# Predictions for bottomonium from a relativistic screened potential model

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**Abstract:** This work conducts a comprehensive analysis of the mass spectra and decay properties of bottomonium states using a relativistic screened potential model. The mass spectrum, decay constants, $E1$ transitions, $M1$ transitions, and annihilation decay widths are evaluated. The interpretation of $\Upsilon(10355)$, $\Upsilon(10580)$,$\Upsilon(10860)$, and $\Upsilon(11020)$ as $S-D$ mixed bottomonium states are analysed. The $\Upsilon(10355)$ state is considered to be $3S-2D$, $\Upsilon(10580)$ and $\Upsilon(10753)$ are considered to be $4S-3D$ mixed states, and the $\Upsilon(10860)$ and $\Upsilon(11020)$ are considered to be $5S-4D$ mixed states.

**Keywords:** bottomonium, relativistic potential model, screened potential, radiative decays, *S-D* mixing

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## I. INTRODUCTION

The study of heavy quarkonium, specifically bottomonium, has emerged as a captivating and influential field in contemporary particle physics. The allure of this research lies not only in the experimental endeavors aimed at unraveling the intricate properties of these heavy quark systems [1, 2] but also its rich theoretical framework, which enables us to understand the intricate interplay of perturbative and non-perturbative quantum chromodynamics (QCD) phenomena across a broad energy spectrum [3]. The history of bottomonium traces back to $\Upsilon(1S)$, first discovered by the E288 Collaboration at Fermilab along with $\Upsilon(2S)$ and $\Upsilon(3S)$ [4, 5]. In 1982, the $\chi_{bJ}(2P)$ states were observed using the CUSB detector at CESR in the reaction $\Upsilon(3S)\to\gamma\chi_{bJ}(2P)$ for $(J=0,1,2)$ [6, 7]. The $\chi_{bJ}(1P)$ states were discovered in 1983 in $\eta_b(2S)\to\gamma\chi_{bJ}(1P)$ and $\chi_{bJ}(1P)\to\gamma\eta_b(1S)$ reactions [8] and were later confirmed in the same year by CESR in $\Upsilon(2S)\to\gamma\chi_{bJ}(1P)\to\gamma\gamma\Upsilon(1S)$ reactions [9]. In 2005, the most precise measurements of branching fractions and photon energies of $\chi_{bJ}(1P)$ and $\chi_{bJ}(2P)$ were conducted by the CLEO Collaboration [10]. For the first time in 1980, CESR observed a peak above the $B\bar{B}$ threshold and suggested it as $\Upsilon(4S)$ [11]. Later in 1985, CLEO at CESR, in addition to $\Upsilon(4S)$, reported the observation of $\Upsilon(10860)$ and $\Upsilon(11020)$ resonances [12]. Most recently, in 2019, the Belle Collaboration measured $e^+e^-\to\Upsilon(1,2,3S)\pi^+\pi^-$ cross sections, determining masses and widths of $\Upsilon(10860)$ and $\Upsilon(11020)$ with improved precision [13]. In 2004, the CLEO Collaboration observed the $\Upsilon(1D)$ state at $10161.1\pm0.6\pm1.6$ MeV via a photon cascade of $\Upsilon(3S)$ decays, identifying it as the $\Upsilon_2(1D)$ state [14]. The BABAR Collaboration later confirmed the $\Upsilon_J(1D)$ triplet in $\Upsilon(3S)\to\gamma\gamma\Upsilon(1D)\to\gamma\gamma\pi^+\pi^-\Upsilon(1S)$, with a significance of $5.8\sigma$ for $\Upsilon_2(1D)$, whereas the significance values for $\Upsilon_1(1D)$ and $\Upsilon_3(1D)$ states were very low [15]. In 2008, BABAR discovered $\eta_b(1S)$ with $10\sigma$ significance via $\Upsilon(3S)\to\gamma\eta_b(1S)$ [16]. It was later confirmed by the CLEO Collaboration [17, 18], which also identified $\eta_b(2S)$ with $5\sigma$ significance in $\Upsilon(2S)\to\gamma\eta_b(2S)$ [18]. The Belle Collaboration observed $\eta_b(2S)$ for the first time in $h_b(2P)\to\gamma\eta_b(2S)$, measuring its mass as $9999.0\pm3.5^{+2.8}_{-1.9}$ MeV and hyperfine splitting as $m[\Upsilon(2S)]-m[\eta_b(2S)]=24.3^{+4.0}_{-4.5}$ MeV [19]. In 2011, the BABAR Collaboration observed $h_b(1P)$ with $3.1\sigma$ significance in $\Upsilon(3S)\to\pi^0h_b(1P)\to\pi^0\gamma\eta_b(1S)$ [20]. The Belle Collaboration later confirmed $h_b(1P)$ in $\Upsilon(5S)\to\pi^+\pi^-h_b(1P)$ and discovered $h_b(2P)$ with $11.2\sigma$ significance, having mass of $10259.8\pm0.6^{+1.4}_{-1.0}$ MeV [21]. In 2012, the ATLAS Collaboration observed $\chi_b(3P)$ in $\chi_b(nP)\to\gamma\Upsilon(1S,2S)$ [22]; it was later confirmed by the D0 Collaboration with a $5.6\sigma$ significance at mass barycentre of $10551\pm14\pm17$ MeV [23]. In 2018, the CMS Collaboration observed $\chi_{b1}(3P)$ and $\chi_{b2}(3P)$ in the $\gamma\Upsilon(3S)$ decay mode, measuring their masses as $10513.42\pm0.41\pm0.18$ MeV and $10524.02\pm0.57\pm0.17$ MeV, respectively, and a mass difference of $10.60\pm0.64\pm0.17$ MeV [24]. In 2012, from the data on $\Upsilon(5S)$ decays to

