Teukolsky-like equations with various spins in spherically symmetric spacetime*

Ya Guo (郭亚)[1] Hiroaki Nakajima[2] Wenbin Lin (林文斌)[1,2†]

[1]School of Physical Science and Technology, Southwest Jiaotong University, Chengdu 610031, China
[2]School of Mathematics and Physics, University of South China, Hengyang 421001, China

Abstract: We study wave equations with various spins on the background of a general spherically symmetric spacetime. We obtain the unified expression of the Teukolsky-like master equations and the corresponding radial equations with the general spins. We also discuss the gauge dependence in the gravitational-wave equations, which have appeared in previous studies.

Keywords: gravitational wave, effective one-body dynamics, Teukolsky equation, wave equations, Klein-Gordon equation, Weyl equation, Maxwell equation

DOI: 10.1088/1674-1137/ad2a61

I. INTRODUCTION

The study of gravitational waves has become a highly important subject in cosmology since the direct observation of gravitational waves by LIGO and VIRGO [1]. Gravitational waves can be described as a solution to the Einstein equation, where the metric is perturbed around a certain background. When the background is chosen as the black hole spacetime, one can, for example, consider the gravitational-wave radiation from a relatively light object rotating around the black hole, which can be studied using the black-hole perturbation theory [2]. However, when the mass of the object is not so light, the contribution to the background spacetime from this object may not be negligible. In such a case, the background must be modified or other methods must be used, such as the post-Newtonian approximation [3]. One systematic method for modifying the background is proposed as effective one-body (EOB) dynamics [4, 5].

The background of EOB dynamics is deformed by the black hole spacetime and hence may not satisfy the vacuum Einstein equation. Motivated by this, we consider a general spherically symmetric spacetime as a simple example of the background, which is not necessarily a vacuum. A particular form of the background appears in EOB dynamics for the spinless binary system [4, 5]. The spherically symmetric spacetime satisfies the Petrov type D condition, which has played a key role in deriving the gravitational-wave equation in previous studies [6−8] using the Newman-Penrose formalism [9], similar to the Teukolsky equation [10] in the vacuum case. The advantage of using the Newman-Penrose formalism is that the role of the Einstein equation is restrictive and is merely used to relate the Ricci tensor to the energy-momentum tensor. Therefore, the extension to the non-vacuum case is relatively easier. It has been found that in order to obtain the decoupled wave equation, the gauge condition must be taken such that some of the coupled degrees of freedom vanish [6−8]. To date, two types of the gravitational-wave equation have been proposed [7, 8] owing to the difference in the gauge conditions.

In this study, we investigate massless wave equations with different spins, that is, the (massless) Klein-Gordon, Weyl, and Maxwell equations on the same background spacetime. To avoid complexities, we first provide a unified expression for these equations consisting of the Newman-Penrose quantities, as in [11−14], and obtain the explicit wave equation and ordinary differential equation for the radial coordinate.

The remainder of this paper is organized as follows. In Section II, we introduce our parameterization of the background of a spherically symmetric spacetime and present the quantities in the Newman-Penrose formalism. In Section III, we observe the wave equation for spin 0, $\pm1/2$, ±1, and ±2 on this background. Subsequently, we provide the unified expression for these equations with general spin s and obtain the explicit Teukolsky-like master equation and the corresponding radial equation. In Section IV, we discuss the gauge dependence in the gravitational-wave equations proposed in previous studies. Fi-

Received 15 January 2024; Accepted 18 February 2024; Published online 19 February 2024

* Supported in part by the National Natural Science Foundation of China (11973025)

† E-mail: lwb@usc.edu.cn

nally, Section V presents a summary and discussion. In Appendix A, we list our notations and conventions.

II. BACKGROUND METRIC AND TETRADS

We consider the general spherically symmetric background as

$$\mathrm{d}s^2 = \mathcal{A}(r)\mathrm{d}t^2 - \mathcal{B}(r)\mathrm{d}r^2 - C(r)r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{1}$$

By choosing appropriate radial coordinate r, we can set$C(r) = 1$, which corresponds to the standard coordinate. We define $\mathcal{D}(r) = \sqrt{\mathcal{A}(r)\mathcal{B}(r)}$, and then background metric (1) becomes

$$\mathrm{d}s^2 = \mathcal{A}(r)\mathrm{d}t^2 - \frac{\mathcal{D}(r)^2}{\mathcal{A}(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{2}$$

The null tetrads corresponding to metric (2) are taken as

$$\begin{aligned} l &= l^A_\mu \mathrm{d}x^\mu = \mathrm{d}t - \frac{\mathcal{D}(r)}{\mathcal{A}(r)}\mathrm{d}r, \\ n &= n^A_\mu \mathrm{d}x^\mu = \frac{\mathcal{A}(r)}{2}\mathrm{d}t + \frac{\mathcal{D}(r)}{2}\mathrm{d}r, \\ m &= m^A_\mu \mathrm{d}x^\mu = -\frac{r}{\sqrt{2}}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi), \\ \bar{m} &= \bar{m}^A_\mu \mathrm{d}x^\mu = -\frac{r}{\sqrt{2}}(\mathrm{d}\theta - \mathrm{i}\sin\theta\mathrm{d}\varphi), \end{aligned} \tag{3}$$

which satisfy

$$\mathrm{d}s^2 = 2\,l n - 2m\bar{m}, \tag{4}$$

where the bar denotes the complex conjugate. Superscript (or subscript) A is used as the symbol of the background quantities [10]. Conversely, for the perturbation quantities of the gravitational field, we use superscipt B, which will appear later.

From the tetrad basis (3), we can compute spin coefficients and the components of the Ricci tensor and Weyl scalars as

$$\kappa^A = \nu^A = \sigma^A = \lambda^A = \pi^A = \tau^A = \epsilon^A = 0, \tag{5}$$

$$\begin{aligned} &\rho^A = -\frac{1}{r\mathcal{D}}, \quad \mu^A = -\frac{\mathcal{A}}{2r\mathcal{D}}, \quad \gamma^A = \frac{\mathcal{A}'}{4\mathcal{D}}, \\ &\alpha^A = -\beta^A = -\frac{\cot\theta}{2\sqrt{2}r}, \end{aligned} \tag{6}$$

$$\Phi^A_{01} = \Phi^A_{10} = \Phi^A_{02} = \Phi^A_{20} = \Phi^A_{12} = \Phi^A_{21} = 0, \tag{7}$$

$$\Phi^A_{00} = \frac{\mathcal{D}'}{r\mathcal{D}^3}, \quad \Phi^A_{22} = \frac{\mathcal{A}^2\mathcal{D}'}{4rD^3}, \tag{8}$$

$$\Phi^A_{11} = \frac{1}{8r^2\mathcal{D}^3}\left[2\mathcal{D}^3 - 2\mathcal{A}\mathcal{D} - r^2(\mathcal{A}'\mathcal{D}' - \mathcal{A}''\mathcal{D})\right], \tag{9}$$

$$\begin{aligned} \Lambda^A = &-\frac{1}{24r^2\mathcal{D}^3}\left[-2\mathcal{D}^3 - r^2\mathcal{A}'\mathcal{D}' + 2\mathcal{A}(\mathcal{D} - 2r\mathcal{D}')\right. \\ &\left.+ r\mathcal{D}(4\mathcal{A}' + r\mathcal{A}'')\right], \end{aligned} \tag{10}$$

