v}(t_f)=\left(1+\frac{m^2}{|\vec{p}(t_f)|^2}\right)^{-\frac{1}{2}}, \tag{64}$$

as it results from Eqs. (34), (35), and (59).

For understanding the role of the conserved quantities in this experiment we must specify that the observers O and O' record different conserved quantities because the translation is an isometry, which changes the components of the conserved quantities apart from the conserved momentum that is not affected by these isometries [6]. Moreover, as the origins of these frames are on the geodesics, both the angular momenta measured in O and O' vanish. We denote by E, $\vec{Q}$ the remaining conserved quantities measured in O and by E', $\vec{Q}'$ those recorded in O' bearing in mind that

$$\vec{P}'=\vec{P}=\vec{p}{\rm e}^{\omega_{\rm H}t_0}\,. \tag{65}$$

The values observed in O' can be deduced from Eqs. (49) and (51) for $\vec{x}'=0$ obtaining the previous mentioned result, $E'=p^0$ and $\vec{Q}'=\vec{p}{\rm e}^{-\omega_{\rm H}t_0}$. The observer O prefers to search for the conserved quantities at t_0 because he knows that these do not change along the geodesic. Thus, he records

$$E = p^0(t_f)\,, \tag{66}$$

$$\vec{Q} = e^{-\omega_H t_0}\left[2\omega_H \vec{d}(t_0)E + \vec{p}\left(1 - \omega_H^2 \vec{d}(t_0)^2\right)\right], \tag{67}$$

as it results from Eqs. (49) and (51) for $\vec{x}_0 = \vec{d}(t_0)$, verifying that $\vec{P}\cdot\vec{Q} = \vec{p}(t_f)^2$ for closing again the identity (45). Note that the relation among the conserved quantities $E, \vec{P}, \cdots$ and $E', \vec{P}', \cdots$ can be derived directly according to the transformation rule under isometries we have discussed recently [18, 19].

C. Null geodesics

The de Sitter null geodesics of the photons with $m = 0$ that read

$$\vec{x}(t) = \vec{x}_0\, e^{\omega_H(t-t_0)} + \vec{n}_P \frac{e^{\omega_H(t-t_0)} - 1}{\omega_H} \tag{68}$$

are interesting being involved in the theory of the redshift. The energy and covariant momentum denoted now by $k^0(t)$ and $\vec{k}(t)$, respectively, are given by

$$k^0(t) = P\, e^{-\omega_H t} = |\vec{\mathfrak{k}}(t)|\,, \tag{69}$$

$$\vec{k}(t) = e^{-\omega_H t} P\left(\vec{n}_P + \omega_H \vec{x}(t)\right) = \vec{\mathfrak{k}}(t) + \vec{\mathfrak{\bar k}}(t)\,, \tag{70}$$

such that we can separate the peculiar momentum, $\vec{\mathfrak{k}}(t) = e^{-\omega_H t}\vec{P}$, and the recessional momentum, $\vec{\mathfrak{\bar k}}(t) = \omega_H \vec{x}(t) P e^{-\omega_H t} = \omega_H \vec{x}(t) k^0(t)$.

Considering again the problem presented in Sec. II.C, we assume that now the observer O' emits a photon of energy k and momentum $\vec{k} = -\vec{n}k$. Under such circumstances, Eq. (59) gives

$$\frac{a(t_0)}{a(t_f)} = e^{-\omega_H(t_f - t_0)} = 1 - \omega_H d(t_0)\,, \tag{71}$$

allowing us to derive the quantities observed by O in his proper frame, namely the energy and covariant momentum,

$$k^0(t_f) = k\left[1 - \omega_H d(t_0)\right], \tag{72}$$

$$\vec{k}(t_f) = \vec{\mathfrak{k}}(t_f) = -\vec{n}\, k^0(t_f)\,, \tag{73}$$

the value of the final time

$$t_f = t_0 - \frac{1}{\omega_H}\ln\left[1 - \omega_H d(t_0)\right], \tag{74}$$

the final distance between O and O' at t_f,

$$d(t_f) = \frac{d(t_0)}{1 - \omega_H d(t_0)}\,, \tag{75}$$

and the redshift z related to the relative energy loss e observed by O,

$$1 - e = \frac{1}{1+z} = 1 - \omega_H d(t_0)\,, \tag{76}$$

resulted from Eq. (37). We recall that the condition $\omega_H d(t_0) < 1$ is mandatory.

On the null geodesics, the conserved quantities have simpler forms as

$$E = k^0(t_f) = k\left[1 - \omega_H d(t_0)\right], \tag{77}$$

$$\vec{P} = \vec{k}\, e^{\omega t_0}\,, \tag{78}$$

$$\vec{Q} = \vec{k}\, e^{-\omega_H t_0}\left[1 - \omega_H d(t_0)\right]^2, \tag{79}$$

such that $\vec{P}\cdot\vec{Q} = E^2$, satisfying the identity (45) with $m = 0$. We observe again that for the special choice $t_0 = 0$ we have $\vec{P} = \vec{k}$ and $\vec{d}(t_0) = \vec{d}$ which simplifies the calculations and their interpretation.