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$\Upsilon(nS)\pi^+\pi^-(n=1,2,3)$ and $h_b(mP)\pi^+\pi^-(m=1,2)$, the Belle Collaboration observed two charged structures, $Z_b(10610)$ and $Z_b(10650)$ [25]. Owing to their charge, they cannot be described in the conventional quarkonium picture and require a four-quark configuration description such as hadronic molecules [26, 27] and tetraquarks [28, 29]. In 2019, the BELLE Collaboration discovered $\Upsilon(10753)$ with $5.2\sigma$ significance in $e^+e^-\to\Upsilon(nS)\pi^+\pi^-$ with a mass of $10752.7\pm5.9^{+0.7}_{-1.1}$ MeV and width of $35.5^{+17.6+3.9}_{-11.3-3.3}$ MeV [13]. It was also identified through cross-section calculations by BABAR and BELLE experiments [30]. Even with significant progress in the experimental domain, key details, such as the total width and mass values of higher resonance and branching ratios for significant decay modes, are still lacking. Unlike the rich charmonium-like *XYZ* sector, only a few unconventional bottomonium-like states (e.g. $Z_b(10610)$, $Z_b(10650)$) have been discovered. No experimental evidence exists for $X_b$, the bottomonium counterpart of $X(3872)$ [31]. The pursuit of similar exotic states in the bottomonium system, such as the $X_b$ whose existence is predicted in multiple models [32, 33], holds promise for understanding the nature of the internal structure of $X(3872)$. Exotic hadron studies have predominantly relied on $e^+e^-$ annihilation experiments, exemplified by BESIII, Belle, BaBar, and CLEO [3]. Belle II at SuperKEKB aims to achieve a peak luminosity of $8\times10^{35}\mathrm{cm}^2\mathrm{s}^{-1}$ by 2025, with operations extending to 2027 to collect over $50\mathrm{ab}^{-1}$ of data [3]. Following the LHCb Upgrade I, Run-3 data will be crucial, whereas the PANDA experiment on antiproton-nucleon interactions and upcoming super $\tau$ - charm factories offer promising avenues for exploring novel states [2, 34]. In view of the potential to discover new states in the bottomonium sector, we have developed a relativistic screened potential model that has proven effective for charmonium [35]. The proposed model provides a robust theoretical framework by considering relativistic effects and the screening of the potential. Our model can facilitate the identification and characterization of new and exotic states within the bottomonium spectrum.

In this paper, we conduct a comprehensive study of bottomonium using a relativistic screened potential model. In Sec. II, we discuss the theoretical model used to describe the bottomonium bound system and the numerical approach used to solve the relativistic Schrodinger equation. Decay constants and various decays are discussed in Sec. III. In Sec. IV, $S-D$ mixing of bottomonium states is discussed. In Sec. V, a thorough investigation of our evaluation and interpretation of bottomonium states are conducted, along with a comparison with experimental results and other theoretical models. In Sec. VI, we present our conclusion.

## II. METHODOLOGY

A relativistic potential model is developed to investigate various bottomonium properties. We utilize the relativistic generalization of the non-relativistic Hamiltonian [36]:

$$H=\sqrt{-\nabla_q^2+m_q^2}+\sqrt{-\nabla_{\bar{q}}^2+m_{\bar{q}}^2}+V(r), \tag{1}$$

where $\vec{r}=\vec{x}_{\bar{q}}-\vec{x}_q$, $\vec{x}_{\bar{q}}$, and $\vec{x}_q$ are the coordinates of the quarks, and operators $\nabla_q^2$ and $\nabla_{\bar{q}}^2$ are the partial derivatives of those coordinates, respectively. $m_q$ and $m_{\bar{q}}$ are the masses of a quark and anti-quark, respectively. The interaction potential $V(r)$ between the quark and antiquark is composed of two components: $V_V(r)$, representing the one-gluon-exchange Coulomb potential term that is dominant at short distance, and $V_S(r)$, which represents the linear confining term adjusted to account for colour screening effects at longer distances [37]:

$$V_V(r)=-\frac{4}{3}\frac{\alpha_s(r)}{r}, \tag{2}$$

$$V_S(r)=\lambda\left(\frac{1-\mathrm{e}^{-\mu r}}{\mu}\right)+V_0, \tag{3}$$

$$V(r)=V_V(r)+V_S(r). \tag{4}$$

Here, $\lambda$ is the linear potential slope, and $\mu$ is the screening factor that regulates the behaviour of the long-range component of $V(r)$, causing it to flatten out as $r$ becomes much larger than $1/\mu$ and exhibit a linear increase as $r$ becomes much smaller than $1/\mu$. $V(r)$ converges to the Cornell potential as $\mu\to0$ [37]. $\alpha_s(r)$ is the running coupling constant in coordinate space obtained via the Fourier transformation of the coupling constant in momentum space $\alpha_s\left(Q^2\right)$ [36] and is given by