$$\Psi^A_0 = \Psi^A_1 = \Psi^A_3 = \Psi^A_4 = 0, \tag{11}$$

$$\begin{aligned} \Psi^A_2 = &\frac{1}{12r^2\mathcal{D}^3}\left[2(\mathcal{A}\mathcal{D} + r\mathcal{A}\mathcal{D}' - \mathcal{D}^3)\right. \\ &\left.- r\mathcal{A}'(2\mathcal{D} + r\mathcal{D}') + r^2\mathcal{A}''\mathcal{D}\right], \end{aligned} \tag{12}$$

where we follow the notations of Newman-Penrose [9] and Pirani [15]. The same notation is also used for Teukolsky [10]. We also list the definitions of the above quantities in the appendix. The prime symbol denotes the derivative with respect to r. Equation (11) implies that the background belongs to Petrov type D, which is important to derive the wave equation on this background. As a special case, for $\mathcal{D}(r) = 1$, we have

$$\Phi^A_{00} = \Phi^A_{22} = 0, \tag{13}$$

and the nonvanishing quantities in the background are simplified as

$$\begin{aligned} &\rho^A = -\frac{1}{r}, \quad \mu^A = -\frac{\mathcal{A}}{2r}, \quad \gamma^A = \frac{\mathcal{A}'}{4}, \\ &\alpha^A = -\beta^A = -\frac{\cot\theta}{2\sqrt{2}r}, \end{aligned} \tag{14}$$

$$\Phi^A_{11} = \frac{1}{8r^2}(2 - 2\mathcal{A} + r^2\mathcal{A}''), \tag{15}$$

$$\Lambda^A = -\frac{1}{24r^2}(-2 + 2\mathcal{A} + 4r\mathcal{A}' + r^2\mathcal{A}''), \tag{16}$$

$$\Psi^A_2 = \frac{1}{12r^2}(-2 + 2\mathcal{A} - 2r\mathcal{A}' + r^2\mathcal{A}''), \tag{17}$$

We also note that when choosing $\mathcal{A} = 1 - 2M/r$ and $\mathcal{D} = 1$, the background reduces to the Schwarzschild spacetime, where M is the mass of the Schwarzschild black hole.

III. WAVE EQUATIONS WITH VARIOUS SPINS

Here, we consider the wave equation with various spins s on the background specified by (5)–(12) in the previous section, namely, the Klein-Gordon ($s = 0$), Weyl

($s=\pm1/2$), and Maxwell equations ($s=\pm1$) and the equation from the Newman-Penrose formalism ($s=\pm2$) under the probe (test field) approximation; therefore, the back reactions from matter and electromagnetic fields to the gravitational background are assumed to be negligible. For simplicity, we also assume that the fields are massless and minimally coupled to the background of the gravitational field, unless otherwise specified.

A. spin 0

The massless Klein-Gordon equation in the gravitational background is

$$\Box\phi=\nabla_\mu(g^{\mu\nu}\partial_\nu\phi)=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)=T,\tag{18}$$

where T is the source. ∇_μ is the covariant derivative with respect to the curved spacetime (not for the local Lorentz transformation), of which the projection by the null tetrads gives

$$D^A=l^\mu_A\nabla_\mu,\quad \Delta^A=n^\mu_A\nabla_\mu,\quad \delta^A=m^\mu_A\nabla_\mu,\quad \bar\delta^A=\bar m^\mu_A\nabla_\mu.\tag{19}$$

By decomposing $g^{\mu\nu}$ in terms of the null tetrads, (18) can be rewritten as

$$\left[(\Delta-2\gamma+2\mu)D+(D-2\rho)\Delta-(\bar\delta-2\alpha)\delta-(\delta-2\alpha)\bar\delta\right]^A\phi=T,\tag{20}$$

where the superscript A outside the parentheses denotes that all the quantities and operators inside the parentheses are background ones, and we use the relation for the spin coefficients,

$$\begin{aligned}&\nabla_\mu l^\mu_A=-2\rho^A,\quad \nabla_\mu n^\mu_A=-2\gamma^A+2\mu^A,\\&\nabla_\mu m^\mu_A=\nabla_\mu\bar m^\mu_A=-2\alpha^A.\end{aligned}\tag{21}$$

For later convenience, we rewrite (20) such that the order of the differential operators is rearranged, satisfying the commutation relations

$$\Delta^AD^A-D^A\Delta^A=2\gamma^AD^A,\quad \bar\delta^A\delta^A-\delta^A\bar\delta^A=2\alpha^A(\delta-\bar\delta)^A.\tag{22}$$

We also use

$$\Delta^A\rho^A=(2\gamma-\mu)^A\rho^A-\Psi^A_2-2\Lambda^A,\tag{23}$$

which is from the background part of that of the Newman-Penrose equation. Then, (20) can be rewritten as

$$\left[(\Delta-2\gamma+\mu)(D-\rho)-(\bar\delta-2\alpha)\delta-\Psi_2-2\Lambda\right]^A\phi=\frac{1}{2}T.\tag{24}$$

Note that the last term $-2\Lambda^A\phi$ on the right hand side is responsible for the minimal coupling. If we consider curvature coupling, there is a contribution proportional to $\mathcal{R}\phi=24\Lambda^A\phi$, where $\mathcal{R}$ is the Ricci scalar (see the appendix).

B. spin $\pm1/2$

The massless Dirac equation can be decomposed into two Weyl equations. For the positive chirality part, the Weyl equation in the gravitational background is

$$(\bar\delta-\alpha)^A\chi_0-(D-\rho)^A\chi_1=0,\tag{25}$$

$$(\Delta-\gamma+\mu)^A\chi_0-(\delta-\alpha)^A\chi_1=0,\tag{26}$$

where χ_0 and χ_1 are the components of the Weyl spinor. We can eliminate χ_0 using the following commutation relation:

$$\begin{aligned}&\left[\Delta+p\gamma-(q-1)\mu\right]^A(\bar\delta+p\alpha)^A\\&\quad-\left(\bar\delta+p\alpha\right)^A(\Delta+p\gamma-q\mu)^A\\&=\nu^AD^A-\lambda^A\delta^A+p(\alpha\lambda+\rho\nu-\Psi_3)^A\\&\quad+q(-D\nu+\delta\lambda-4\alpha\lambda+2\Psi_3)^A\\&=0,\end{aligned}\tag{27}$$

where p and q are arbitrary constants, and we just use $\nu^A=\lambda^A=\Psi^A_3=0$ for the last equality. Hence, (27) holds not only in the vacuum, but also for the background specified by (5)–(12). We obtain the wave equation for χ_1 in a similar way via the method used to derive the Teukolsky equation [10]. We operate $(\Delta-\gamma+2\mu)^A$ on (25) and $(\bar\delta-\alpha)^A$ on (26) and then obtain the difference of them. The terms with χ_0 are canceled from (27) with $p=q=-1$, and the remaining part is

$$\left[(\Delta-\gamma+2\mu)(D-\rho)-(\bar\delta-\alpha)(\delta-\alpha)\right]^A\chi_1=0,\tag{28}$$

which gives the wave equation for $s=-1/2$. In a similar way, the wave equation for χ_0 (for $s=1/2$) is obtained as

$$\left[(D-2\rho)(\Delta-\gamma+\mu)-(\delta-\alpha)(\bar\delta-\alpha)\right]^A\chi_0=0.\tag{29}$$

We note that (28) and (29) take the same forms as in the vacuum case [10]. For later convenience, we rewrite (29) as in the previous subsection. We use (22), (23), and

$$D^A\gamma^A \quad = \Psi_2^A + \Phi_{11}^A - \Lambda^A, \tag{30}$$

$$(\delta+\bar{\delta})^A\alpha^A = \mu^A\rho^A + 4(\alpha^A)^2 - \Psi_2^A + \Phi_{11}^A + \Lambda^A, \tag{31}$$

$$D^A\mu^A = \mu^A\rho^A + \Psi_2^A + 2\Lambda^A. \tag{32}$$

Then, (29) can be rewritten as

$$\left[(\Delta-3\gamma+\mu)(D-2\rho)-(\bar{\delta}-3\alpha)(\delta+\alpha)-3\Psi_2\right]^A\chi_0 = 0, \tag{33}$$