IV. MILNE-TYPE UNIVERSE

Let us finish with an example of a manifold whose kinematics was never studied. This is the spatially flat FLRW manifold M with the Milne type scale factor $a(t) = \omega_M t$ defined on the domain $t \in (0, \infty)$, whose constant (frequency) ω_M is introduced from dimensional reasons [20, 21]. Then, after substituting the Hubble function $\frac{\dot a(t)}{a(t)} = \frac{1}{t}$ in Eq. (6), which is independent of ω_M, we may write the line element in the physical frame $\{t, \vec{x}\}$ as

$$ds^2 = \left(1 - \frac{1}{t^2}\vec{x}^2\right)dt^2 + 2\vec{x}\cdot d\vec{x}\frac{dt}{t} - d\vec{x}\cdot d\vec{x}. \tag{80}$$

The conformal time $t_c \in (-\infty, \infty)$ is defined as

$$t_c = \int \frac{dt}{a(t)} = \frac{1}{\omega_M}\ln(\omega_M t)\,. \tag{81}$$

We obtain

$$a(t_c) = e^{\omega_M t_c}\,, \tag{82}$$

a function of the line element (4) of the conformal frame

$\{t_c, \vec{x}_c\}$.

Here, the constant ω_M is a useful free parameter representing the expansion speed of M. We remind the reader that in the case of the genuine Milne universe (of negative space curvature but globally flat), one must set $\omega_M = 1$ for eliminating the gravitational sources [2]. In contrast, our spacetime M is produced by isotropic gravitational sources, i.e., the density ρ and pressure $\underline{p}$, evolving in time as [20]

$$\rho = \frac{3}{8\pi G}\frac{1}{t^2}, \quad \underline{p} = -\frac{1}{8\pi G}\frac{1}{t^2}, \tag{83}$$

and vanishing for $t \to \infty$. These sources govern the expansion of M that can be better observed in the frame $\{t, \vec{x}\}$ where the line element (80) lays out an expanding horizon at $|\vec{x}_h| = t$. For $t \to \infty$, M tends to the Minkowski spacetime and the gravitational sources vanish.

We deduce first the equation of the time-like geodesics, solving the integral of Eq. (21), which leads to the final form

$$\vec{x}(t) = \frac{t}{t_0}\vec{x}_0 + \vec{n}_P\, t \ln\left(\frac{t}{t_0}\frac{P + \sqrt{P^2 + \omega_M^2 m^2 t_0^2}}{P + \sqrt{P^2 + \omega_M^2 m^2 t^2}}\right), \tag{84}$$

for $m = 0$, we obtain the equation of the null geodesics

$$\vec{x}(t) = \frac{t}{t_0}\vec{x}_0 + \vec{n}_P\, t \ln\left(\frac{t}{t_0}\right). \tag{85}$$

The energy, momentum, and velocity need to be derived according to Eqs. (19), (15), and (27). These are complicated formulas that can be manipulated in applications by using algebraic codes on computer.

Furthermore, coming back to the problem of two observers formulated in Sec. II.C, we solve Eq. (33) for deriving the final time t_f and the ratio

$$\frac{a(t_0)}{a(t_f)} = \frac{t_0}{t_f} = \frac{1}{2p}\left(\nu - \frac{m^2}{\nu}\right), \tag{86}$$

where

$$\nu = (p^0 + m)\mathrm{e}^{-\omega_M d}. \tag{87}$$

We recall that p is the scalar initial momentum of the particle of mass m lunched by O' at t_0 while p^0 is the corresponding energy (29). Then, according to Eqs. (31) and (32), we obtain the final energy and covariant momentum

$$p^0(t_f) = \frac{1}{2}\left(\nu + \frac{m^2}{\nu}\right), \tag{88}$$

$$\vec{p}(t_f) = -\vec{n}\,\frac{1}{2}\left(\nu - \frac{m^2}{\nu}\right), \tag{89}$$

and the energy loss,

$$1 - e = \frac{1}{2p}\left(\nu + \frac{m^2}{\nu}\right). \tag{90}$$

The final distance between O and O' when the particle arrives in O,

$$d(t_f) = d(t_0)\frac{1}{1-e}, \tag{91}$$

where $d(t_0) = d\,a(t_0) = \omega_M t_0 d$. As in this geometry, the observer's horizon is at t, we must impose the restriction $d(t_0) < t_0 \to \omega_M d < 1$ in order to keep the final distance inside the observer's horizon, $d(t_f) < t_f$.

When O and O' observe a photon, then they record $t_f = t_0\,\mathrm{e}^{\omega_M d}$, $k^0(t_f) = |\vec{k}(t_f)| = k\,\mathrm{e}^{-\omega_M d}$ and the redshift $1 + z = \mathrm{e}^{\omega_M d}$, which for small values of $\omega_M d$ can be confused with the de Sitter redshift since the expansion

$$\frac{1}{1+z} = \mathrm{e}^{-\omega_M d} = 1 - \omega_M d + O(\omega_M^2 d^2), \tag{92}$$

is somewhat similar to Eq. (76). However, for larger distances, the discrepancy between the linear behavior of the de Sitter redshift and the exponential one in the spacetime M becomes obvious.