$$\alpha_s(r)=\sum_i\alpha_i\frac{2}{\sqrt{\pi}}\int_0^{\gamma_i r}\mathrm{e}^{-x^2}\mathrm{d}x, \tag{5}$$

where $\alpha_i's$ are the free parameters to imitate the short-distance behaviour of $\alpha_s\left(Q^2\right)$ as predicted using QCD. The parameters values are taken as $\alpha_1=0.15$, $\alpha_2=0.15$, $\alpha_3=0.20$, and $\gamma_1=1/2$, $\gamma_2=\sqrt{10}/2$, $\gamma_3=\sqrt{1000}/2$ [38]. The Hamiltonian $H$ is solved as an eigenvalue equation using the method developed in [38, 39]. The Hamiltonian Eq. (1) can be solved as an eigenvalue equation

$$E\Psi(\vec{r})=\left[\sqrt{-\nabla_q^2+m_q}+\sqrt{-\nabla_{\bar{q}}^2+m_{\bar{q}}}+V(r)\right]\Psi(\vec{r}). \tag{6}$$

The wave function can be expanded using spectral integration, which enables us to express the wave function as an integral over the eigenstates of the Hamiltonian $H$:

$$\Psi(\vec{r}) = \int \mathrm{d}^3 r' \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\vec{k}(\vec{r}-\vec{r}')}\Psi(\vec{r}'). \quad (7)$$

Eq. (1) can be rewritten as

$$E\Psi\left(\vec{r}\right) = \int \mathrm{d}^3 r' \frac{\mathrm{d}^3 k}{(2\pi)^3}\left(\sqrt{k^2+m_q} + \sqrt{k^2+m_{\bar{q}}}\right)\mathrm{e}^{\mathrm{i}\vec{k}(\vec{r}-\vec{r}')}\Psi\left(\vec{r}'\right) + V(r)\Psi\left(\vec{r}\right). \quad (8)$$

The exponential term can be expanded in terms of spherical harmonics as

$$e^{\mathrm{i}\vec{k}\cdot\vec{r}} = 4\pi\sum_{nl} Y^*_{nl}\left(\hat{k}\right) Y_{nl}(\hat{r}) j_l(kr) i^l, \quad (9)$$

where $j_l$ is the spherical Bessel function, $Y^*_{nl}(\hat{k})$ and $Y_{nl}(\hat{r})$ are the spherical harmonics with the normalization condition $\int \mathrm{d}\Omega Y_{n_1 l_1}(\hat{k}) Y_{n_2 l_2}(\hat{r}) = \delta_{n_1 n_2}\delta_{l_1 l_2}$, and $\hat{k}$ and $\hat{r}$ are unit vectors along the $\vec{k}$ and $\vec{r}$ directions, respectively. The wave function can be factorized into radial $R_l(r)$ and angular $Y_{nl}(r)$ parts. Substituting Eq. (9) in (8) and simplifying, we obtain [38, 39]

$$Eu_l(r) = \frac{2}{\pi}\int \mathrm{d}k k^2 \int \mathrm{d}r' r r'\left(\sqrt{k^2+m_q^2} + \sqrt{k^2+m_{\bar{q}}^2}\right) \times j_l(kr) j_l(kr') u_l(r') + V(r) u_l(r), \quad (10)$$

where $u_l(r)$ is the reduced radial wave function ($R_l(r) = u_l(r)/r$). When the separation distance grows for a quark-antiquark bound state, the wavefunction gradually decreases and eventually approaches zero at a sufficiently large distance. To represent this behaviour, we introduce a characteristic distance scale $L$, confining the bound state's wavefunction within the spatial interval of $0<r<L$. Next, we can expand the reduced wavefunction $u_l(r)$ in terms of the spherical Bessel function for angular momentum $l$ as

$$u_l(r) = \sum_{n=1}^{\infty} c_n \frac{a_n r}{L} j_l\left(\frac{a_n r}{L}\right), \quad (11)$$

where $c_n$ represents the expansion coefficients, $a_n$ is the $n$-th root of the spherical Bessel function, and $j_l(a_n) = 0$. For large values of $N$, Eq. (11) can be truncated. The momentum is discretized as a result of confinement of space, which enables us to replace $a_n/L \leftrightarrow k$, and the integration in Eq. (10) can be replaced by $\int \mathrm{d}k \to \sum_n \Delta a_n/L$, where $\Delta a_n = a_n - a_{n-1}$. For a finite space interval, $0<r,r'<L$, incorporating all the changes in the Eq. (10), we obtain the final equation in terms of the coefficients $c_n$ as [38, 39]

$$Ec_m = \sum_{n=1}^{N}\frac{a_n}{N_m^2 a_m}\int_0^L \mathrm{d}r V(r) r^2 j_l\left(\frac{a_m r}{L}\right)\left(\frac{a_n r}{L}\right)c_n + \frac{2}{\pi L^3}\left[\sqrt{\left(\frac{a_m}{L}\right)^2+m_q^2} + \sqrt{\left(\frac{a_m}{L}\right)^2+m_{\bar{q}}^2}\right]\Delta a_m a_m^2 N_m^2 c_m. \quad (12)$$

where $N_m$ is the module of spherical Bessel function:

$$N_m^2 = \int_0^L \mathrm{d}r' r'^2 j_l\left(\frac{a_m r'}{L}\right)^2. \quad (13)$$

