Chinese Physics C Vol. 45, No. 9 (2021) 093112

Lax connections in $T\bar{T}$-deformed integrable field theories*

Bin Chen(陈斌)[1,2,3†] Jue Hou(侯爵)[1‡] Jia Tian(田佳)[4§]

[1]School of Physics, Peking University, No.5 Yiheyuan Rd, Beijing 100871, China
[2]Center for High Energy Physics, Peking University, No.5 Yiheyuan Rd, Beijing 100871, China
[3]Peng Huanwu Center for Fundamental Theory, Hefei 230026, China
[4]Kavli Institute for Theoretical Sciences (KITS), University of Chinese Academy of Science, Beijing 100190, China

Abstract: In this work, we attempt to construct the Lax connections of $T\bar{T}$-deformed integrable field theories in two different ways. With reasonable assumptions, we make an ansatz and find the Lax pairs in the $T\bar{T}$-deformed affine Toda theories and the principal chiral model by solving the Lax equations directly. This method is straightforward, but it may be difficult to apply for general models. We then make use of a dynamic coordinate transformation to read the Lax connection in the deformed theory from the undeformed one. We find that once the inverse of the transformation is available, the Lax connection can be read easily. We show the construction explicitly for a few classes of scalar models and find consistency with those determined using the first method.

Keywords: integrable field theory, Latex connection, $T\bar{T}$ deformation

DOI: 10.1088/1674-1137/ac0ee4

I. INTRODUCTION

The $T\bar{T}$-deformation of two-dimensional field theories [1, 2] has recently attracted much attention. It is a type of solvable irrelevant deformation and induces a flow in the space of field theories that satisfies the differential equation

$$\partial_t \mathcal{L}^{(t)} = \det\left(T^{(t)}_{\mu\nu}\right), \tag{1}$$

where $T^{(t)}_{\mu\nu}$ is the stress-energy tensor and $T\bar{T} = -\pi^2 \det T^{(t)}_{\mu\nu}$. A remarkable property of this flow is that it preserves integrability if the undeformed theory is integrable. In the original paper [1], the preservation of integrability under a $T\bar{T}$-deformation or its generations has been supported by showing that the infinite conserved charges of the undeformed theory are still conserved under the flow. Another piece of evidence for integrability arises from the fact that the S-matrix in the deformed theory is only modified by adding a CDD-like factor [1, 3]. A word of caution is that the solvability of the deformed theories does not rely on integrability crucially, and it can be understood from various aspects [4-9]. Nevertheless, integrability may provide additional convenient handles on the theory.

It is well-known that integrability can be described in other frameworks, such as the Lax pair formulation and the Bäcklund transformation formulation. In particular, the existence of the Lax connection is usually taken as a hallmark of classical integrability, and it also paves the way to quantization [10]. However the Lax connection is notoriously difficult to find. Most of the time, it requires the art of guessing and trial and error. In this work, we will derive the Lax connections of several $T\bar{T}$-deformed integrable theories with two different methods. The first method is rather straightforward. We start from a reasonable ansatz and find the connection by imposing equations of motion and solving the Lax equation. This method is suggestive but can be limited to specific models. The second method is more systematic, and it relies on the fact that the $T\bar{T}$-deformation can be realized as a dynamic coordinate transformation [9]. It is reminiscent of the method for deriving the Lax connections of γ-deformed superstring theory [11]. This similarity is also expected, considering the fact that the holographic $T\bar{T}$-deformation [12], also known as the single trace $T\bar{T}$-de-

Received 3 June 2021; Accepted 28 June 2021; Published online 11 August 2021

* Supported by the National Natural Science Foundation of China (NSFC) (11735001). JT is also supported by the UCAS program of special research associate and by the internal funds of the KITS

† E-mail: bchen01@pku.edu.cn
‡ E-mail: houjue@pku.edu.cn
§ E-mail: wukongjiaozi@ucas.ac.cn

formation,aswellastheγdeformations,canberelatedtoaTsT deformation [13-15]. The difference is that the single trace $T\bar{T}$-deformation is a field redefinition, while the $T\bar{T}$-deformation of field theories involves a change of coordinates.

The paper is organized as follows. In Sect. II and Sect. III, we derive the Lax connection of (affine) Toda field theories and the principle chiral model directly with a reasonable ansatz. In Sect. IV, we first review the dynamic coordinate transformation approach of $T\bar{T}$-deformation, after which we reproduce the results obtained in Sects. II and III, and we finally derive the Lax connection for a $T\bar{T}$-deformed non-relativistic non-linear Schrödinger theory. In Sect. V, we provide conclusions.

II. $T\bar{T}$-DEFORMED (AFFINE) TODA FIELD THEORIES

In this section, we consider the (affine) Toda field theories and their $T\bar{T}$-deformations. The integrability of (affine) Toda field theories can be studied from the point of view of the Lax connections. As examples of N scalar theories, the $T\bar{T}$-deformed Lagrangians of these models have been derived in [16]. Here, we derive the deformed Lax connections with some reasonable ansatz.

A. Undeformed theories

The Lagrangian of a rank-r affine Toda field theory (for a review on Toda field theory, see [17]) is given by

$$\mathcal{L}^{(0)} \equiv \partial\vec{\phi}\cdot\bar{\partial}\vec{\phi}+V \tag{2}$$

with

$$V=-\frac{m^2}{\beta^2}\sum_{i=0}^{r} n_i \mathrm{e}^{\beta\vec{\alpha}_i\cdot\vec{\phi}}, \tag{3}$$

where $\vec{\phi}$ is a vector field of r components, the set of integer number $\{n_i\}$ characterizes the theory, $\{\vec{\alpha}_i,\ i=1,\cdots,r\}$ are positive simple roots of the underlying Lie algebra, and $\vec{\alpha}_0=-\sum_i^r n_i\vec{\alpha}_i$. If in the summation the term $i=0$ is omitted, then the theory reduces to the conformal Tada field theory. The generators of the Cartan subalgebra $\vec{H}=\{H_a,\ a=1,2,\cdots,r\}$, and the simple roots $\{E_{\vec{\alpha}_i},E_{-\vec{\alpha}_i},\ i=0,1,\cdots r\}$ satisfy the standard commutation relations

$$\begin{aligned}
&H_a,H_b=0,\quad \vec{H},E_{\pm\vec{\alpha}_i}=\pm\vec{\alpha}_i E_{\pm\vec{\alpha}_i},\\
&E_{\vec{\alpha}_i},E_{-\vec{\alpha}_j}=\delta_{ij}\frac{2\vec{\alpha}_j\cdot\vec{H}}{|\vec{\alpha}_j|^2},\\
&E_{\vec{\alpha}_i},E_{\vec{\alpha}_j}=\begin{cases} \mathcal{N}_{\vec{\alpha}_i+\vec{\alpha}_j}E_{\vec{\alpha}_i+\vec{\alpha}_j}, & \text{if } \vec{\alpha}_i+\vec{\alpha}_j \text{ is a root,}\\ 0, & \text{if } \vec{\alpha}_i+\vec{\alpha}_j \text{ is not a root.}\end{cases}
\end{aligned} \tag{4}$$