C. spin ±1

The Maxwell equation in the gravitational background is

$$(D-2\rho)^A\phi_1 - (\bar{\delta}-2\alpha)^A\phi_0 = J_l, \tag{34}$$

$$\delta^A\phi_1 - (\Delta+\mu-2\gamma)^A\phi_0 = J_m, \tag{35}$$

$$(D-\rho)^A\phi_2 - \bar{\delta}^A\phi_1 = J_{\bar{m}}, \tag{36}$$

$$(\delta-2\alpha)^A\phi_2 - (\Delta+2\mu)^A\phi_1 = J_n, \tag{37}$$

where ϕ_0, ϕ_1 , and ϕ_2 are complex and constructed from the field strength (the Faraday tensor) $F_{\mu\nu}$ as [1)]

$$\phi_0 = F_{\mu\nu}l_A^\mu m_A^\nu, \quad \phi_1 = \tfrac{1}{2}F_{\mu\nu}(l_A^\mu n_A^\nu + \bar{m}_A^\mu m_A^\nu), \quad \phi_2 = F_{\mu\nu}\bar{m}_A^\mu n_A^\nu. \tag{38}$$

J_l, J_n, J_m , and $J_{\bar{m}}$ are the projections of the current J^μ by the null tetrads as $J_l = J^\mu l_\mu^A$, etc. From (36) and (37), we can construct the wave equation for ϕ_2 via a similar procedure to that used in the previous subsection. Using the commutation relation (27) with $p=0$ and $q=-2$, we have

$$\left[(\Delta+3\mu)(D-\rho)-\bar{\delta}(\delta-2\alpha)\right]^A\phi_2 = J_2, \tag{39}$$

where J_2 is defined by

$$J_2 = (\Delta+3\mu)^A J_{\bar{m}} - \bar{\delta}^A J_n. \tag{40}$$

In a similar way, we can obtain the wave equation for ϕ_0 from (34) and (35) as

$$\left[(D-3\rho)(\Delta-2\gamma+\mu)-\delta(\bar{\delta}-2\alpha)\right]^A\phi_0 = J_0, \tag{41}$$

where J_0 is defined by

$$J_0 = \delta^A J_l - (D-3\rho)^A J_m. \tag{42}$$

Note that (39) and (41) take the same forms as in the vacuum case [10]. For later convenience, we rewrite (41) using (22), (23), and (30)–(32) as

$$\left[(\Delta-4\gamma+\mu)(D-3\rho)-(\bar{\delta}-4\alpha)(\delta+2\alpha)-6\Psi_2\right]^A\phi_0 = J_0, \tag{43}$$

D. spin ±2

The wave equations for the spin ±2 are obtained from the perturbed Einstein equation or the perturbed Newman-Penrose equation. In a previous paper [8], we studied and obtained the wave equations, and in the next section, we revisit the derivation to discuss the gauge dependence. Then, we provide the result of the equations.

Here, we take the gauge such that the following quantities vanish [8]:

$$\lambda^B = \sigma^B = 0, \tag{44}$$

$$(\bar{\delta}-\bar{\tau}+2\alpha+2\bar{\beta})^B\Phi_{22}^A = 0, \tag{45}$$

$$(\delta+\bar{\pi}-2\bar{\alpha}-2\beta)^B\Phi_{00}^A = 0, \tag{46}$$

where the superscript B denotes the perturbation part. Under the above gauge, the wave equation for the perturbation part of the Weyl scalar Ψ_4^B is

$$\left[(\Delta+2\gamma+5\mu)(D-\rho)-(\bar{\delta}+2\alpha)(\delta-4\alpha)-3\Psi_2+2\Phi_{11}\right]^A\Psi_4^B = T_4, \tag{47}$$

where the source T_4 is defined by

$$\begin{aligned} T_4 =& (\Delta+2\gamma+5\mu)^A\left[(\bar{\delta}+2\alpha)^A\Phi_{21}^B - (\Delta+\mu)^A\Phi_{20}^B\right] \\ &- (\bar{\delta}+2\alpha)^A\left[\bar{\delta}^A\Phi_{22}^B - (\Delta+2\gamma+2\mu)^A\Phi_{21}^B\right]. \end{aligned} \tag{48}$$

In a similar way, we can also obtain the wave equation for Ψ_0^B as

$$\begin{aligned} &\left[(D-5\rho)(\Delta-4\gamma+\mu)-(\delta+2\alpha)(\bar{\delta}-4\alpha)\right. \\ &\left.-3\Psi_2+2\Phi_{11}\right]^A\Psi_0^B = T_0, \end{aligned} \tag{49}$$

1) As in the vacuum case, we will consider the wave equation just for ϕ_0 and ϕ_2, from which ϕ_1 can be obtained using (34)–(37).

where the source T_0 is defined by

$$T_0 = (\delta+2\alpha)^A\left[(D-2\rho)^A\Phi^B_{01} - \delta^A\Phi^B_{00}\right] - (D-5\rho)^A\left[(D-\rho)^A\Phi^B_{02} - (\delta+2\alpha)^A\Phi^B_{01}\right]. \quad (50)$$

For later convenience, we rewrite (49) using (22), (23), and (30)–(32) as

$$\left[(\Delta-6\gamma+\mu)(D-5\rho) - (\bar{\delta}-6\alpha)(\delta+4\alpha) - 15\Psi_2 + 2\Phi_{11}\right]^A\Psi^B_0 = T_0. \quad (51)$$

E. Unified wave equation for general spin s

We can unify the above wave equations (24), (28), (33), (39), (43), (47), and (51) and express the unified equation for general spin s as

$$\Big\{\left[\Delta-2(1+s)\gamma+(1-s+|s|)\mu\right]\left[D-(1+s+|s|)\rho\right] - \left[\bar{\delta}-2(1+s)\alpha\right](\delta+2s\alpha) - (1+3s+2s^2)\Psi_2 + \frac{1}{3}(|s|-3|s|^2+2|s|^3)\Phi_{11} - 2\delta_s\Lambda\Big\}^A\tilde{\psi}_{(s)} = \tilde{T}_{(s)}, \quad (52)$$

where we collectively denote the fields and sources as

$$\tilde{\psi}_{(s)} = \begin{cases} \Psi^B_4 & \text{for } s=-2 \\ \phi_2 & \text{for } s=-1 \\ \chi_1 & \text{for } s=-1/2 \\ \phi & \text{for } s=0 \\ \chi_0 & \text{for } s=1/2 \\ \phi_0 & \text{for } s=1 \\ \Psi^B_0 & \text{for } s=2 \end{cases}, \qquad \tilde{T}_{(s)} = \begin{cases} T_4 & \text{for } s=-2 \\ J_2 & \text{for } s=-1 \\ 0 & \text{for } s=-1/ \\ T/2 & \text{for } s=0 \\ 0 & \text{for } s=1/2 \\ J_0 & \text{for } s=1 \\ T_0 & \text{for } s=2 \end{cases} \quad (53)$$

δ_s is defined by

$$\delta_s = \begin{cases} 1 & \text{for } s=0 \\ 0 & \text{otherwise.} \end{cases} \quad (54)$$