When $L$ and $N$ attain sufficiently large values, the solution tends to become nearly stationary [38, 39]. The spin dependent interaction potential is given by [40, 41]

$$V_{SD}(r) = V_{SS}(r)\vec{S}_q\cdot\vec{S}_{\bar{q}} + V_{LS}(r)\vec{L}\cdot\vec{S} + V_T(r) S_{12}. \quad (14)$$

$V_{SS}$ is the spin singlet-triplet hyperfine splitting term given by

$$V_{SS}(r) = \frac{32\pi\alpha_s(r)}{9m_q^2}\tilde{\delta}_\sigma(r). \quad (15)$$

Here, $\tilde{\delta}_\sigma(r) = \left(\frac{\sigma}{\sqrt{\pi}}\right)^3 \mathrm{e}^{-\sigma^2 r^2}$ is the smeared delta function [42, 43]. To regularize the non-zero hyperfine splitting, smearing of the delta function as a Gaussian of width $1/\sigma$ is necessary [42, 43]. The spin orbit term $V_{LS}$ and tensor term $V_T$, which describe the fine structure splitting of the states, are given by

$$V_{LS}(r) = \frac{1}{2m_q^2 r}\left(3V_V'(r) - V_S'(r)\right),$$
$$V_T(r) = \frac{1}{m_q^2}\left(\frac{V_V'(r)}{r} - V_V''(r)\right). \quad (16)$$

The tensor operator $S_{12} = 3\left(\vec{S}_q\cdot\hat{r}\right)\left(\vec{S}_{\bar{q}}\cdot\hat{r}\right) - \vec{S}_q\cdot\vec{S}_{\bar{q}}$ has non-vanishing diagonal matrix elements only between $L>0$ spin-triplet states. The spin-dependent interactions are diagonal in a $|J,L,S\rangle$ basis with matrix elements given by [42–44]

$$\langle\vec{S}_q\cdot\vec{S}_{\bar{q}}\rangle = \frac{1}{2}S^2 - \frac{3}{4},$$
$$\langle\vec{L}\cdot\vec{S}\rangle = \frac{1}{2}\left[J(J+1) - L(L+1) - S(S+1)\right],$$
$$\langle S_{12}\rangle = \begin{cases} -\dfrac{L}{6(2L+3)} & J = L+1 \\ \dfrac{1}{6} & J = L \\ -\dfrac{L+1}{6(2L-1)} & J = L-1. \end{cases} \quad (17)$$

Eq. (12) represents an eigenvalue equation in matrix form, which is solved numerically. The eigenvalues correspond to the masses of spin-averaged states, and the eigenvectors represent their wave functions. Using the obtained normalized wave functions for the spin-averaged states, we evaluate the spin-dependent corrections perturbatively. The model parameters are determined using the $\chi^2$ fit method through minimizing $\chi^2$, defined as

$$\chi^2 = \sum_i \left( \frac{M^i_{\rm Exp} - M^i_{\rm Th}}{M^i_{\rm Er}} \right)^2 , \tag{18}$$

where $M^i_{\rm Exp}$ and $M^i_{\rm Th}$ are the experimental and predicted masses, respectively, and $M^i_{\rm Er}$ is the error in $M^i_{\rm Exp}$. The errors of the observed masses $M^i_{\rm Er}$ are taken as 0.1% of the masses of the respective states, $M^i_{\rm Exp}$. These errors are different from their corresponding experimental uncertainties, which are too small for some states and are unevenly distributed. This approach ensures balanced weighting in the fitting process and prevents states that have lower experimental errors from disproportionately influencing the fit [45]. For fitting, we have considered the well established four $S$-wave states $\eta_b(1S,2S)$, $\Upsilon(1S,2S)$, four $P$-wave states $h_c(1P,2P)$,$\chi_{b1}(1P,2P)$, and one $D$-wave state, $1^3D_2$. Using this approach, we obtain a $\chi^2$ value of 14.1. The fitted parameters are listed in Table 1. The masses of $S,P,D,F$, and $G$ states are presented in Tables 2−5, respectively.

**Table 1.** Parameters used in our model.

| $m_q$/GeV | $\sigma$ (/GeV$^2$) | $\lambda$/GeV | $\mu$/GeV | $\Lambda$/GeV |
|---|---|---|---|---|
| 4.744 | 4.967 | 0.240 | 0.039 | 0.17 |