The equations of motion are simply given by

$$2\partial\bar{\partial}\vec{\phi}-\frac{\delta V}{\delta\vec{\phi}}=0, \tag{5}$$

and the Lax connections are

$$\begin{aligned}
L&=-\frac{\beta}{2}\partial\vec{\phi}\cdot\vec{H}-\lambda\sum_{i=0}^{r}m_i\mathrm{e}^{\beta\vec{\alpha}_i\cdot\vec{\phi}/2}E_{\vec{\alpha}_i},\\
\bar{L}&=\frac{\beta}{2}\bar{\partial}\vec{\phi}\cdot\vec{H}+\frac{1}{\lambda}\sum_{i=0}^{r}m_i\mathrm{e}^{\beta\vec{\alpha}_i\cdot\vec{\phi}/2}E_{-\vec{\alpha}_i},
\end{aligned} \tag{6}$$

where $\lambda\in\mathbf{C}$ is the spectral parameter and $m_i^2=\frac{1}{4}|\vec{\alpha}_i|^2m^2n_i$. For a classical integrable system, the equations of motion are equivalent to the Lax equation

$$\partial\bar{L}-\bar{\partial}L-L,\bar{L}=0. \tag{7}$$

For later convenience, we introduce two new combinations

$$\begin{aligned}
E_+&=\sum_{i=0}^{r}m_i\mathrm{e}^{\beta\vec{\alpha}_i\cdot\vec{\phi}/2}E_{\vec{\alpha}_i},\\
E_-&=\sum_{i=0}^{r}m_i\mathrm{e}^{\beta\vec{\alpha}_i\cdot\vec{\phi}/2}E_{-\vec{\alpha}_i}
\end{aligned} \tag{8}$$

satisfying

$$\begin{aligned}
[E_+,E_-]&=-\frac{\beta}{2}\vec{\nabla}V\cdot\vec{H},\\
\left[\vec{H},E_\pm\right]&=\pm\frac{2}{\beta}\vec{\nabla}E_\pm,
\end{aligned} \tag{9}$$

where $\vec{\nabla}f$ denotes $\dfrac{\delta f}{\delta\vec{\phi}}$. Then, the Lax connections (6) can be rewritten as

$$L=-\frac{\beta}{2}\partial\vec{\phi}\cdot\vec{H}-\lambda E_+,\quad \bar{L}=\frac{\beta}{2}\bar{\partial}\vec{\phi}\cdot\vec{H}+\frac{1}{\lambda}E_-. \tag{10}$$

B. $T\bar{T}$-deformed theories

The $T\bar{T}$-deformed Lagrangian of an N-scalar theory is [16, 18]

$$\mathcal{L}^{(t)}=\frac{V}{1-tV}+\frac{1}{2t(1-tV)}(\Omega_T-1), \tag{11}$$

where

$$\Omega_T=\sqrt{1+Y+Z},\quad Y=4t(1-tV)\left(\partial\vec{\phi}\cdot\bar{\partial}\vec{\phi}\right),$$
$$Z=-4t^2(1-tV)^2\left(\partial\vec{\phi}\cdot\partial\vec{\phi}\right)\left(\bar{\partial}\vec{\phi}\cdot\bar{\partial}\vec{\phi}\right)-\left(\partial\vec{\phi}\cdot\bar{\partial}\vec{\phi}\right)^2. \tag{12}$$

The equations of motion are given by

$$\vec{A}_e\equiv\partial_\mu\frac{\delta\mathcal{L}^{(t)}}{\delta\partial_\mu\vec{\phi}}-\frac{\delta\mathcal{L}^{(t)}}{\delta\vec{\phi}}=0. \tag{13}$$

Substituting (11) and (12) into (13), one can obtain

$$\begin{aligned}\vec{A}_e=&\left\{\frac{1}{\Omega_T}[\bar{\partial}\vec{\phi}-2t(1-tV)(\partial\vec{\phi}(\bar{\partial}\vec{\phi}\cdot\bar{\partial}\vec{\phi})-\bar{\partial}\vec{\phi}(\partial\vec{\phi}\cdot\bar{\partial}\vec{\phi}))]\right\}\\&+\bar{\partial}\left\{\frac{1}{\Omega_T}[\partial\vec{\phi}-2t(1-tV)(\bar{\partial}\vec{\phi}(\partial\vec{\phi}\cdot\partial\vec{\phi})-\partial\vec{\phi}(\partial\vec{\phi}\cdot\bar{\partial}\vec{\phi}))]\right\}\\&-\frac{\vec{\nabla}V}{4\Omega_T(1-tV)^2}\left[(\Omega_T+1)^2-Z\right].\end{aligned} \tag{14}$$

Given these equations of motions, we propose a simple ansatz for the Lax connection:

$$\begin{aligned}L=&-\frac{\beta}{2}\vec{a}_1\cdot\vec{H}-\lambda b_1E_++\frac{1}{\lambda}c_1E_-,\\\bar{L}=&\frac{\beta}{2}\vec{a}_2\cdot\vec{H}-\lambda b_2E_++\frac{1}{\lambda}c_2E_-,\end{aligned} \tag{15}$$

where $\vec{a}_1,b_1,c_1,\vec{a}_2,b_2,c_2$ are the functions of $\vec{\phi}$ and their derivatives and will be determined by imposing the Lax equation and the equations of motion. Notice that in our ansatz (15), the Lax connection depends uniformly on the simple roots $E_{\vec{\alpha}}$. Plugging (15) into (7) directly gives rise to a set of linear differential equations $\vec{A}_H,A'_+,A'_-$, corresponding to the components $\vec{H}$, E_+, and E_-, respectively. In principle, $\vec{A}_H,A'_+,A'_-$, should vanish separately. However, because terms such as $\nabla E_\pm$ are not uniformly dependent on the simple roots $E_{\vec{\alpha}}$, we require that the coefficients of terms such as $\nabla E_\pm$ vanish separately. Consequently, we obtain five sets of linear equations

$$\begin{cases}\vec{A}_H\equiv\partial\vec{a}_2+\bar{\partial}\vec{a}_1-\vec{\nabla}V(b_1c_2-b_2c_1)=0,\\A_+\equiv-\partial b_2+\bar{\partial}b_1=0,\\A_-\equiv\partial c_2-\bar{\partial}c_1=0,\\\vec{A}_{p+}\equiv-\partial\vec{\phi}b_2+\bar{\partial}\vec{\phi}b_1-(\vec{a}_1b_2+\vec{a}_2b_1)=0,\\\vec{A}_{p-}\equiv\partial\vec{\phi}c_2-\bar{\partial}\vec{\phi}c_1-(\vec{a}_1c_2+\vec{a}_2c_1)=0.\end{cases} \tag{16}$$

To solve these equations, we make another assumption that they can be written as linear combinations of the equations of motion, i.e.,

$$\begin{aligned}&\vec{A}_H=f_H\vec{A}_e,\quad A_+=\vec{f}_+\cdot\vec{A}_e,\quad A_-=\vec{f}_-\cdot\vec{A}_e,\\&\vec{A}_{p+}=f_{p+}\vec{A}_e,\quad \vec{A}_{p-}=f_{p-}\vec{A}_e.\end{aligned} \tag{17}$$

To ensure the equivalence between the Lax equation (16) and the equations of motion (13), there should be no common zero of $f_H,\vec{f}_+,\vec{f}_-,f_{p+},f_{p-}$. Indeed, we are making quite strong assumptions here, but we will show that a consistent solution does exist.

For the undeformed theory, by (10), one can find that

$$f_H=1,\quad f_{p+}=0,\quad f_{p-}=0, \tag{18}$$

and there are no $\vec{f}_+,\vec{f}_-$ terms. We assume that (18) is still true for the deformed theory and observe that if we take

$$\vec{f}_+=-t\bar{\partial}\vec{\phi}, \tag{19}$$

then

$$\vec{f}_+\cdot\vec{A}_e=-\partial\left[\frac{t}{\Omega_T}\left(\bar{\partial}\vec{\phi}\cdot\bar{\partial}\vec{\phi}\right)\right]+\bar{\partial}\left[\frac{(\Omega_T+1)^2-Z}{4\Omega_T(1-tV)}\right] \tag{20}$$

suggesting that we can identify

$$b_1=\frac{(\Omega_T+1)^2-Z}{4\Omega_T(1-tV)},\quad b_2=\frac{t}{\Omega_T}\left(\bar{\partial}\vec{\phi}\cdot\bar{\partial}\vec{\phi}\right) \tag{21}$$

up to some constants, which can be fixed to be zero after considering other equations. Similarly, by taking $\vec{f}_-=-t\partial\vec{\phi}$, we can determine c_1 and c_2

$$c_1=\frac{t}{\Omega_T}\left(\partial\vec{\phi}\cdot\partial\vec{\phi}\right),\quad c_2=\frac{(\Omega_T+1)^2-Z}{4\Omega_T(1-tV)}. \tag{22}$$

Finally, from $\vec{f}_H\vec{A}_e$, we fix all the remaining functions in our ansatz

$$\begin{aligned}\vec{a}_1=&\frac{1}{\Omega_T}\left[\partial\vec{\phi}-2t(1-tV)\left(\bar{\partial}\vec{\phi}\left(\partial\vec{\phi}\cdot\partial\vec{\phi}\right)-\partial\vec{\phi}\left(\partial\vec{\phi}\cdot\bar{\partial}\vec{\phi}\right)\right)\right],\\\vec{a}_2=&\frac{1}{\Omega_T}\left[\bar{\partial}\vec{\phi}-2t(1-tV)\left(\partial\vec{\phi}\left(\bar{\partial}\vec{\phi}\cdot\bar{\partial}\vec{\phi}\right)-\bar{\partial}\vec{\phi}\left(\partial\vec{\phi}\cdot\bar{\partial}\vec{\phi}\right)\right)\right].\end{aligned} \tag{23}$$

Plugging (21), (22), and (23) into (16), one can check that (16) is indeed equivalent to the equations of motion (13).