We rewrite (52) in a slightly simpler form using the following redefinitions:

$$\psi_{(s)} = \exp\left[(|s|-s)f\right]\tilde{\psi}_{(s)}, \quad T_{(s)} = \exp\left[(|s|-s)f\right]\tilde{T}_{(s)}, \quad (55)$$

where f is a function of r, which will be determined soon. By substituting (55) into (52), we have

$$\Big\{\left[\Delta-2(1+s)\gamma+\mu+(|s|-s)(\mu-\Delta f)\right]\left[D-(1+2s)\rho-(|s|-s)\right. - \left[\bar{\delta}-2(1+s)\alpha\right](\delta+2s\alpha) - (1+3s+2s^2)\Psi_2 + \frac{1}{3}(|s|-3|s|^2+2|s|^3)\Phi_{11} - 2\delta_s\Lambda\Big\}^A\psi_{(s)} = T_{(s)}. \quad (56)$$

We find that $\mu^A-\Delta^A f$ and $\rho^A+D^A f$ can simultaneously vanish by choosing $f=\ln r$, namely, [1)]

$$\psi_{(s)} = r^{|s|-s}\tilde{\psi}_{(s)}, \quad T_{(s)} = r^{|s|-s}\tilde{T}_{(s)}. \quad (57)$$

Then, (56) is simplified as

$$\Big\{\left[\Delta-2(1+s)\gamma+\mu\right]\left[D-(1+2s)\rho\right] - \left[\bar{\delta}-2(1+s)\alpha\right](\delta+2s\alpha) - (1+3s+2s^2)\Psi_2 + \frac{1}{3}(|s|-3|s|^2+2|s|^3)\Phi_{11} - 2\delta_s\Lambda\Big\}^A\psi_{(s)} = T_{(s)}. \quad (58)$$

The advantage of the above form is that in the vacuum $\Phi^A_{11}=\Lambda^A=0$, the equation (58) depends on s but not $|s|$. The same transformation (57) has been performed in the vacuum case [10]. Similar equations to (52) and (58) were studied in [11−14]; however, these were considered for positive and negative s separately, or restricted to positive s. Here, we obtain the completely unified expression for both positive and negative s. Moreover, we find the contributions of Φ^A_{11} and Λ^A to the wave equation.

By substituting the background, the explicit form of the unified wave equation (58) is

1) The additive integration constant for f is irrelevant here, and is chosen in convenience.

$$\begin{aligned}&\frac{r^2}{\mathcal{A}}\frac{\partial^2\psi_{(s)}}{\partial t^2}-\frac{1}{\mathcal{D}(r^2\mathcal{A})^s}\frac{\partial}{\partial r}\left[\frac{(r^2\mathcal{A})^{s+1}}{\mathcal{D}}\frac{\partial\psi_{(s)}}{\partial r}\right]-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_{(s)}}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2\psi_{(s)}}{\partial\varphi^2}\\&+\left(\frac{2sr}{\mathcal{D}}-\frac{sr^2\mathcal{A}'}{\mathcal{A}\mathcal{D}}\right)\frac{\partial\psi_{(s)}}{\partial t}-\frac{2\mathrm{i}s\cot\theta}{\sin\theta}\frac{\partial\psi_{(s)}}{\partial\varphi}+\left[s^2\cot^2\theta-s-\frac{s(2s+1)r\mathcal{A}\mathcal{D}'}{\mathcal{D}^3}\right.\\&+\frac{1}{3}(1-\delta_s+3s+2s^2)\left(1-\frac{\mathcal{A}}{\mathcal{D}^2}-\frac{2r\mathcal{A}'}{\mathcal{D}^2}+\frac{2r\mathcal{A}\mathcal{D}'}{\mathcal{D}^3}+\frac{r^2\mathcal{A}'\mathcal{D}'}{2\mathcal{D}^3}-\frac{r^2\mathcal{A}''}{2\mathcal{D}^2}\right)\\&\left.+\frac{1}{6}(|s|-3|s|^2+2|s|^3)\left(1-\frac{\mathcal{A}}{\mathcal{D}^2}-\frac{r^2\mathcal{A}'\mathcal{D}'}{2\mathcal{D}^3}+\frac{r^2\mathcal{A}''}{2\mathcal{D}^2}\right)\right]\psi_{(s)}=2r^2T_{(s)}.\end{aligned}\tag{59}$$

Note that (59) with $s=\pm2$ is different from the equation obtained in our previous study [8]. However, this is simply because of the difference in the transformation (57), and they are equivalent. We also note that in the case of $\mathcal{A}=1-2M/r$ and $\mathcal{D}=1$, (59) reduces to the Teukolsky master equation with spin s on the background of the Schwarzschild spacetime. In the case of $\mathcal{D}=1$, the above is simplified as

$$\begin{aligned}&\frac{r^2}{\mathcal{A}}\frac{\partial^2\psi_{(s)}}{\partial t^2}-\frac{1}{(r^2\mathcal{A})^s}\frac{\partial}{\partial r}\left[(r^2\mathcal{A})^{s+1}\frac{\partial\psi_{(s)}}{\partial r}\right]-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_{(s)}}{\partial\theta}\right)\\&-\frac{1}{\sin^2\theta}\frac{\partial^2\psi_{(s)}}{\partial\varphi^2}+\left(2sr-\frac{sr^2\mathcal{A}'}{\mathcal{A}}\right)\frac{\partial\psi_{(s)}}{\partial t}-\frac{2\mathrm{i}s\cot\theta}{\sin\theta}\frac{\partial\psi_{(s)}}{\partial\varphi}\\&+\left[s^2\cot^2\theta-s+\frac{1}{3}(1-\delta_s+3s+2s^2)\left(1-\mathcal{A}-2r\mathcal{A}'-\frac{1}{2}r^2\mathcal{A}''\right)\right.\\&\left.+\frac{1}{6}(|s|-3|s|^2+2|s|^3)\left(1-\mathcal{A}+\frac{1}{2}r^2\mathcal{A}''\right)\right]\psi_{(s)}=2r^2T_{(s)}.\end{aligned}\tag{60}$$

First, we consider the homogeneous case. The equation allows the separation of the variables, and we assume the product form of the solution to be

$$\psi_{(s)}=\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}m\varphi}R(r)S(\theta),\tag{61}$$

where ω is the frequency of the waves, and m is constant. Then, the separated equations are

$$\begin{aligned}&\frac{1}{\mathcal{D}(r^2\mathcal{A})^s}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{(r^2\mathcal{A})^{s+1}}{\mathcal{D}}\frac{\mathrm{d}R}{\mathrm{d}r}\right]+\left[\frac{r^2\omega^2}{\mathcal{A}}+\mathrm{i}\omega\left(\frac{2sr}{\mathcal{D}}-\frac{sr^2\mathcal{A}'}{\mathcal{A}\mathcal{D}}\right)+\frac{s(2s+1)r\mathcal{A}\mathcal{D}'}{\mathcal{D}^3}\right.\\&-\frac{1}{3}(1-\delta_s+3s+2s^2)\left(1-\frac{\mathcal{A}}{\mathcal{D}^2}-\frac{2r\mathcal{A}'}{\mathcal{D}^2}+\frac{2r\mathcal{A}\mathcal{D}'}{\mathcal{D}^3}+\frac{r^2\mathcal{A}'\mathcal{D}'}{2\mathcal{D}^3}-\frac{r^2\mathcal{A}''}{2\mathcal{D}^2}\right)\\&\left.-\frac{1}{6}(|s|-3|s|^2+2|s|^3)\left(1-\frac{\mathcal{A}}{\mathcal{D}^2}-\frac{r^2\mathcal{A}'\mathcal{D}'}{2\mathcal{D}^3}+\frac{r^2\mathcal{A}''}{2\mathcal{D}^2}\right)-\lambda_{(s)}\right]R=0,\end{aligned}\tag{62}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)+\left(-\frac{m^2}{\sin^2\theta}-\frac{2sm\cot\theta}{\sin\theta}-s^2\cot^2+s+\lambda_{(s)}\right)S=0,\tag{63}$$