**Table 2.** $S$ wave mass spectra of $b\bar{b}$ states (in MeV).

| States | Ours | Exp [66] | [45] | [53] | [58] | [56] | [85] |
|---|---|---|---|---|---|---|---|
| $1^1S_0$ | 9406.4 | 9398.7 | 9398 | 9402 | 9423 | 9412.22 | 9390 |
| $2^1S_0$ | 9998.9 | 9999.0 | 9989 | 9976 | 9983 | 9995.48 | 9990 |
| $3^1S_0$ | 10374.9 | | 10336 | 10336 | 10342 | 10339.00 | 10326 |
| $4^1S_0$ | 10671.8 | | 10597 | 10635 | 10638 | 10572.49 | 10584 |
| $5^1S_0$ | 10924.5 | | 10810 | 10869 | 10901 | 10746.76 | 10800 |
| $6^1S_0$ | 11147.9 | | 10991 | 11097 | 11140 | 11064.47 | 10988 |
| $1^3S_1$ | 9451.1 | 9460.3 | 9463 | 9465 | 9463 | 9460.75 | 9460 |
| $2^3S_1$ | 10023.8 | 10023.3 | 10017 | 10003 | 10001 | 10026.22 | 10015 |
| $3^3S_1$ | 10394.2 | 10355.1 | 10356 | 10354 | 10354 | 10364.65 | 10343 |
| $4^3S_1$ | 10688.1 | 10579.4 | 10612 | 10635 | 10650 | 10594.47 | 10597 |
| $5^3S_1$ | 10938.9 | 10885.2 | 10822 | 10878 | 10912 | 10766.14 | 10811 |
| $6^3S_1$ | 11160.9 | 11000.0 | 11001 | 11102 | 11151 | 11081.70 | 10997 |

**Table 3.** $P$ wave mass spectra of $b\bar{b}$ states (in MeV).

| States | Ours | Exp[66] | [45] | [53] | [58] | [56] | [85] |
|---|---|---|---|---|---|---|---|
| $1^1P_1$ | 9872.9 | 9899.3 | 9894 | 9882 | 9899 | 9874.56 | 9909 |
| $2^1P_1$ | 10271.7 | 10259.8 | 10259 | 10250 | 10268 | 10270.00 | 10254 |
| $3^1P_1$ | 10582.7 | | 10530 | 10541 | 10570 | 10526.50 | 10519 |
| $4^1P_1$ | 10845.6 | | 10751 | 10790 | | 10714.80 | |
| $5^1P_1$ | 11077.1 | | 10938 | 11016 | | 10863.00 | |
| $1^3P_0$ | 9838.7 | 9859.4 | 9858 | 9847 | 9874 | 9849.61 | 9864 |
| $2^3P_0$ | 10244.9 | 10232.5 | 10235 | 10226 | 10248 | 10252.54 | 10220 |
| $3^3P_0$ | 10559.4 | | 10513 | 10522 | 10551 | 10512.88 | 10490 |
| $4^3P_0$ | 10824.5 | | 10736 | 10775 | | 10703.56 | |
| $5^3P_0$ | 11057.4 | | 10926 | 11004 | | 10853.38 | |
| $1^3P_1$ | 9865.7 | 9892.8 | 9889 | 9876 | 9894 | 9871.47 | 9903 |
| $2^3P_1$ | 10266.2 | 10255.5 | 10255 | 10246 | 10265 | 10267.86 | 10249 |
| $3^3P_1$ | 10578.1 | 10513.4 | 10527 | 10538 | 10567 | 10524.84 | 10515 |
| $4^3P_1$ | 10841.5 | | 10749 | 10788 | | 10713.44 | |
| $5^3P_1$ | 11073.3 | | 10936 | 11014 | | 10861.83 | |
| $1^3P_2$ | 9885.6 | 9912.2 | 9910 | 9897 | 9907 | 9881.40 | 9921 |
| $2^3P_2$ | 10282.3 | 10268.6 | 10269 | 10261 | 10274 | 10274.77 | 10264 |
| $3^3P_2$ | 10592.3 | 10524.0 | 10539 | 10550 | 10576 | 10530.21 | 10528 |
| $4^3P_2$ | 10854.6 | | 10758 | 10798 | | 10717.86 | |
| $5^3P_2$ | 11085.6 | | 10944 | 11022 | | 10865.62 | |

**Table 4.** $D$ wave mass spectra of $b\bar{b}$ states (in MeV).

| States | Ours | Exp[66] | [45] | [53] | [58] | [56] | [85] |
|---|---|---|---|---|---|---|---|
| $1^1D_2$ | 10149.1 | | 10163 | 10148 | 10149 | 10153.80 | 10153 |
| $2^1D_2$ | 10476.3 | | 10450 | 10450 | 10465 | 10456.60 | 10432 |
| $3^1D_2$ | 10749.9 | | 10681 | 10706 | 10740 | 10664.70 | |
| $4^1D_2$ | 10989.2 | | 10876 | 10935 | 10988 | 10823.00 | |
| $5^1D_2$ | 11204.1 | | 11046 | | | 10952.60 | |
| $1^3D_1$ | 10139.4 | | 10153 | 10138 | 10145 | 10144.99 | 10146 |
| $2^3D_1$ | 10467.3 | | 10442 | 10441 | 10462 | 10450.23 | 10425 |
| $3^3D_1$ | 10741.3 | 10752.7 | 10675 | 10698 | 10736 | 10659.68 | |
| $4^3D_1$ | 10981.0 | | 10871 | 10928 | 10985 | 10818.83 | |
| $5^3D_1$ | 11196.1 | | 11041 | | | 10949.01 | |
| $1^3D_2$ | 10147.9 | 10163.7 | 10162 | 10147 | 10149 | 10152.77 | 10153 |
| $2^3D_2$ | 10475.0 | | 10450 | 10449 | 10465 | 10455.86 | 10432 |
| $3^3D_2$ | 10748.7 | | 10681 | 10705 | 10740 | 10664.12 | |
| $4^3D_2$ | 10987.9 | | 10876 | 10934 | 10988 | 10822.52 | |
| $5^3D_2$ | 11202.8 | | 11045 | | | 10951.59 | |
| $1^3D_3$ | 10154.2 | | 10170 | 10155 | 10150 | 10158.31 | 10157 |
| $2^3D_3$ | 10481.1 | | 10456 | 10455 | 10466 | 10459.85 | 10436 |
| $3^3D_3$ | 10754.6 | | 10686 | 10711 | 10741 | 10667.25 | |
| $4^3D_3$ | 10993.7 | | 10880 | 10939 | 10990 | 10825.12 | |
| $5^3D_3$ | 11208.5 | | 11049 | | | 10954.42 | |