To summarize, the Lax connections of the $T\bar{T}$-deformed (affine) Toda field theories are of the forms expressed in (15), with the functions being given by (21), (22), and (23). We want to stress that after we assume (18) and (19), the solutions can be determined directly, without solving any other equations.

C. Examples

To compare with the existing results in the literature, let us consider some specific examples. The first one is the Liouville field theory, which corresponds to the Toda field theory of sl_2 Lie algebra with parameters

$$\beta=\frac{1}{2},\quad m_0=0,\quad m_1=-\frac{\sqrt{\mu}}{2}. \tag{24}$$

The undeformed Lagrangian is

$$\mathcal{L}^{(0)}=\partial\phi\bar{\partial}\phi-\mu e^{\phi}. \tag{25}$$

The $T\bar{T}$-deformed Liouville field theory was studied in [19], where infinite conserved currents were constructed from some ansatz without using the Lax connection. From the discussion in the last subsection, we can present the deformed Lax connections explicitly

$$\begin{aligned}L&=-\frac{1}{4}\frac{\partial\phi}{\Omega_T}H+\sqrt{\mu}\lambda Be^{\phi/2}E_{\alpha_1}-\frac{\sqrt{\mu}}{\lambda}(\partial\phi)^2Ce^{\phi/2}E_{-\alpha_1},\\ \bar{L}&=\frac{1}{4}\frac{\bar{\partial}\phi}{\Omega_T}H-\frac{\sqrt{\mu}}{\lambda}Be^{\phi/2}E_{-\alpha_1}+\sqrt{\mu}\lambda(\bar{\partial}\phi)^2Ce^{\phi/2}E_{\alpha_1},\end{aligned} \tag{26}$$

where

$$\begin{aligned}B&=\frac{(\Omega_T+1)^2}{8\Omega_T(1-tV)},\quad C=\frac{t}{2\Omega_T},\\ \Omega_T&=\sqrt{1+4t(1-tV)\left(\partial\phi\bar{\partial}\phi\right)}.\end{aligned} \tag{27}$$

Let the generators of sl_2 Lie algebra be

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad E_{\alpha_1}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad E_{-\alpha_1}=\begin{pmatrix}0&0\\1&0\end{pmatrix}, \tag{28}$$

then, if we take the undeforming limit, $t\to 0$, the Lax connections become

$$L=\begin{pmatrix}-\frac{1}{4}\partial\phi & \frac{\sqrt{\mu}\lambda}{2}e^{\phi/2}\\ 0 & \frac{1}{4}\partial\phi\end{pmatrix},\quad \bar{L}=\begin{pmatrix}\frac{1}{4}\bar{\partial}\phi & 0\\ -\frac{\sqrt{\mu}}{2\lambda}e^{\phi/2} & -\frac{1}{4}\bar{\partial}\phi\end{pmatrix}, \tag{29}$$

which are the Lax connections of the Liouville field theory.

Our next example is the sine-Gordon model, which corresponds to the affine Toda field of affine sl_2 algebra with parameters

$$\beta=\frac{i}{2},\ m_0=m_1=-\frac{i}{2},\ n_0=n_1=1, \alpha_0=-2,\ \ \alpha_1=2. \tag{30}$$

The undeformed Lagrangian is given by

$$\mathcal{L}^{(0)}=\partial\phi\bar{\partial}\phi-2\cos\phi. \tag{31}$$

By setting

$$E_{\alpha_0}=E_{-\alpha_1},\quad E_{-\alpha_0}=E_{\alpha_1}, \tag{32}$$

we find that the deformed Lax connections are

$$\begin{aligned}L=&-\frac{i}{4}\frac{\partial\phi}{\Omega_T}H+\left(i\lambda Be^{i\phi/2}+\frac{1}{i\lambda}(\partial\phi)^2Ce^{-i\phi/2}\right)E_{\alpha_1}\\ &+\left(i\lambda Be^{-i\phi/2}+\frac{1}{i\lambda}(\partial\phi)^2Ce^{i\phi/2}\right)E_{-\alpha_1},\\ \bar{L}=&\frac{i}{4}\frac{\bar{\partial}\phi}{\Omega_T}H+\left(\frac{1}{i\lambda}Be^{-i\phi/2}+i\lambda(\bar{\partial}\phi)^2Ce^{i\phi/2}\right)E_{\alpha_1}\\ &+\left(\frac{1}{i\lambda}Be^{i\phi/2}+i\lambda(\bar{\partial}\phi)^2Ce^{-i\phi/2}\right)E_{-\alpha_1},\end{aligned} \tag{33}$$

which are the same as those found in [18]. In the undeforming limit, $t\to 0$, the Lax connections reduce to the Lax connections of the sine-Gordon model

$$L=\begin{pmatrix}-\frac{i}{4}\partial\phi & \frac{i\lambda}{2}e^{i\phi/2}\\ \frac{i\lambda}{2}e^{-i\phi/2} & \frac{i}{4}\partial\phi\end{pmatrix},\quad \bar{L}=\begin{pmatrix}\frac{i}{4}\bar{\partial}\phi & \frac{1}{2i\lambda}e^{-i\phi/2}\\ \frac{1}{2i\lambda}e^{i\phi/2} & -\frac{i}{4}\bar{\partial}\phi\end{pmatrix}. \tag{34}$$

III. PRINCIPAL CHIRAL MODEL

In this section, we consider the principal chiral model (PCM), which is an integrable sigma model. The $T\bar{T}$-deformed Lagrangian of PCM has been obtained in [16, 18]. We will use a strategy similar to that used in the previous section to derive the deformed Lax connection.

A. Undeformed theory

The PCM is a field theory whose field takes values in some Lie group manifold. Its action is [1)]

1) In the section, we consider the theory in the flat space and take the Euclidean signature, that is, $g^{\mu\nu}=diag(1,1)$. The coordinate is (x^0,x^1) and the Levi-Civita symbol is $\epsilon^{0,1}=-\epsilon^{1,0}=1$.

$$S_0 = \int \mathrm{d}x^2 g^{\mu\nu}\mathrm{Tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g\right), \quad g\in G. \tag{35}$$

Usually, the Lie group is chosen to be semisimple, but we will leave it to be arbitrary because our focus is on integrability. The model has symmetry group $G_L\times G_R$. The equation of motion of the PCM is simply

$$\partial^\mu\left(g^{-1}\partial_\mu g\right)=0, \tag{36}$$

which is equivalent to the current conservation equation

$$\partial_\mu j^\mu = 0, \quad j^\mu \equiv g^{-1}\partial^\mu g. \tag{37}$$

Here, j^μ is the conserved current corresponding to the G_R symmetry. In addition to (37), the conserved current also satisfies the flatness condition:

$$\partial_0 j_1 - \partial_1 j_0 = -[j_0, j_1]. \tag{38}$$

Equations (37) and (38) are equivalent to the Lax equation with the Lax connections

$$L_0 =; -\frac{1}{\lambda^2+1}(\lambda j_1 + j_0), \quad L_1 = -\frac{1}{\lambda^2+1}(-\lambda j_0 + j_1), \tag{39}$$

where $\lambda\in\mathbf{C}$ is the spectral parameter.