where $\lambda_{(s)}$ is the separation constant. From (63), we can find that $S(\theta)\mathrm{e}^{\mathrm{i}m\varphi}$ coincides with the spin-weighted spherical harmonics ${}_sY_{lm}(\theta,\varphi)$ with spin s, where l and m take the values of

$$l=|s|,\ |s|+1,\ |s|+2,\ \ldots,\quad m=-l,\ -l+1,\ \ldots,\ l-1,\ l,\tag{64}$$

respectively. $\lambda_{(s)}$ becomes the eigenvalue of ${}_sY_{lm}(\theta,\varphi)$, which is given by

$$\lambda_{(s)}=(l-s)(l+s+1).\tag{65}$$

For the nonhomogeneous case, we expand $\psi_{(s)}$ and $T_{(s)}$ in terms of ${}_sY_{lm}(\theta,\varphi)$ as

$$\psi_{(s)}=\int\mathrm{d}\omega\sum_{l,m}R^{(s)}_{lm\omega}(r)\,{}_sY_{lm}(\theta,\varphi)\mathrm{e}^{-\mathrm{i}\omega t},\tag{66}$$

$$-2r^2T_{(s)}=\int\mathrm{d}\omega\sum_{l,m}G^{(s)}_{lm\omega}(r)\,{}_sY_{lm}(\theta,\varphi)\mathrm{e}^{-\mathrm{i}\omega t}.\tag{67}$$

Then, $R^{(s)}_{lm\omega}(r)$ satisfies

$$\frac{1}{\mathcal{D}(r^2\mathcal{A})^s}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{(r^2\mathcal{A})^{s+1}}{\mathcal{D}}\frac{\mathrm{d}R^{(s)}_{lm\omega}}{\mathrm{d}r}\right]+\left[\frac{r^2\omega^2}{\mathcal{A}}+\mathrm{i}\omega\left(\frac{2sr}{\mathcal{D}}-\frac{sr^2\mathcal{A}'}{\mathcal{A}\mathcal{D}}\right)+\frac{s(2s+1)r\mathcal{A}\mathcal{D}'}{\mathcal{D}^3}\right.$$
$$-\frac{1}{3}(1-\delta_s+3s+2s^2)\left(1-\frac{\mathcal{A}}{\mathcal{D}^2}-\frac{2r\mathcal{A}'}{\mathcal{D}^2}+\frac{2r\mathcal{A}\mathcal{D}'}{\mathcal{D}^3}+\frac{r^2\mathcal{A}'\mathcal{D}'}{2\mathcal{D}^3}-\frac{r^2\mathcal{A}''}{2\mathcal{D}^2}\right)$$
$$\left.-\frac{1}{6}(|s|-3|s|^2+2|s|^3)\left(1-\frac{\mathcal{A}}{\mathcal{D}^2}-\frac{r^2\mathcal{A}'\mathcal{D}'}{2\mathcal{D}^3}+\frac{r^2\mathcal{A}''}{2\mathcal{D}^2}\right)-\lambda_{(s)}\right]R^{(s)}_{lm\omega}=G^{(s)}_{lm\omega}. \tag{68}$$

We again note that in the case of $\mathcal{A}=1-2M/r$ and $\mathcal{D}=1$, (68) reduces to the Teukolsky radial equation with spin s on the background of the Schwarzschild spacetime. In the case of $\mathcal{D}=1$, (68) reduces to

$$\frac{1}{(r^2\mathcal{A})^s}\frac{\mathrm{d}}{\mathrm{d}r}\left[(r^2\mathcal{A})^{s+1}\frac{\mathrm{d}R^{(s)}_{lm\omega}}{\mathrm{d}r}\right]+\left[\frac{r^2\omega^2}{\mathcal{A}}+\mathrm{i}\omega\left(2sr-\frac{sr^2\mathcal{A}'}{\mathcal{A}}\right)\right.$$
$$-\frac{1}{3}(1-\delta_s+3s+2s^2)\left(1-\mathcal{A}-2r\mathcal{A}'-\frac{1}{2}r^2\mathcal{A}''\right)$$
$$\left.-\frac{1}{6}(|s|-3|s|^2+2|s|^3)\left(1-\mathcal{A}+\frac{1}{2}r^2\mathcal{A}''\right)-\lambda_{(s)}\right]R^{(s)}_{lm\omega}=G^{(s)}_{lm\omega}, \tag{69}$$

which reduces to [11] for positive s.

IV. GAUGE DEPENDENCE IN GRAVITATIONAL-WAVE EQUATIONS

In previous studies, two types of the gravitational-wave equations have appeared. One can be found in [6, 8], and the other in [7]. Because these equations are obtained from the same set of coupled equations in the Newman-Penrose formalism but with different gauges, both equations should describe the gravitational wave correctly. Here, one question can be raised: although the unknown variables Ψ_4^B and Ψ_0^B are gauge-invariant quantities, why can we have two (or more, in principle) forms of the wave equations for each variable? To answer this question, we revisit the derivation of the gravitational-wave equation on the background, with emphasis on the gauge dependence.

We focus on the wave equation for Ψ_4^B and consider the case of $\mathcal{D}=1$ because in [7], only this case is considered. Here, our gauge conditions (44) and (45) reduce to $\lambda^B=0$ [8]. We begin with the following three equations in the Newman-Penrose formalism:

$$(\delta+4\beta-\tau)\Psi_4-(\Delta+4\mu+2\gamma)\Psi_3+3\nu\Psi_2$$
$$=(\bar\delta-\bar\tau+2\bar\beta+2\alpha)\Phi_{22}-(\Delta+2\gamma+2\bar\mu)\Phi_{21}$$
$$-2\lambda\Phi_{12}+2\nu\Phi_{11}+\bar\nu\Phi_{20}, \tag{70}$$

$$(D+4\epsilon-\rho)\Psi_4-(\bar\delta+4\pi+2\alpha)\Psi_3+3\lambda\Psi_2$$
$$=(\bar\delta-2\bar\tau+2\alpha)\Phi_{21}-(\Delta+2\gamma-2\bar\gamma+\bar\mu)\Phi_{20}$$
$$+\bar\sigma\Phi_{22}-2\lambda\Phi_{11}+2\nu\Phi_{10}, \tag{71}$$

$$(\Delta+\mu+\bar\mu+3\gamma-\bar\gamma)\lambda-(\bar\delta+\pi-\bar\tau+\bar\beta+3\alpha)\nu+\Psi_4=0. \tag{72}$$

We split all the quantities in the above into the background (A) and perturbation parts (B); for instance, $\Psi_4=\Psi_4^A+\Psi_4^B$, etc. We keep the first order of the perturbation only. The background part of the above equations is satisfied, and the perturbation part becomes

$$(\delta-4\alpha)^A\Psi_4^B-(\Delta+2\gamma+4\mu)^A\Psi_3^B+3\nu^B\Psi_2^A$$
$$=\bar\delta^A\Phi_{22}^B-(\Delta+2\gamma+2\mu)^A\Phi_{21}^B+2\nu^B\Phi_{11}^A, \tag{73}$$

$$(D-\rho)^A\Psi_4^B-(\bar\delta+2\alpha)^A\Psi_3^B+3\lambda^B\Psi_2^A$$
$$=(\bar\delta+2\alpha)^A\Phi_{21}^B-(\Delta+\mu)^A\Phi_{20}^B-2\lambda^B\Phi_{11}^A, \tag{74}$$

$$(\Delta+2\gamma+2\mu)^A\lambda^B-(\bar\delta+2\alpha)^A\nu^B+\Psi_4^B=0. \tag{75}$$