Finally, we observe that the Milne-type and ds Sitter universes behave somewhat complementary such that the cosmic time of one of these manifolds behaves as the conformal time of the other one. The self explanatory Table 1 completes this image [21].

Table 1. The two complementary behaviors were observed.

	M	de Sitter
t	$0 < t < \infty$	$-\infty < t < \infty$
t_c	$-\infty < t_c < \infty$	$-\infty < t_c < 0$
$a(t)$	$\omega_M t$	$\mathrm{e}^{\omega_{\mathrm{H}} t}$
$a(t_c)$	$\mathrm{e}^{\omega_M t_c}$	$-\frac{1}{\omega_{\mathrm{H}} t_c}$
transl.	$\omega_M d < 1$	$\omega_{\mathrm{H}} d < 1$
$1+z$	$\mathrm{e}^{\omega_M d}$	$[1 - \omega_{\mathrm{H}} d(t_0)]^{-1}$

The only similarity is the condition satisfied by the translation parameter d for remaining inside the observer's horizon.

V. CONCLUDING REMARKS

We presented the complete kinematics in physical frames on spatially flat FLRW spacetimes based on the conserved quantities; among them, the conserved momentum is the central piece of our approach. In these frames, the geodesics are determined completely by the initial condition and conserved momentum. Moreover, this allows us to separate the peculiar motion from the recessional one such that the energy and peculiar momentum satisfy the mass-shell condition of special relativity. In this framework, we discussed the problem of two observers pointing out the relative energy loss during propagation which in the massless case gives the well-known redshift effect.

The first example is the kinematics of the de Sitter expanding universe related to our previous results concerning the geodesics of this manifold [8]. Here, we presented the measurable quantities on geodesics in physical comoving frames for the first time, showing how these are related to the rich set of the conserved quantities of this geometry. We observed that only the conserved energy is related directly to the measured one, while the conserved momentum and its dual momentum help each other in closing the mass-shell relation. In addition, we pointed out that the meaning of the conserved momentum depends on the choice of the initial time that we can set such that the conserved momentum coincides with the covariant initial momentum. This observation is important because the momentum operator in the de Sitter quantum mechanics is related exclusively to the conserved momentum [17].

The second example we present here for the first time is the kinematics on a new manifold we considered recently in quantum theory [20, 21]. This is a spatially flat FLRW spacetime with a Milne type scale factor produced by gravitational sources proportional with t^{-2} that could be of interest in inflation scenarios. The geodesic motion on this manifold is studied in physical frames deriving the kinetic quantities on geodesics and outlining the results of the experiment of two observers, including the redshift. We argued that this manifold is mathematically interesting as it behaves complementary to the de Sitter one because the conformal time of one of them behaves as the cosmic time of the other. We have thus at least two different examples of FLRW kinematics that can be studied in detail.

In conclusion, we may say that the physical coordinates and the conserved momentum offer a suitable framework in which we can clearly distinguish between the recessional motion due to the background expansion and the peculiar one that behaves as in special relativity. Thus, we may obtain a new perspective in interpreting the astrophysical measurements in our actual expanding universe.

References

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of relativity* (J. Wiley & Sons, New York 1972)
[2] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge 1982)
[3] B. Carter and R. G. McLenaghan, Phys. Rev. D **19**, 1093 (1979)
[4] I. I. Cotăescu, J. Phys. A: Math. Gen. **33**, 9177 (2000)
[5] I. I. Cotăescu, Europhys. Lett. **86**, 20003 (2009)
[6] I. I. Cotăescu, Eur. Phys. J. C **77**, 485 (2017)
[7] I. I. Cotăescu, Eur. Phys. J. C **78**, 95 (2018)
[8] I. I. Cotăescu, Mod. Phys. Lett. A **32**, 1750223 (2017)
[9] P. Painleve, C. R. Acad. Sci. (Paris) **173**, 677 (1921)
[10] A. Gullstrand, Arkiv. Mat. Astron. Fys. **16**, 1 (1922)
[11] E. R. Harrison, *Cosmology: The Science of the Universe* (New York: Cambridge Univ. Press, 1981)
[12] E. Harrison, Astrophys. J. **403**, 28 (1993)
[13] G. E. Lemaître, Ann. Soc. Sci. de Bruxelles **47A**, 49 (1927)
[14] G. E. Lemaître, MNRAS **91**, 483 (1931)
[15] E. Hubble, Proc. Nat. Acad. Sci. **15**, 168 (1929)
[16] G. Lemaître, J. Math. Phys. (MIT) **4**, 188 (1925)
[17] I. I. Cotăescu, GRG **43**, 1639 (2011)
[18] I. I. Cotăescu, Mod. Phys. Lett. **38**, 2150162 (2021)
[19] I. I. Cotăescu, Eur. Phys. J. C **81**, 32 (2021)
[20] I. I. Cotăescu, Int. J. Mod. Phys. A **34**, 1950024 (2019)
[21] I. I. Cotăescu and D. Popescu, Chinese Phys. C **44**, 055104 (2020)