B. $T\bar{T}$-deformed theory

The $T\bar{T}$-deformed Lagrangian of the PCM is given by [16]

$$\mathcal{L}^{(t)}_{\mathrm{PCM}} = \frac{1}{2t}(-1+\Omega_P), \tag{40}$$

where

$$\begin{aligned}\Omega_P &= \sqrt{1+4t\mathrm{Tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\mu g\right)+8t^2\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\mathrm{Tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\rho g\right)\mathrm{Tr}\left(g^{-1}\partial_\nu g g^{-1}\partial_\sigma g\right)}\\ &= \sqrt{1+4t\mathrm{Tr}\left(j_\mu j^\mu\right)+8t^2\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\mathrm{Tr}\left(j_\mu j_\rho\right)\mathrm{Tr}\left(j_\nu j_\sigma\right)}.\end{aligned} \tag{41}$$

The equation of motion, $A_{e\mathrm{PCM}}=0$, can also be cast into the form of a conservation law:

$$A_{e\mathrm{PCM}} \equiv \partial_\mu \frac{\delta\mathcal{L}^{(t)}_{\mathrm{PCM}}}{\delta\partial_\mu\vec{\phi}} - \frac{\delta\mathcal{L}^{(t)}_{\mathrm{PCM}}}{\delta\vec{\phi}} = 2(\partial_\mu J^\mu)g^{-1}. \tag{42}$$

Here, the conserved current J^μ is defined as

$$J^\mu = \frac{1}{\Omega_P}\left(j^\mu + 4t\epsilon^{\mu\nu}\epsilon^{\rho\sigma} j_\rho \mathrm{Tr}(j_\nu j_\sigma)\right), \tag{43}$$

which satisfies the following useful identities

$$\begin{aligned}[][J_0,J_1] &=[j_0,j_1],\\ [J_0,j_0] &=\frac{1}{\Omega_P}4t\mathrm{Tr}(j_0j_1)[j_0,j_1],\\ [J_0,j_1] &=\frac{1}{\Omega_P}(1+4t\mathrm{Tr}(j_1j_1))[j_0,j_1],\\ [J_1,j_0] &=-\frac{1}{\Omega_P}(1+4t\mathrm{Tr}(j_0j_0))[j_0,j_1],\\ [J_1,j_1] &=-\frac{1}{\Omega_P}4t\mathrm{Tr}(j_0j_1)[j_0,j_1].\end{aligned} \tag{44}$$

Note that the current j_μ still satisfies the flatness condition (38), so a reasonable ansatz for the Lax connections could be that they are the linear combination of the new conserved current (43) and j_μ:

$$\begin{aligned}L_0 &=a_0J_1+b_0j_0+c_0j_1,\\ L_1 &=a_1J_0+b_1j_0+c_1j_1,\end{aligned} \tag{45}$$

where a_0,a_1,b_0,b_1,c_0,c_1 are constants to be determined. Again, we assume that the Lax equation is linearly dependent on the equation of motion:

$$\begin{aligned}A_{\mathrm{LPCM}} &\equiv \partial_0L_1-\partial_1L_0-[L_0,L_1],\\ A_{\mathrm{LPCM}} &= A_{e\mathrm{PCM}}\cdot f_{\mathrm{PCM}}.\end{aligned} \tag{46}$$

For the undeformed theory, using (39) and the definitions of A_{LPCM} and $A_{e\mathrm{PCM}}$, we obtain

$$f_{\mathrm{PCM}} = \frac{1}{2}\frac{\lambda}{\lambda^2+1}g. \tag{47}$$

Assuming that (47) is still true in the deformed case, we end up with

$$A_{\mathrm{LPCM}} = \frac{\lambda}{\lambda^2+1}\partial_\mu J^\mu. \tag{48}$$

Plugging (45) into (46) and matching it with (48), we have

$$a_0=-a_1=-\frac{\lambda}{\lambda^2+1}, \quad b_0=c_1=-\frac{1}{\lambda^2+1}, \quad b_1=c_0=0, \tag{49}$$

where we have used the identity, $\partial_0 j_1 - \partial_1 j_0 = -[j_0, j_1]$.

In summary, the Lax connection of the $T\bar{T}$-deformed PCM is given by

$$L_0 = -\frac{1}{\lambda^2+1}(\lambda J_1 + j_0), \quad L_1 = -\frac{1}{\lambda^2+1}(-\lambda J_0 + j_1), \tag{50}$$

where J_μ has been defined by (43). This result is expected considering the identities (44). Given the Lax connection, we can define the monodromy matrix as the holonomy along a constant time slice

$$M(x^0;\lambda) = \mathcal{P}\exp\left(\int_{-\infty}^{\infty} \mathrm{d}x^1\, L_1(x^0, x^1, \lambda)\right). \tag{51}$$

The set of (non-local) infinite conserved charges can be generated by expanding the monodromy matrix with respect to the spectral parameter as

$$\begin{aligned} M(\lambda) =& \exp\left(\sum_{n=1}^{\infty} \frac{Q_n}{\lambda^n}\right) = 1 + \frac{1}{\lambda}\int_{-\infty}^{+\infty} \mathrm{d}x^1 J_0 \\ &- \frac{1}{\lambda^2}\left(\int_{-\infty}^{+\infty} \mathrm{d}x^1\, j_1 - \int_{-\infty}^{+\infty} \mathrm{d}x^1 \int_{-\infty}^{x^1} \mathrm{d}y^1 J_0(x) J_0(y)\right) \\ &+ O\left(\frac{1}{\lambda^3}\right). \end{aligned} \tag{52}$$

For the undeformed PCM, these non-local charges span the classical Yangian algebra [20]. Under the $T\bar{T}$-deformation, the algebra becomes deformed in a very complicated way.

IV. LAX CONNECTIONS FROM DYNAMIC COORDINATE TRANSFORMATION

The solvability of the $T\bar{T}$-deformation can be understood in various ways. From the point of view of integrability[1], the most transparent approach is to realize the $T\bar{T}$-deformation as a dynamic coordinate transformation. As shown in [3, 9, 21], the $T\bar{T}$ deformation can be interpreted as a space-time deformation. In Euclidean signature, the deformed and undeformed space-time are related via the following (state dependent or dynamic) coordinate transformation

$$\mathrm{d}x^\mu = \left(\delta^\mu_\nu + t\,\tilde{T}^\mu_\nu(\mathbf{y})\right)\mathrm{d}y^\nu, \quad \mathbf{y} = (y^1, y^2), \tag{53}$$

$$\mathrm{d}y^\mu = \left(\delta^\mu_\nu + t\,(\tilde{T}^{(\tau)})^\mu_\nu(\mathbf{x})\right)\mathrm{d}x^\nu, \quad \mathbf{x} = (x^1, x^2), \tag{54}$$

with $\tilde{T}^\mu_\nu = -\epsilon^\mu_\rho \epsilon^\sigma_\nu T^\rho_\sigma$ and $(\tilde{T}^{(\tau)})^\mu_\nu = -\epsilon^\mu_\rho \epsilon^\sigma_\nu (T^\tau)^\rho_\sigma$, where $T = T^{(0)}$ and $T^{(\tau)}$ are the undeformed and deformed stress-energy tensors in the coordinates $\mathbf{y}$ and $\mathbf{x}$, respectively. Using this map, we can obtain the solutions of the deformed equation of motion as

$$\phi^{(\tau)}(\mathbf{x}) = \phi^{(0)}(\mathbf{y}(\mathbf{x})). \tag{55}$$

In addition to the solutions of the equation of motion, the deformed conserved currents can also be obtained from the undeformed ones using the above coordinate transformations [21]. First, let us switch to complex coordinates, defined by