Now, we obtain the wave equation for Ψ_4^B using the same procedure as in the previous section. We operate $(\Delta+2\gamma+5\mu)^A$ to (74) and $(\bar\delta+2\alpha)^A$ to (73) and then find the difference of them. The terms with Ψ_3^B are canceled by (27) with $p=2$, $q=-4$, and the remainder becomes

$$\left[(\Delta+2\gamma+5\mu)(D-\rho)-(\bar\delta+2\alpha)(\delta-4\alpha)\right]^A\Psi_4^B$$
$$+(3\Psi_2+2\Phi_{11})^A(\Delta+2\gamma+5\mu)^A\lambda^B$$
$$-(3\Psi_2-2\Phi_{11})^A(\bar\delta+2\alpha)^A\nu^B$$
$$=T_4-\lambda^B\Delta^A(3\Psi_2+2\Phi_{11})^A+\nu^B\bar\delta^A(3\Psi_2-2\Phi_{11})^A, \tag{76}$$

where T_4 is defined by

$$T_4 = (\Delta+2\gamma+5\mu)^A\left[(\bar{\delta}+2\alpha)^A\Phi^B_{21}-(\Delta+\mu)^A\Phi^B_{20}\right] - (\bar{\delta}+2\alpha)^A\left[\bar{\delta}^A\Phi^B_{22}-(\Delta+2\gamma+2\mu)^A\Phi^B_{21}\right]. \quad (77)$$

For the third line in (76), we have

$$\Delta^A(3\Psi_2-2\Phi_{11})^A = -3\mu^A(3\Psi_2+2\Phi_{11})^A+8\mu^A\Phi^A_{11}, \quad (78)$$

$$\bar{\delta}^A(3\Psi_2+2\Phi_{11})^A = -3\pi^A(3\Psi_2-2\Phi_{11})^A, \quad (79)$$

and then substituting the above into (76), we obtain

$$\begin{aligned}&\left[(\Delta+2\gamma+5\mu)(D-\rho)-(\bar{\delta}+2\alpha)(\delta-4\alpha)-3\Psi_2\right]^A\Psi^B_4\\ &\quad+2\Phi^A_{11}\left[(\Delta+2\gamma+2\mu)^A\lambda^B+(\bar{\delta}+2\alpha)^A\nu^B\right]\\ &= T_4-4\lambda^B[(\Delta+2\mu)\Phi_{11}]^A,\end{aligned} \quad (80)$$

where we use (75) and $\bar{\delta}^A\Phi^A_{11}=0$ from the spherical symmetry. By eliminating the terms with ν^B using (75) again, we have

$$\begin{aligned}&\left[(\Delta+2\gamma+5\mu)(D-\rho)-(\bar{\delta}+2\alpha)(\delta-4\alpha)-3\Psi_2+2\Phi_{11}\right]^A\Psi^B_4\\ &= T_4-4(\Delta+2\gamma+4\mu)^A(\Phi^A_{11}\lambda^B).\end{aligned} \quad (81)$$

Moreover, using (78) and (79), the above can be rewritten as

$$\begin{aligned}&\left[(\Delta+2\gamma+5\mu)(D-\rho)-(\bar{\delta}+2\alpha)(\delta-4\alpha)\right.\\ &\quad\left.-3\Psi_2+2\Phi_{11}\right]^A\Psi^B_4\\ &= T_4+(3\Psi_2-2\Phi_{11})^A(\Delta+2\gamma+2\mu)^A\lambda^B\\ &\quad-(\Delta+2\gamma+5\mu)^A\left[(3\Psi_2+2\Phi_{11})^A\lambda^B\right],\end{aligned} \quad (82)$$

So far, we have not used a gauge condition. If we take the gauge condition as $\lambda^B=0$, Eqs. (80) and (82) are reduced to the wave equation (47) in [6, 8] as

$$\begin{aligned}&\left[(\Delta+2\gamma+5\mu)(D-\rho)-(\bar{\delta}+2\alpha)(\delta-4\alpha)-3\Psi_2+2\Phi_{11}\right]^A\Psi^B_4\\ &= T_4,\end{aligned} \quad (83)$$

which has a similar form to the vacuum case $\Phi^A_{11}=\Lambda^A=0$. However, from (74), λ^B can be expressed in terms of Ψ^B_3 as

$$\lambda^B = \frac{1}{(3\Psi_2+2\Phi_{11})^A}\left[-(D-\rho)^A\Psi^B_4+(\bar{\delta}+2\alpha)^A(\Psi_3+\Phi_{21})^B -(\Delta+\mu)^A\Phi^B_{20}\right]. \quad (84)$$

Substituting the above into (82) gives

$$\begin{aligned}&\left[F_2^{-1}(\Delta+2\gamma+2\mu-F_1)(D-\rho)\right.\\ &\quad\left.-(\bar{\delta}+2\alpha)(\delta-4\alpha)-3\Psi_2+2\Phi_{11}\right]^A\Psi^B_4\\ &= F_2^{-1}(\Delta+2\gamma+2\mu-F_1)^A\\ &\quad\times\left[(\bar{\delta}+2\alpha)^A(\Psi_3+\Phi_{21})^B-(\Delta+\mu)^A\Phi^B_{20}\right]\\ &\quad-(\bar{\delta}+2\alpha)^A\left[\bar{\delta}^A\Phi^B_{22}-(\Delta+2\gamma+2\mu)^A\Phi^B_{21}\right],\end{aligned} \quad (85)$$

or

$$\begin{aligned}&\left[(\Delta+2\gamma+2\mu-F_1)(D-\rho)-F_2(\bar{\delta}+2\alpha)(\delta-4\alpha)-3\Psi_2\right.\\ &\quad\left.-2\Phi_{11}\right]^A\Psi^B_4\\ &= (\Delta+2\gamma+2\mu-F_1)^A\left[(\bar{\delta}+2\alpha)^A(\Psi_3+\Phi_{21})^B-(\Delta+\mu)^A\Phi^B_{20}\right]\\ &\quad-F_2(\bar{\delta}+2\alpha)^A\left[\bar{\delta}^A\Phi^B_{22}-(\Delta+2\gamma+2\mu)^A\Phi^B_{21}\right],\end{aligned} \quad (86)$$

where F_1 and F_2 are defined by

$$F_1 = \Delta\left[\ln(3\Psi_2+2\Phi_{11})^A\right],\quad F_2 = \left(\frac{3\Psi_2+2\Phi_{11}}{3\Psi_2-2\Phi_{11}}\right)^A. \quad (87)$$

Thus, if we take the gauge condition as $\Psi^B_3=0$, (86) is reduced to the wave equation in [7] as

$$\begin{aligned}&\left[(\Delta+2\gamma+2\mu-F_1)(D-\rho)-F_2(\bar{\delta}+2\alpha)(\delta-4\alpha)\right.\\ &\quad\left.-3\Psi_2-2\Phi_{11}\right]^A\Psi^B_4 = \tilde{T}_4,\end{aligned} \quad (88)$$

where $\tilde{T}_4$ is defined by

$$\tilde{T}_4 = (\Delta+2\gamma+2\mu-F_1)^A\left[(\bar{\delta}+2\alpha)^A\Phi^B_{21}-(\Delta+\mu)^A\Phi^B_{20}\right] - F_2(\bar{\delta}+2\alpha)^A\left[\bar{\delta}^A\Phi^B_{22}-(\Delta+2\gamma+2\mu)^A\Phi^B_{21}\right]. \quad (89)$$