**Table 5.** $F$ and $G$ wave mass spectra of $b\bar{b}$ states (in MeV).

| States | Ours | [45] | [53] | States | Ours | [45] | [53] |
|---|---|---|---|---|---|---|---|
| $1^1F_3$ | 10366.8 | 10366 | 10355 | $1^1G_4$ | 10552.9 | 10534 | 10530 |
| $2^1F_3$ | 10652.2 | 10609 | 10619 | $2^1G_4$ | 10809.8 | 10747 | 10770 |
| $3^1F_3$ | 10900.0 | 10812 | 10853 | $3^1G_4$ | 11038.2 | 10929 | |
| $4^1F_3$ | 11121.5 | 10988 | | $4^1G_4$ | 11245.3 | | |
| $5^1F_3$ | 11323.0 | | | $5^1G_4$ | 11435.6 | | |
| $1^3F_2$ | 10363.6 | 10362 | 10350 | $1^3G_3$ | 10552.8 | 10533 | 10529 |
| $2^3F_2$ | 10648.8 | 10605 | 10615 | $2^3G_3$ | 10809.2 | 10745 | 10769 |
| $3^3F_2$ | 10896.5 | 10809 | 10850 | $3^3G_3$ | 11037.3 | 10928 | |
| $4^3F_2$ | 11117.8 | 10985 | | $4^3G_3$ | 11244.1 | | |
| $5^3F_2$ | 11319.2 | | | $5^3G_3$ | 11434.3 | | |
| $1^3F_3$ | 10366.8 | 10366 | 10355 | $1^3G_4$ | 10553.4 | 10535 | 10531 |
| $2^3F_3$ | 10652.2 | 10609 | 10619 | $2^3G_4$ | 10810.1 | 10747 | 10770 |
| $3^3F_3$ | 10899.9 | 10812 | 10853 | $3^3G_4$ | 11038.4 | 10929 | |
| $4^3F_3$ | 11121.3 | 10988 | | $4^3G_4$ | 11245.5 | | |
| $5^3F_3$ | 11322.7 | | | $5^3G_4$ | 11435.8 | | |
| $1^3F_4$ | 10368.5 | 10369 | 10358 | $1^3G_5$ | 10552.6 | 10536 | 10532 |
| $2^3F_4$ | 10654.1 | 10612 | 10622 | $2^3G_5$ | 10809.8 | 10748 | 10772 |
| $3^3F_4$ | 10902.1 | 10815 | 10856 | $3^3G_5$ | 11038.5 | 10931 | |
| $4^3F_4$ | 11123.7 | 10990 | | $4^3G_5$ | 11245.9 | | |
| $5^3F_4$ | 11325.3 | | | $5^3G_5$ | 11436.4 | | |

## III. DECAY PROPERTIES

Bottomonium decays are important for understanding internal structures, revealing underlying dynamics, and distinguishing states. A comparison of mass spectra and decay properties with experimental data helps to validate theoretical models. Using the obtained wave functions, we calculate various decay properites of bottomonium.

Decay constants are fundamental parameters that characterize the strength of the weak interaction responsible for the decay processes, and it measures the probability amplitude to decay into lighter hadrons. The decay constant of pseudoscalar ($f_P$) and vector ($f_V$) states can be calculated using the Van Royen Weisskopf formula [46]:

$$f_{P/V} = \sqrt{\frac{3|R_{P/V}(0)|^2}{\pi M_{P/V}}}\bar{C}(\alpha_s), \tag{19}$$

where $R_{P/V}(0)$ is the radial wavefunction at the origin for pseudoscalar (vector) meson state, $M_{P/V}$ is the mass of the pseudoscalar (vector) meson state, and $\bar{C}(\alpha_s)$ is the QCD correction given by [47]

$$\bar{C}^2(\alpha_s) = 1 - \frac{\alpha_s(\mu)}{\pi}\left(\delta^{P,V} - \frac{m_q - m_{\bar{q}}}{m_q + m_{\bar{q}}}\ln\frac{m_q}{m_{\bar{q}}}\right), \tag{20}$$

where $\delta^P = 2$ and $\delta^V = 8/3$. The decay constant of $P$-wave states can be evaluated using [48, 49]

$$f_{\chi_0} = 12\sqrt{\frac{3}{8\pi m_q}}\left(\frac{|R'_{\chi_0}(0)|}{M_{\chi_0}}\right),$$
$$f_{\chi_0} = 8\sqrt{\frac{9}{8\pi m_q}}\left(\frac{|R'_{\chi_1}(0)|}{M_{\chi_1}}\right). \tag{21}$$

Here ,$M_{\chi_0}$ and $M_{\chi_1}$ are the masses of $\chi_0$ and $\chi_1$ states, respectively. The decay constants for the pseudoscalar $f_P$ and vector $f_V$ are presented in Table 6 and decay constants for $f_{\chi_0}$ and $f_{\chi_1}$ are presented in Table 7. Bottomonium annihilation decays leave distinct signals in experimental data, enabling bottomonium states to be identified and characterized in high-energy collider experiments and precision spectroscopic investigations.