$$z = x^1 + \mathrm{i}x^2, \quad \bar{z} = x^1 - \mathrm{i}x^2, \tag{56}$$

$$w = y^1 + \mathrm{i}y^2, \quad \bar{w} = y^1 - \mathrm{i}y^2. \tag{57}$$

Starting from the 1-forms in the $\mathbf{w}$ coordinates

$$\begin{aligned} \mathcal{J}_k =& T_{k+1}(\mathbf{w})\mathrm{d}w + \Theta_{k-1}(\mathbf{w})\mathrm{d}\bar{w}, \\ \bar{\mathcal{J}}_k =& \bar{T}_{k+1}(\mathbf{w})\mathrm{d}\bar{w} + \bar{\Theta}_{k-1}(\mathbf{w})\mathrm{d}w, \end{aligned} \tag{58}$$

where T_{k+1}, Θ_{k-1}, and their complex conjugates are the higher conserved currents of underformed theory. Under the change of coordinates, we have

$$\begin{pmatrix} \mathrm{d}w \\ \mathrm{d}\bar{w} \end{pmatrix} = \mathcal{J}^T \begin{pmatrix} \mathrm{d}z \\ \mathrm{d}\bar{z} \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} \partial w & \partial \bar{w} \\ \bar{\partial} w & \bar{\partial}\bar{w} \end{pmatrix}. \tag{59}$$

where ∂ and $\bar{\partial}$ denote the derivative with respect to z and $\bar{z}$, respectively. Now, the Jacobian is of the form

$$\mathcal{J} = \frac{1}{\Delta(\mathbf{w})}\begin{pmatrix} 1 + 2t\Theta_0(\mathbf{w}) & -2tT_2(\mathbf{w}) \\ -2t\bar{T}_2(\mathbf{w}) & 1 + 2t\bar{\Theta}_0(\mathbf{w}) \end{pmatrix} \tag{60}$$

with

$$\Delta(\mathbf{w}) = (1 + 2t\Theta_0(\mathbf{w}))(1 + 2t\bar{\Theta}_0(\mathbf{w})) - 4t^2 T_2(\mathbf{w})\bar{T}_2(\mathbf{w}). \tag{61}$$

Substituting (59) and (60) into (58), one can read off the components of the currents in $\mathbf{z}$ coordinates:

$$T_{k+1}(\mathbf{z}, t) = \frac{T_{k+1}(\mathbf{w}(\mathbf{z})) + 2t(T_{k+1}(\mathbf{w}(\mathbf{z}))\Theta_0(\mathbf{w}(\mathbf{z})) - \Theta_{k-1}(\mathbf{w}(\mathbf{z}))T_2(\mathbf{w}(\mathbf{z})))}{\Delta(\mathbf{w}(\mathbf{z}))}, \tag{62}$$

1) Here specifically by integrability we mean there exist infinite conserved charges.

$$\Theta_{k-1}(\mathbf{z},t)=\frac{\Theta_{k-1}(\mathbf{w}(\mathbf{z}))+2t(\Theta_{k-1}(\mathbf{w}(\mathbf{z}))\bar{\Theta}_0(\mathbf{w}(\mathbf{z}))-T_{k+1}(\mathbf{w}(\mathbf{z}))\bar{T}_2(\mathbf{w}(\mathbf{z})))}{\Delta(\mathbf{w}(\mathbf{z}))}. \tag{63}$$

In a similar way, we can read the Lax connection of the deformed model. If the Lax connection of the undeformed model is

$$L(w,\bar{w})=\mathcal{L}\mathrm{d}w+\bar{\mathcal{L}}\mathrm{d}\bar{w}, \tag{64}$$

one can expect the deformed Lax pair should be given by

$$\begin{aligned}L=&\mathcal{L}(z,\bar{z})\mathrm{d}z+\bar{\mathcal{L}}(z,\bar{z})\mathrm{d}\bar{z}\\=&\mathcal{L}(w,\bar{w})\left(\frac{\partial w}{\partial z}\mathrm{d}z+\frac{\partial w}{\partial\bar{z}}\mathrm{d}\bar{z}\right)\\&+\bar{\mathcal{L}}(w,\bar{w})\left(\frac{\partial\bar{w}}{\partial z}\mathrm{d}z+\frac{\partial\bar{w}}{\partial\bar{z}}\mathrm{d}\bar{z}\right),\end{aligned}$$

which leads to the transformation law for the Lax connections:

$$\begin{aligned}&\mathcal{L}(\mathbf{z},t)\\=&\frac{\mathcal{L}_w(\mathbf{w}(\mathbf{z}))+2t(\mathcal{L}_w(\mathbf{w}(\mathbf{z}))\Theta_0(\mathbf{w}(\mathbf{z}))-\mathcal{L}_{\bar{w}}(\mathbf{w}(\mathbf{z}))T_2(\mathbf{w}(\mathbf{z})))}{\Delta(\mathbf{w}(\mathbf{z}))},\\&\bar{\mathcal{L}}(\mathbf{z},t)\\=&\frac{\mathcal{L}_{\bar{w}}(\mathbf{w}(\mathbf{z}))+2t(\mathcal{L}_{\bar{w}}(\mathbf{w}(\mathbf{z}))\bar{\Theta}_0(\mathbf{w}(\mathbf{z}))-\mathcal{L}_w(\mathbf{w}(\mathbf{z}))\bar{T}_2(\mathbf{w}(\mathbf{z})))}{\Delta(\mathbf{w}(\mathbf{z}))}.\end{aligned} \tag{65}$$

In the following, we will verify the above relations in a free scalar theory and the sine-Gordon model, whose deformed Lax pairs are explicitly given in the literature [18]. Moreover, we will attempt to reproduce the Lax connections of affine Toda field theory and the PCM, which we found in previous sections.

A. Free scalar

Consider the free scalar with the Lagrangian

$$L(\mathbf{w})=\partial_w\phi\partial_{\bar{w}}\phi. \tag{66}$$

The model is integrable with the trivial Lax pair

$$\mathcal{L}_w=\partial_w\phi,\quad \mathcal{L}_{\bar{w}}=-\partial_{\bar{w}}\phi \tag{67}$$

such that the Lax equation

$$\partial_{\bar{w}}\mathcal{L}_w-\partial_w\mathcal{L}_{\bar{w}}=2\partial_w\partial_{\bar{w}}\phi=0 \tag{68}$$

coincides with the equation of motion. The stress-energy tensor is simply

$$\begin{aligned}&T_2(\mathbf{w})=-\frac{1}{2}(\partial_w\phi)^2,\quad \Theta_0(\mathbf{w})=0,\\&\Delta=1-4t^2T_2(\mathbf{w})\bar{T}_2(\mathbf{w}),\end{aligned} \tag{69}$$

which leads to the following transformation

$$\partial_w\phi=\partial\phi-\frac{1}{4\tau}\left(\frac{-1+\Omega_T}{\bar{\partial}\phi}\right)^2\bar{\partial}\phi, \tag{70}$$

$$\partial_{\bar{w}}\phi=\bar{\partial}\phi-\frac{1}{4t}\left(\frac{-1+\Omega_T}{\partial\phi}\right)^2\partial\phi, \tag{71}$$

with $\Omega_T=\sqrt{1+4t\partial\phi\bar{\partial}\phi}$. Therefore, the deformed Lax connection is given by

$$\mathcal{L}(\mathbf{z},\tau)=\frac{\partial_w\phi(\mathbf{w}(\mathbf{z}))+2t\partial_{\bar{w}}\phi(\mathbf{w}(\mathbf{z}))T_2(\mathbf{w}(\mathbf{z}))}{1-4t^2T_2(\mathbf{w}(\mathbf{z}))\bar{T}_2(\mathbf{w}(\mathbf{z}))}=\frac{\partial\phi}{\Omega_T}, \tag{72}$$

$$\bar{\mathcal{L}}(\mathbf{z},\tau)=\frac{-\partial_{\bar{w}}\phi(\mathbf{w}(\mathbf{z}))-2t\partial_w\phi(\mathbf{w}(\mathbf{z}))\bar{T}_2(\mathbf{w}(\mathbf{z}))}{1-4t^2T_2(\mathbf{w}(\mathbf{z}))\bar{T}_2(\mathbf{w}(\mathbf{z}))}=-\frac{\bar{\partial}\phi}{\Omega_T}. \tag{73}$$