Next, we consider the gauge transformation in the wave equation, for which we take the following tetrad rotations [1] [16]:

1) Since we here consider the case $\mathcal{D}=1$ only, the tetrad rotations are enough. For the genetral case $\mathcal{D}\neq 1$, we have to take into account the general coordinate transformation as well [8].

$$l^\mu \to l^\mu, \quad m^\mu \to m^\mu + a l^\mu, \quad \bar{m}^\mu \to \bar{m}^\mu + \bar{a} l^\mu,$$
$$n^\mu \to n^\mu + \bar{a} m^\mu + a \bar{m}^\mu + a\bar{a} l^\mu, \tag{90}$$

$$n^\mu \to n^\mu, \quad m^\mu \to m^\mu + b n^\mu, \quad \bar{m}^\mu \to \bar{m}^\mu + \bar{b} n^\mu,$$
$$l^\mu \to l^\mu + \bar{b} m^\mu + b \bar{m}^\mu + b\bar{b} n^\mu, \tag{91}$$

$$l^\mu \to e^{-c} l^\mu, \quad n^\mu \to e^c n^\mu, \quad m^\mu \to \mathrm{e}^{\mathrm{i}\vartheta} m^\mu, \quad \bar{m}^\mu \to \mathrm{e}^{-\mathrm{i}\vartheta} \bar{m}^\mu. \tag{92}$$

To avoid changing the background, we assume that parameters a, b, c, and ϑ are in the first order of the perturbation, and hence for the perturbation quantities, the transformation of the first order is sufficient. In the wave equation (80), the left hand side is gauge-invariant because Ψ_4^B is so, which imples that the right hand side must also be invariant. Using

$$\lambda^B \to \lambda^B + (\bar{\delta} + 2\alpha)^A \bar{a}, \quad \Phi_{21}^B \to \Phi_{21}^B + 2\Phi_{11}^A \bar{a},$$

$$\Phi_{20}^B \to \Phi_{20}^B, \quad \Phi_{22}^B \to \Phi_{22}^B, \tag{93}$$

source term T_4 transforms as

$$T_4 \to T_4 + 4\left[(\bar{\delta} + 2\alpha)(\Delta + 2\gamma + 3\mu)\right]^A (\Phi_{11}^A \bar{a}). \tag{94}$$

We can find that this transformation is cancelled by that of other terms on the right hand side of (80), where we use (27) and $\bar{\delta}^A \Phi_{11}^A = 0$. We can also show that $\tilde{T}_4$, defined by (89), has a nontrivial gauge transformation under (90)–(92), which is cancelled by that of Ψ_3^B as

$$\Psi_3^B \to \Psi_3^B + 3\Psi_2^A \bar{a}. \tag{95}$$

Thus, the origin of the gauge dependence of the gravitational-wave equation is due to that of the source term, particularly Φ_{21}^B. Note that in the vacuum case, this dependence does not appear because $\Phi_{11}^A = 0$. We can also show that the two gravitational-wave equations (83) and (88) coincide in the vacuum background because of

$$F_1 = -3\mu^A, \quad F_2 = 1, \tag{96}$$

in the vacuum.

V. SUMMARY AND DISCUSSION

In this study, we investigated the wave equations with various spins on the background of a general spherically symmetric spacetime. By introducing spin variable s, we unified these equations using s itself, $|s|$, and δ_s. The transformation (57) to simplify the equation was possible, and the form of (57), in turn, became the same as in the vacuum case although the background, particularly spin coefficient ρ^A, was deformed.

We also discussed the gauge dependence in the form of the gravitational-wave equations from a previous study. The gauge dependence of the wave equation originates from that of the source term, and hence it cannot be avoided, except in the vacuum case. If we take another gauge, the form of the gravitational-wave equation changes, which also affects the unified expressions (52) and (58).

A similar analysis using the metric perturbation was performed in [17–19], where the wave equations resembled the Regge-Wheeler and the Zerilli equations [20, 21], and its generalization to general spin s has also been studied [22]. In the vacuum case, there is a transformation between the Teukolsky and Regge-Wheeler equations, known as the Chandrasekhar transformation [23]. Moreover, for $s \neq 0$, the relationship between $R_{lm\omega}^{(-|s|)}(r)$ and $R_{lm\omega}^{(|s|)}(r)$ can be regarded as a special case of the Chandrasekhar transformation [10, 24]. It would be interesting to investigate whether similar relations hold in the current case as in [25].

A possible generalization would be to extend the results to the axisymmetric background, which contains the Kerr black hole as an example. In the vacuum case, the backgrounds satisfying the type D condition are fully classified as the so-called the Kinnersley metric [26]. It would be interesting to find the non-vacuum extension of the Kinnersley metric and the wave equation on that background with general spin. The gravitational-wave equation on a certain non-vacuum axisymmetric background was proposed in [27]. However, this equation does not allow for the separation of the variables, hence, further modification is needed [28].

The study of the (gravitational) wave equation using the Newman-Penrose formalism is applicable to not only EOB dynamics but also modified gravitational theories, as long as the gravitational degrees of freedom are described by the metric tensor, such as the scalar-tensor, Horndeski [29], and degenerated higher order scalar-tensor theories [30]. In these theories, the equations of motion become complicated. However, in the Newman-Penrose formalism, the role of the Einstein equation is rather restrictive. In particular, the original Newman-Penrose equations (70)–(72) for the wave equation originate from the Bianchi identity, which also takes the same form in these theories. Then, one can perform a similar analysis to ours. It would be interesting to obtain wave equations in these theories, especially the equations for other degrees of freedom, for example, the perturbed scalar field in the above theories.

APPENDIX A NOTATIONS AND CONVENTIONS

In this paper, we follow the notations and conventions of Newman-Penrose [9] and Pirani [15]. Here, we list some of the definitions for convenience. The metric has the sign $(+---)$, and the Riemann curvature is decomposed as

$$R_{\mu\nu\alpha\beta}=C_{\mu\nu\alpha\beta}-\frac{1}{2}\left(g_{\mu\alpha}R_{\nu\beta}-g_{\mu\beta}R_{\nu\alpha}+g_{\nu\beta}R_{\mu\alpha}-g_{\nu\alpha}R_{\mu\beta}\right)+\frac{1}{6}\mathcal{R}(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha}),\tag{A1}$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor. Ricci tensor $R_{\mu\nu}$ and Ricci scalar $\mathcal{R}$ are defined by

$$R_{\mu\nu}=R^{\rho}{}_{\mu\nu\rho},\quad \mathcal{R}=g^{\mu\nu}R_{\mu\nu}.\tag{A2}$$