The leptonic decay formula for $S$-wave ($n^3S_1$) and $D$-wave ($n^3D_1$) states are calculated using the Van Royen-Weisskopf formula along with the QCD correction factor [46, 50−52]:

$$\Gamma\left(n^3S_1 \to l^+l^-\right) = \frac{4\alpha^2 e_q^2}{M(n^3S_1)^2}|R_{nS}(0)|^2\left[1 - \frac{16\alpha_s(\mu)}{3\pi}\right],$$
$$\Gamma\left(n^3D_1 \to l^+l^-\right) = \frac{25\alpha^2 e_q^2}{2m_q^4 M(n^3D_1)^2}|R''_{nD}(0)|^2, \tag{22}$$

where $R'_{nL}(0)$ is the value of radial wavefunction at origin for $nL$ state, ($'$) represents the order of derivative, and $M(n^{2S+1}L_J)$ is the mass of the $n^{2S+1}L_J$ state.

The annihilation decays for the $S$-wave ($n^1S_0$) and $P$-wave ($n^3P_0$ and $n^3P_2$) into two photons ($\gamma\gamma$) and $S$-wave ($n^3S_1$) states into three photons ($\gamma\gamma\gamma$) with first order QCD correction factors are given by [50, 51]

$$\Gamma\left(n^1S_0 \to \gamma\gamma\right) = \frac{2^2 3\alpha^2 e_q^4}{M(n^1S_0)^2}|R_{nS}(0)|^2\left[1 - \frac{3.4\alpha_s(\mu)}{\pi}\right],$$
$$\Gamma\left(n^3P_0 \to \gamma\gamma\right) = \frac{2^4 27\alpha^2 e_q^4}{M(n^3P_0)^4}|R'_{nP}(0)|^2\left[1 + \frac{0.2\alpha_s(\mu)}{\pi}\right],$$
$$\Gamma\left(n^3P_2 \to \gamma\gamma\right) = \frac{2^4 36\alpha^2 e_q^4}{5M(n^3P_2)^4}|R'_{nP}(0)|^2\left[1 - \frac{16\alpha_s(\mu)}{3\pi}\right],$$
$$\Gamma\left(n^3S_1 \to \gamma\gamma\gamma\right) = \frac{2^2 4(\pi^2 - 9)\alpha^3 e_q^6}{3\pi M(n^3S_1)^2}|R_{nS}(0)|^2\left[1 - \frac{12.6\alpha_s(\mu)}{\pi}\right]. \tag{23}$$

The annihilation decays for $S$-wave ($n^1S_0$), $P$-wave ($n^3P_0$ and $n^3P_2$), $D$-wave ($n^1D_2$), $F$-wave ($n^3F_2, n^3F_3$ and $n^3F_4$), and $G$-wave ($n^1G_4$) states into two gluons ($gg$) with first order QCD correction factors are given by [50, 51, 53]

$$
\begin{aligned}
\Gamma\left(n^1S_0\to gg\right) &= \frac{2^2 2\alpha_s^2(\mu)}{3M(n^1S_0)^2}|R_{nS}(0)|^2\left[1+\frac{4.8\alpha_s(\mu)}{\pi}\right],\\
\Gamma\left(n^3P_0\to gg\right) &= \frac{2^4 6\alpha_s^2(\mu)}{M(n^3P_0)^4}|R'_{nP}(0)|^2\left[1+\frac{10\alpha_s(\mu)}{\pi}\right],\\
\Gamma\left(n^3P_2\to gg\right) &= \frac{2^4 8\alpha_s^2(\mu)}{5M(n^3P_2)^4}|R'_{nP}(0)|^2\left[1-\frac{0.1\alpha_s(\mu)}{\pi}\right],\\
\Gamma\left(n^1D_2\to gg\right) &= \frac{2^6 2\alpha_s^2(\mu)}{3\pi M(n^1D_2)^6}|R''_{nD}(0)|^2\left[1-\frac{2.2\alpha_s(\mu)}{\pi}\right],\\
\Gamma\left(n^3F_2\to gg\right) &= \frac{2^8 919\alpha_s^2(\mu)}{135M(n^3F_2)^8}|R'''_{nF}(0)|^2,\\
\Gamma\left(n^3F_3\to gg\right) &= \frac{2^8 20\alpha_s^2(\mu)}{27M(n^3F_3)^8}|R'''_{nF}(0)|^2,\\
\Gamma\left(n^3F_4\to gg\right) &= \frac{2^8 20\alpha_s^2(\mu)}{27M(n^3F_4)^8}|R'''_{nF}(0)|^2,\\
\Gamma\left(n^1G_4\to gg\right) &= \frac{2^{10} 2\alpha_s^2(\mu)}{3\pi M(n^1G_4)^{10}}|R''''_{nG}(0)|^2.
\end{aligned}
\tag{24}
$$