Indeed, the Lax equation matches the equation of motion of the $T\bar{T}$-deformed free scalar:

$$\partial\left(\frac{\bar{\partial}\phi}{\Omega_T}\right)+\bar{\partial}\left(\frac{\partial\phi}{\Omega_T}\right)=0. \tag{74}$$

B. Sine-Gordon model

Next, we turn to the $T\bar{T}$-deformed sine-Gordon model, whose Lax pair has been given in [18]. As a first step, we need to find the Jacobian (60), which is determined by the stress-energy tensor in $\mathbf{w}$ space-time. The Lagrangian of the sine-Gordon model is given by adding the potential[1)]

$$V=4\sin^2\left(\frac{\phi}{2}\right) \tag{75}$$

to the free scalar Lagrangian (66). From the standard procedure, one can find the expression of the stress-energy

1) Here we have chosen the convention used in [18] in order to make the comparison.

tensor

$$T_2(\mathbf{w}) = -\frac{1}{2}(\partial_w\phi)^2, \quad \bar{T}_2(\mathbf{w}) = -\frac{1}{2}(\partial_{\bar{w}}\phi)^2,$$
$$\Theta_0(\mathbf{w}) = -2\sin^2\left(\frac{\phi}{2}\right), \tag{76}$$

which leads to the following transformations

$$\partial_w\phi = \frac{-1+\Omega_T}{2t\bar{\partial}\phi}, \quad \partial_{\bar{w}}\phi = \frac{-1+\Omega_T}{2t\partial\phi},$$
$$\Omega_T = \sqrt{1+4t(1-tV)\partial\phi\bar{\partial}\phi}. \tag{77}$$

Recall that the undeformed Lax connection is

$$\mathcal{L}_w = -\frac{\mathrm{i}}{4}\partial_w\phi H + \frac{\lambda}{2}\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}E_+ + \frac{\lambda}{2}\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}E_-, \tag{78}$$

$$\mathcal{L}_{\bar{w}} = \frac{\mathrm{i}}{4}\partial_{\bar{w}}\phi H + \frac{1}{2\lambda}\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}E_+ + \frac{1}{2\lambda}\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}E_-. \tag{79}$$

The deformed Lax connection can be expanded with respect to these three generators as

$$\mathcal{L}(\mathbf{z},t) = \mathcal{L}^0 H + \mathcal{L}^+ E_+ + \mathcal{L}^- E_-,$$
$$\bar{\mathcal{L}}(\mathbf{z},\tau) = \bar{\mathcal{L}}^0 H + \bar{\mathcal{L}}^+ E_+ + \bar{\mathcal{L}}^- E_-. \tag{80}$$

From transformation (65), we have the deformed Lax connections

$$\mathcal{L}^0 = -\frac{\mathrm{i}\partial\phi}{4\Omega_T}, \quad \bar{\mathcal{L}}^0 = \frac{\mathrm{i}\bar{\partial}\phi}{4\Omega_T},$$
$$\mathcal{L}^+ = \frac{\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}}{\lambda}\frac{(\partial\phi)^2 t}{2\Omega_T} + \lambda\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}\frac{(\Omega_T+1)^2}{8\Omega_T(1-tV)},$$
$$\bar{\mathcal{L}}^+ = \lambda\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}\frac{(\bar{\partial}\phi)^2 t}{2\Omega_T} + \frac{\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}}{\lambda}\frac{(\Omega_T+1)^2}{8\Omega_T(1-tV)},$$
$$\mathcal{L}^- = \frac{\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}}{\lambda}\frac{(\partial\phi)^2 t}{2\Omega_T} + \lambda\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}\frac{(\Omega_T+1)^2}{8\Omega_T(1-tV)},$$
$$\bar{\mathcal{L}}^- = \lambda\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}\frac{(\bar{\partial}\phi)^2 t}{2\Omega_T} + \frac{\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}}{\lambda}\frac{(\Omega_T+1)^2}{8\Omega_T(1-tV)}, \tag{81}$$

which coincide with those found in [18].

C. Liouville field theory

The Lagrangian of the classical Liouville field theory is

$$\mathcal{L}(\mathbf{w}) = \partial_w\phi\partial_{\bar{w}}\phi - \mu\mathrm{e}^{\phi}, \quad V = -\mu\mathrm{e}^{\phi} \tag{82}$$

with the Lax connection

$$\mathcal{L}_w = -\partial_w\phi H + 2\lambda\sqrt{\mu}\mathrm{e}^{\frac{\phi}{2}}E_+,$$
$$\mathcal{L}_{\bar{w}} = \partial_{\bar{w}}\phi H - \frac{1}{2\lambda}\sqrt{\mu}\mathrm{e}^{\frac{\phi}{2}}E_-. \tag{83}$$

The field transformation is also given by (77). Decomposing the Lax connection as (80), we again find

$$\mathcal{L}^0 = -\frac{\partial\phi}{\Omega_T}, \quad \bar{\mathcal{L}}^0 = \frac{\bar{\partial}\phi}{\Omega_T},$$
$$\mathcal{L}^+ = \frac{\lambda\sqrt{\mu}\mathrm{e}^{\frac{\phi}{2}}(1+\Omega_T)^2}{2\Omega_T(1-tV)}, \quad \bar{\mathcal{L}}^+ = \frac{2t\lambda\sqrt{\mu}(\bar{\partial}\phi)^2}{\Omega_T},$$
$$\mathcal{L}^- = -\frac{t\sqrt{\mu}\mathrm{e}^{\frac{\phi}{2}}(\partial\phi)^2}{2\lambda\Omega_T}, \quad \bar{\mathcal{L}}^- = -\frac{\sqrt{\mu}\mathrm{e}^{\frac{\phi}{2}}(1+\Omega_T)^2}{8\lambda\Omega_T(1-tV)}. \tag{84}$$

These differ from the ones in (25) up to numerical factors because of the different conventions.

With these deformed Lax connections, one can derive infinite conserved charges. Conversely, the (anti)-holomorphic currents are simply given by taking powers of the modified traceless stress-energy tensor

$$T_{2n} = \left((\partial_w\phi)^2 - 2\partial_w^2\phi\right)^n, \quad \bar{T}_{2n} = \left((\bar{\partial}_w\phi)^2 - 2\bar{\partial}_w^2\phi\right)^n. \tag{85}$$

From (77) and (62), one can read the deformed currents

$$T_{2n}(\mathbf{z}) = -\frac{\Omega_T + (2t(1-tV)\partial\phi\bar{\partial}\phi+1)}{2\Omega_T(1-\tau V)}T_{2n}(\mathbf{w}(\mathbf{z})),$$
$$\Theta_{2n}(\mathbf{z}) = \frac{t(\bar{\partial}\phi)^2}{\Omega_T}T_{2n}(\mathbf{w}(\mathbf{z})). \tag{86}$$

The explicit expressions of these currents have been derived in [19] using a different method.