The twelve spin coefficients are defined by

$$\begin{aligned}
&\kappa=m^{\mu}l^{\nu}\nabla_{\nu}l_{\mu},\quad \tau=m^{\mu}n^{\nu}\nabla_{\nu}l_{\mu},\\
&\epsilon=\frac{1}{2}(n^{\mu}l^{\nu}\nabla_{\nu}l_{\mu}-\bar{m}^{\mu}l^{\nu}\nabla_{\nu}m_{\mu}),\\
&\sigma=m^{\mu}m^{\nu}\nabla_{\nu}l_{\mu},\quad \rho=m^{\mu}\bar{m}^{\nu}\nabla_{\nu}l_{\mu},\\
&\gamma=\frac{1}{2}(n^{\mu}n^{\nu}\nabla_{\nu}l_{\mu}-\bar{m}^{\mu}n^{\nu}\nabla_{\nu}m_{\mu}),\\
&\nu=-\bar{m}^{\mu}n^{\nu}\nabla_{\nu}n_{\mu},\quad \mu=-\bar{m}^{\mu}m^{\nu}\nabla_{\nu}n_{\mu},\\
&\beta=\frac{1}{2}(n^{\mu}m^{\nu}\nabla_{\nu}l_{\mu}-\bar{m}^{\mu}m^{\nu}\nabla_{\nu}m_{\mu}),\\
&\lambda=-\bar{m}^{\mu}\bar{m}^{\nu}\nabla_{\nu}n_{\mu},\quad \pi=-\bar{m}^{\mu}l^{\nu}\nabla_{\nu}n_{\mu},\\
&\alpha=\frac{1}{2}(n^{\mu}\bar{m}^{\nu}\nabla_{\nu}l_{\mu}-\bar{m}^{\mu}\bar{m}^{\nu}\nabla_{\nu}m_{\mu}),
\end{aligned}\tag{A3}$$

where ∇_{μ} is the covariant derivative with respect to the curved spacetime. The Ricci tensor is decomposed into the following components:

$$\begin{aligned}
&\Phi_{00}=-\frac{1}{2}R_{\mu\nu}l^{\mu}l^{\nu},\quad \Phi_{01}=-\frac{1}{2}R_{\mu\nu}l^{\mu}m^{\nu},\\
&\Phi_{02}=-\frac{1}{2}R_{\mu\nu}m^{\mu}m^{\nu},\quad \Phi_{10}=-\frac{1}{2}R_{\mu\nu}l^{\mu}\bar{m}^{\nu},\\
&\Phi_{11}=-\frac{1}{4}R_{\mu\nu}(l^{\mu}n^{\nu}+m^{\mu}\bar{m}^{\nu}),\quad \Phi_{12}=-\frac{1}{2}R_{\mu\nu}n^{\mu}m^{\nu},\\
&\Phi_{20}=-\frac{1}{2}R_{\mu\nu}\bar{m}^{\mu}\bar{m}^{\nu},\quad \Phi_{21}=-\frac{1}{2}R_{\mu\nu}n^{\mu}\bar{m}^{\nu},\\
&\Phi_{22}=-\frac{1}{2}R_{\mu\nu}n^{\mu}n^{\nu},
\end{aligned}$$

$$\Lambda=\frac{1}{24}\mathcal{R}=\frac{1}{12}R_{\mu\nu}(l^{\mu}n^{\nu}-m^{\mu}\bar{m}^{\nu}).\tag{A4}$$

Finally, the Weyl scalars are defined by

$$\begin{aligned}
&\Psi_0=-C_{\mu\nu\lambda\rho}l^{\mu}m^{\nu}l^{\lambda}m^{\rho},\\
&\Psi_1=-C_{\mu\nu\lambda\rho}l^{\mu}n^{\nu}l^{\lambda}m^{\rho},\\
&\Psi_2=-\frac{1}{2}C_{\mu\nu\lambda\rho}(l^{\mu}n^{\nu}l^{\lambda}n^{\rho}-l^{\mu}n^{\nu}m^{\lambda}\bar{m}^{\rho}),\\
&\Psi_3=-C_{\mu\nu\lambda\rho}l^{\mu}n^{\nu}\bar{m}^{\lambda}n^{\rho},\\
&\Psi_4=-C_{\mu\nu\lambda\rho}n^{\mu}\bar{m}^{\nu}n^{\lambda}\bar{m}^{\rho}.
\end{aligned}\tag{A5}$$

ACKNOWLEDGEMENTS

The authors thank Prof. Ruffini for useful discussions.

References

[1] B. Abbott *et al.* (LIGO Scientific and Virgo), Phys. Rev. Lett. **116**(6), 061102 (2016), arXiv:1602.03837[gr-qc]
[2] Y. Mino, M. Sasaki, M. Shibata *et al.*, Prog. Theor. Phys. Suppl. **128**, 1 (1997), arXiv:gr-qc/9712057[gr-qc]
[3] E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*, (Cambridge University Press, 2014) doi: 10.1017/CBO9781139507486.
[4] A. Buonanno and T. Damour, Phys. Rev. D **62**, 064015 (2000), arXiv:gr-qc/0001013[gr-qc]
[5] T. Damour, Phys. Rev. D **94**(10), 104015 (2016), arXiv:1609.00354[gr-qc]
[6] J. Jing, S. Chen, M. Sun *et al.*, Sci. China Phys. Mech. Astron. **65**(6), 260411 (2022), arXiv:2112.09838[gr-qc]
[7] J. Jing, S. Long, W. Deng *et al.*, Sci. China Phys. Mech. Astron. **65**(10), 100411 (2022), arXiv:2208.02420[gr-qc]
[8] Y. Guo, H. Nakajima, and W. Lin, Sci. China Phys. Mech. Astron. **66**(7), 270412 (2023), arXiv:2301.08318[gr-qc]
[9] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962)
[10] S. A. Teukolsky, Astrophys. J. **185**, 635 (1973)
[11] C. M. Harris and P. Kanti, JHEP **10**, 014 (2003), arXiv:hep-ph/0309054[hep-ph]
[12] E. C. Vagenas, A. Farag Ali, M. Hemeda *et al.*, Annals Phys. **432**, 168574 (2021), arXiv:2008.09853[hep-th]
[13] A. Arbey, J. Auffinger, M. Geiller *et al.*, Phys. Rev. D **103**(10), 104010 (2021), arXiv:2101.02951[gr-qc]
[14] G. Q. Li and J. X. Mo, Astrophys. Space Sci. **336**, 441 (2011)
[15] F. A. E. Pirani, *Lectures on General Relativity*, Brandeis Summer Institute in Theoretical Physics, 1964.
[16] A. I. Janis and E. T. Newman, J. Math. Phys. **6**, 902 (1965)
[17] J. E. Thompson, B. F. Whiting, and H. Chen, Class. Quant. Grav. **34**(17), 174001 (2017), arXiv:1611.06214[gr-qc]
[18] M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**(8), 084053 (2021), arXiv:2108.08668[gr-qc]
[19] W. Liu, X. Fang, J. Jing *et al.*, Sci. China Phys. Mech. Astron. **66**(1), 210411 (2023), arXiv:2201.01259[gr-qc]

[20] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957)
[21] F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970)
[22] E. W. Leaver, Phys. Rev. D **34**, 384 (1986)
[23] S. Chandrasekhar, Proc. R. Soc. Lond. A **A343**, 289 (1975)
[24] A. A. Starobinsky and S. M. Churilov, Zh. Eksp. Teor. Fiz. **65**, 3 (1973); A. A. Starobinsky and S. M. Churilov, Sov. Phys. JETP 38, 1 (1974).
[25] A. Arbey, J. Auffinger, M. Geiller *et al*., Phys. Rev. D **104**(8), 084016 (2021), arXiv:2107.03293[gr-qc]
[26] W. Kinnersley, J. Math. Phys. **10**, 1195 (1969)
[27] J. Jing, W. Deng, S. Long *et al*., Sci. China Phys. Mech. Astron. **66**(7), 270411 (2023), arXiv:2305.03225[gr-qc]
[28] Y. Guo, H. Nakajima, and W. Lin, *Teukolsky-like equations in a non-vacuum axisymmetric type D spacetime*, arXiv: 2309.06237[gr-qc]
[29] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974)
[30] D. Langlois and K. Noui, JCAP **02**, 034 (2016), arXiv:1510.06930[gr-qc]