D. *N* bosonic scalars with arbitrary potential

To construct the deformed Lax connections for the (affine) Toda field theories, let us first consider N free scalars with arbitrary potential [16, 18]

$$\mathcal{L}_N = \sum_i^N \partial_w\phi_i\partial_{\bar{w}}\phi_i + V(\phi_i). \tag{87}$$

From the relationships

$$\frac{\partial x^1}{\partial y^1} = 1 + tT_2^2(\mathbf{y}), \quad \frac{\partial x^2}{\partial y^2} = 1 + tT_1^1(\mathbf{y}),$$
$$\frac{\partial x^1}{\partial y^2} = \frac{\partial x^2}{\partial y^1} - = -tT_2^1(\mathbf{y}), \tag{88}$$

we can compute the inverse of the Jacobian

$$\mathcal{J}_N^{-1}=\begin{pmatrix}\partial_w z & \partial_w \bar{z}\\ \partial_{\bar{w}} z & \partial_{\bar{w}}\bar{z}\end{pmatrix}=\begin{pmatrix}1-tV & -t\sum_i(\partial_w\phi_i)^2\\ -t\sum_i(\partial_{\bar{w}}\phi_i)^2 & 1-tV\end{pmatrix}. \tag{89}$$

The main technical difficulty of this method is to solve $\partial_w\phi_i$ and $\partial_{\bar{w}}\phi_i$ from

$$\begin{pmatrix}\partial_w\phi_i\\ \partial_{\bar{w}}\phi_i\end{pmatrix}=\mathcal{J}_N^{-1}\begin{pmatrix}\partial\phi_i\\ \bar{\partial}\phi_i\end{pmatrix} \tag{90}$$

in terms of $\partial\phi_i$ and $\bar{\partial}\phi_i$. For this particular example, we find the following solution

$$\partial_w\phi_i=\frac{1}{2t}\frac{\bar{\partial}\phi_i(-1+\Omega_T)+\tilde{t}\dfrac{\partial B}{\partial\partial\phi_i}}{\bar{K}}, \tag{91}$$

$$\partial_{\bar{w}}\phi=\frac{1}{2t}\frac{\partial\phi_i(-1+\Omega_T)+\tilde{t}\dfrac{\partial B}{\partial\bar{\partial}\phi_i}}{K}, \tag{92}$$

with

$$\tilde{t}=t(1-tV),\quad \Omega_T=\sqrt{1+4\tilde{t}(\mathcal{L}^{(0)}-\tilde{t}B)}, \tag{93}$$

$$\mathcal{L}^{(0)}=\sum_{i=1}^N\partial\phi_i\bar{\partial}\phi_i,\quad K=\sum_i^N(\partial\phi_i)^2,\quad \bar{K}=\sum_i^N(\bar{\partial}\phi_i)^2, \tag{94}$$

$$B=\sum_{i=1}^N(\partial\phi_i)^2\sum_{j=1}^N(\bar{\partial}\phi_j)^2-\left(\sum_{i=1}^N\partial\phi_i\bar{\partial}\phi_i\right)^2. \tag{95}$$

The stress-energy tensor is given by

$$K_w=\sum_i^N(\partial_w\phi_i)^2,\quad \bar{K}_{\bar{w}}=\sum_i^N(\partial_{\bar{w}}\phi_i)^2, \tag{96}$$

$$T_2=-\frac{1}{2}K_w,\quad \bar{T}_2=-\frac{1}{2}\bar{K}_{\bar{w}},\quad \Theta_0=-\frac{1}{2}V. \tag{97}$$

Therefore, the deformed Lax connection is directly given by (65)

$$\mathcal{L}=\frac{(1-\tau V)\mathcal{L}_w+\tau K_w\mathcal{L}_{\bar{w}}}{(1-\tau V)^2-\tau^2K_w\bar{K}_{\bar{w}}}, \tag{98}$$

$$\bar{\mathcal{L}}=\frac{(1-\tau V)\mathcal{L}_{\bar{w}}+\tau\bar{K}_{\bar{w}}\mathcal{L}_w}{(1-\tau V)^2-\tau^2K_w\bar{K}_{\bar{w}}}. \tag{99}$$

Using the identities (90), we can find the relations among K_w, $\bar{K}_{\bar{w}}$, K, and $\bar{K}$

$$K_w=(1-\tau V)^2K+\tau^2K_w^2\bar{K}-2\tau K_w(1-\tau V)\mathcal{L}^{(0)}, \tag{100}$$

$$\bar{K}_{\bar{w}}=(1-\tau V)^2\bar{K}+\tau^2K_{\bar{w}}^2K-2\tau\bar{K}_{\bar{w}}(1-\tau V)\mathcal{L}^{(0)}. \tag{101}$$

These are quadratic equations, whose solutions are[1)]

$$K_w=\frac{2\tilde{t}\mathcal{L}^{(0)}+1-\Omega_T}{2t^2\bar{K}},\quad \bar{K}_{\bar{w}}=\frac{2\tilde{t}\mathcal{L}^{(0)}+1-\Omega_T}{2\tau^2K}, \tag{102}$$

where we used the identity

$$B=K\bar{K}-\mathcal{L}^{(0)}\mathcal{L}^{(0)}. \tag{103}$$

Substituting (102) into (98) gives

$$\mathcal{L}=-\frac{\Omega_T+(2\tilde{t}\mathcal{L}^{(0)}+1)}{2\Omega_T(1-tV)}\mathcal{L}_w-\frac{tK}{\Omega_T}\mathcal{L}_{\bar{w}}, \tag{104}$$

$$\bar{\mathcal{L}}=-\frac{\Omega_T+(2\tilde{t}\mathcal{L}^{(0)}+1)}{2\Omega_T(1-tV)}\mathcal{L}_{\bar{w}}-\frac{t\bar{K}}{\Omega_T}\mathcal{L}_w. \tag{105}$$

For the affine Toda theories, whose Lax connections are known, it is straightforward to read the deformed Lax connection from (105). They turn out to match the ones we derived previously in Section II.B. Furthermore, we can use the relations to derive the deformed Lax connection of the PCM if we make the following identification

$$j_\mu=j_\mu^iT_i,\quad j_\mu^i\equiv\partial_\mu\phi_i, \tag{106}$$

where T_i are the generators of the Lie algebra with the Killing metric $\mathrm{Tr}(T_iT_j)=\delta_{ij}$.

E. Nonlinear Schrödinger model

As our last example, let us consider the $T\bar{T}$-deformed nonlinear Schrödinger model, which is a non-relativistic complex field theory. The $T\bar{T}$-deformed Lagrangian was recently derived in [22-24]. Here, we derive the deformed Lax connection from the dynamic coordinate transformation.

For the undeformed model, the Lagrangian is

$$\mathcal{L}_{\rm NS}(y_1,y_2)=\frac{\mathrm{i}}{2}\left(\bar{q}\partial_{y_1}q-q\partial_{y_1}\bar{q}\right)-\frac{\partial_{y_2}q\partial_{y_2}\bar{q}}{2m}-g|q\bar{q}|^2, \tag{107}$$

1) There are two branches of solutions, here we only keep the one which is consistent with our results in previous section. The other branch gives equivalent result up to a gauge transformation.

which has the following equations of motion

$$-\mathrm{i}\partial_{y_1}q=\frac{1}{2m}\partial_{y_2}^2q-2gq^2\bar{q},\quad \mathrm{i}\partial_{y_1}\bar{q}=\frac{1}{2m}\partial_{y_2}^2\bar{q}-2gq\bar{q}^2,\tag{108}$$

and the stress-energy tensor

$$T_{y_2y_2}=-\frac{1}{m}\partial_{y_2}q\partial_{y_2}\bar{q}-\mathcal{L}_{NS}(y_1,y_2),\tag{109}$$

$$T_{y_2y_1}=-\frac{1}{2m}\left(\partial_{y_2}\bar{q}\partial_{y_1}q+\partial_{y_2}q\partial_{y_1}\bar{q}\right),\tag{110}$$

$$T_{y_1y_2}=\frac{\mathrm{i}}{2}\left(\bar{q}\partial_{y_2}q-q\partial_{y_2}\bar{q}\right),\tag{111}$$

$$T_{y_1y_1}=\frac{\mathrm{i}}{2}\left(\bar{q}\partial_{y_1}q-q\partial_{y_1}\bar{q}\right)-\mathcal{L}_{\rm NS}(y_1,y_2).\tag{112}$$

The corresponding Lax connection is

$$U_{y_2}=-\mathrm{i}\lambda\sigma_3+\mathrm{i}\sqrt{2gm}Q,\tag{113}$$

$$V_{y_1}=-\frac{\mathrm{i}\lambda^2}{m}\sigma_3+\mathrm{i}\sqrt{\frac{2g}{m}}\lambda Q+\sqrt{\frac{g}{2m}}\partial_{y_2}Q\sigma_3+\mathrm{i}gQ^2\sigma_3,\tag{114}$$

where

$$\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad Q=\begin{pmatrix}0&q\\-\bar{q}&0\end{pmatrix}.\tag{115}$$

Solving (88), one can find the following rules of transformation [24]:

$$\partial_{y_1}q=\frac{2m(B-S)\partial_{x_1}\bar{q}+2\tilde{t}\bar{A}C}{2t\bar{A}^2},\quad \partial_{y_2}q=\frac{2m(B-S)}{2t\bar{A}},$$
$$\partial_{y_1}\bar{q}=\frac{2m(B-S)\partial_{x_1}q-2\tilde{t}AC}{2tA^2},\quad \partial_{y_2}\bar{q}=\frac{2m(B-S)}{2tA},\tag{116}$$

where we have defined

$$\tilde{t}=t(1+tV),\quad C=\partial_{x_2}\bar{q}\partial_{x_1}q-\partial_{x_1}\bar{q}\partial_{x_2}q,\tag{117}$$

$$B=1+\frac{\mathrm{i}t}{2}(\bar{q}\partial_{x_1}q-q\partial_{x_1}\bar{q}),\quad S=\sqrt{B^2-\frac{2\tilde{t}}{m}A\bar{A}},\tag{118}$$

$$A=\partial_{x_2}q+\frac{\mathrm{i}t}{2}qC,\quad \bar{A}=\partial_{x_2}\bar{q}+\frac{\mathrm{i}t}{2}\bar{q}C.\tag{119}$$

Substituting (108) and (116) into (65), and after some manipulation, we end up with the final results for the deformed Lax connection

$$\begin{pmatrix}V_{x_1}\\U_{x_2}\end{pmatrix}=\begin{pmatrix}J_{11}&J_{12}\\J_{21}&J_{22}\end{pmatrix}\begin{pmatrix}V_{y_1}\\U_{y_2}\end{pmatrix},\tag{120}$$

where

$$J_{11}=\frac{tB(B+S)}{2S\tilde{t}},\quad J_{12}=-\frac{t(A\partial_{x_1}\bar{q}+\bar{A}\partial_{x_1}q)}{2mS},$$
$$J_{21}=\frac{\mathrm{i}t^2(B+S)(\bar{q}\partial_{x_2}q-q\partial_{x_2}\bar{q})}{4S\tilde{t}},\tag{121}$$

$$J_{22}=\frac{2tA\bar{A}}{2mS(B-S)}-\frac{t(A\partial_{x_2}\bar{q}+\bar{A}\partial_{x_2}q)}{2mS}.\tag{122}$$

V. CONCLUSION

In this work, we constructed the Lax connections of several $T\bar{T}$-deformed integrable models in two different ways and found a consistent picture. The first method is based on a proper ansatz, which assumes that the Lax equation is linearly dependent on the equation of motion. In the discussion, we also assumed that some proportional functions or parameters are invariant under the deformation. We obtained the Lax connections for the affine Toda theories and the principal chiral model. This method is suggestive, but its potential is not clear to us.

The other method relies on a dynamic coordinate transformation between the $T\bar{T}$-deformed theory and its predecessor. This method is systematic, but it may be difficult to implement in some models because of the complexity of the dynamic coordinate transformation. We showed the power of the coordinate transformation in several models, including the free scalar theory, sine-Gordon model, Liouville field theory, N-scalar theory, and non-linear Schrödinger model.

We want to stress that the dynamic coordinate transformation is not a diffeomorphism. Because the coordinate transformation depends on the dynamic fields, the inverse of the transformation cannot be obtained in a closed form. Actually, we attempted to derive the Lax connection of the $T\bar{T}$-deformed KdV equation. In this case, the coordinate transformation depends on higher order derivatives, so the closed form of the inverse of the transformation is unlikely to exist. It is interesting to investigate the effectiveness of the coordinate transformation in other models, for example, fermionic ones [25]. In addition to the two methods discussed in this work, it would be interesting to study the Lax connection from other aspects regarding $T\bar{T}$-deformation, e.g., from the light–cone gauge approach in [7].

Given the explicit form of the Lax connection, there are various applications. The first one involves the construction of infinite conserved charges, as we show for the PCM. Expression (52) indicates that the conserved charges become deformed in a very complicated way, and it would be very interesting to study how the algebra is deformed. The other application involves constructing solitonic surfaces, following [9]. Most importantly, we hope our construction of the Lax connection can shed light on the quantization of the $T\bar{T}$-deformation.

ACKNOWLEDGEMENTS

We would like to thank Huajia Wang for valuable discussions.

References

[1] F. A. Smirnov and A. B. Zamolodchikov, Nucl. Phys. B **915**, 363-383 (2017), arXiv:1608.05499[hep-th]

[2] A. Cavaglià, S. Negro, I. M. Szécsényi *et al.*, JHEP **10**, 112 (2016), arXiv:1608.05534[hep-th]

[3] S. Dubovsky, V. Gorbenko, and M. Mirbabayi, JHEP **09**, 136 (2017), arXiv:1706.06604[hep-th]

[4] A. B. Zamolodchikov, arXiv: hep-th/0401146 [hep-th]

[5] J. Cardy, JHEP **10**, 186 (2018), arXiv:1801.06895[hepth]

[6] S. Dubovsky, V. Gorbenko, and G. Hernández-Chifflet, JHEP **09**, 158 (2018), arXiv:1805.07386[hep-th]

[7] S. Frolov, Proc. Steklov Inst. Math. **309**, 107-126 (2020), arXiv:1905.07946[hepth]

[8] N. Callebaut, J. Kruthoff, and H. Verlinde, JHEP **04**, 084 (2020), arXiv:1910.13578[hep-th]

[9] R. Conti, S. Negro, and R. Tateo, JHEP **02**, 085 (2019), arXiv:1809.09593[hep-th]

[10] E. K. Sklyanin, J. Sov. Math. **19**, 1546-1596 (1982)

[11] S. Frolov, JHEP **05**, 069 (2005), arXiv:hep-th/0503201[hep-th]

[12] A. Giveon, N. Itzhaki, and D. Kutasov, JHEP **07**, 122 (2017), arXiv:1701.05576[hep-th]

[13] M. Baggio and A. Sfondrini, Phys. Rev. D **98**(2), 021902 (2018), arXiv:1804.01998[hep-th]

[14] A. Sfondrini and S. J. van Tongeren, Phys. Rev. D **101**(6), 066022 (2020), arXiv:1908.09299[hep-th]

[15] T. Araujo, E. Ó. Colgáin, Y. Sakatani *et al.*, JHEP **03**, 168 (2019), arXiv:1811.03050[hepth]

[16] G. Bonelli, N. Doroud, and M. Zhu, JHEP **06**, 149 (2018), arXiv:1804.10967[hepth]

[17] E. Corrigan, arXiv: hep-th/9412213 [hep-th]

[18] R. Conti, L. Iannella, S. Negro *et al.*, JHEP **11**, 007 (2018), arXiv:1806.11515[hepth]

[19] M. Leoni, JHEP **07**(07), 230 (2020), arXiv:2005.08906[hepth]

[20] N. J. MacKay, Phys. Lett. B **281**, 90-97 (1992), Erratum: Phys. Lett. B **308**, 444 (1993)

[21] R. Conti, S. Negro, and R. Tateo, JHEP **11**, 120 (2019), arXiv:1904.09141[hepth]

[22] B. Chen, J. Hou, and J. Tian, Phys. Rev. D **104**(2), 025004 (2021), arXiv:2012.14091 [hep-th]

[23] D. Hansen, Y. Jiang, and J. Xu, JHEP **04**, 186 (2021), arXiv:2012.12290 [hep-th]

[24] P. Ceschin, R. Conti, and R. Tateo, JHEP **04**, 121 (2021), arXiv:2012.12760 [hep-th]

[25] E. A. Coleman, J. Aguilera-Damia, D. Z. Freedman *et al.*, JHEP **10**, 080 (2019), arXiv:1906.05439[hepth]