The annihilation decays for $S$-wave ($n^3S_1$), $P$-wave ($n^1P_1$), and $D$-wave ($n^3D_1, n^3D_2$ and $n^3D_3$) states into three gluons ($ggg$) with first order QCD correction factors are given by [50, 51, 54]

$$
\begin{aligned}
\Gamma\left(n^3S_1\to ggg\right) &= \frac{2^2 10(\pi^2-9)\alpha_s^3(\mu)}{81\pi M(n^3S_1)^2}|R_{nS}(0)|^2\left[1-\frac{4.9\alpha_s(\mu)}{\pi}\right],\\
\Gamma\left(n^1P_1\to ggg\right) &= \frac{2^4 20\alpha_s^3(\mu)}{9\pi M(n^1P_1)^4}|R'_{nP}(0)|^2\ln\left(m_q\langle r\rangle\right),\\
\Gamma\left(n^3D_1\to ggg\right) &= \frac{2^6 760\alpha_s^3(\mu)}{81\pi M(n^3D_1)^6}|R''_{nD}(0)|^2\ln\left(4m_q\langle r\rangle\right),\\
\Gamma\left(n^3D_2\to ggg\right) &= \frac{2^6 10\alpha_s^3(\mu)}{9\pi M(n^3D_2)^6}|R''_{nD}(0)|^2\ln\left(4m_q\langle r\rangle\right),\\
\Gamma\left(n^3D_3\to ggg\right) &= \frac{2^6 40\alpha_s^3(\mu)}{9\pi M(n^3D_3)^6}|R''_{nD}(0)|^2\ln\left(4m_q\langle r\rangle\right).
\end{aligned}
\tag{25}
$$

The annihilation decays for $S$-wave ($n^3S_1$) states via strong and electromagnetic interactions into a photon and two gluons ($\gamma gg$) [43, 50, 55] and the $P$ - wave ($n^3P_1$) into a light flavour meson and a gluon ($q\bar{q}g$) are given by [50, 51]

$$
\begin{aligned}
\Gamma\left(n^3S_1\to\gamma gg\right) &= \frac{2^2 8(\pi^2-9)e_q^2\alpha\alpha_s^3(\mu)}{9\pi M(n^3S_1)^2}|R_{nS}(0)|^2\times\left[1-\frac{7.4\alpha_s(\mu)}{\pi}\right],\\
\Gamma\left(n^3P_1\to q\bar{q}g\right) &= \frac{2^4 8n_f\alpha_s^3(\mu)}{9\pi M(n^3P_1)^4}|R'_{nP}(0)|^2\ln\left(m_q\langle r\rangle\right),
\end{aligned}
\tag{26}
$$

where $n_f$ is the number of flavors. For all decays, the strong coupling constant $\alpha_s(\mu)$ is calculated using the expression

$$
\alpha_s(\mu) = \frac{4\pi}{\beta_0\ln\frac{\mu^2}{\Lambda^2}}\left[1-\frac{\beta_1\ln\left(\ln\frac{\mu^2}{\Lambda^2}\right)}{\beta_0^2\ln\frac{\mu^2}{\Lambda^2}}\right],
\tag{27}
$$

where $\beta_0 = 11-(2/3)n_f$, $\beta_1 = 102-(18/3)n_f$, $\Lambda$ is the QCD constant taken from [51], and $\mu$ is the reduced mass. All annihilation decay widths for $b\bar{b}$ bound system are presented in Tables 8−14, respectively.

The bottomonium states have significantly heavier masses, and their intrinsic compactness is more pronounced [56]. Thus, radiative transitions in bottomonium are expected to be dominant because of the favourable conditions for photon emission or absorption [56]. Radiative transitions serve as effective means for detection, particularly for states with higher quantum numbers that are difficult to observe with traditional techniques. The $E1$ radiative partial widths between the states $\left(n_i^{2S+1}L^i_{J_i}\to\gamma+n_f^{2S+1}L^f_{J_f}\right)$ are given by [45, 57]

$$
\Gamma_{E1}(i\to\gamma+f) = \frac{4\alpha e_q^2}{3}E_\gamma^3\frac{E_f}{M_i}C_{fi}|\epsilon_{fi}|^2\delta_{S_fS_i}.
\tag{28}
$$

**Table 6.** Pseudoscalar and vector decay constants (in MeV).

| States | $f_{P/V}$ | Exp[66] | [58] | [56] | [77] | [78] |
|---|---|---|---|---|---|---|
| $1^1S_0$ | 655.9 | | 529 | 578.21 | 646.025 | 744 |
| $2^1S_0$ | 489.2 | | 317 | 499.48 | 518.803 | 577 |
| $3^1S_0$ | 431.8 | | 280 | 450.35 | 474.954 | 511 |
| $4^1S_0$ | 398.6 | | 264 | 413.93 | 449.654 | 471 |
| $5^1S_0$ | 375.5 | | 255 | 385.68 | 432.072 | 443 |
| $6^1S_0$ | 357.6 | | 249 | 360.93 | 418.645 | 422 |
| $1^3S_1$ | 640.2 | 715±5 | 530 | 551.53 | 647.250 | 706 |
| $2^3S_1$ | 478.0 | 498±8 | 317 | 477.05 | 519.436 | 547 |
| $3^3S_1$ | 422.0 | 430±4 | 280 | 430.42 | 475.440 | 484 |
| $4^3S_1$ | 389.7 | 336±18 | 265 | 395.80 | 450.066 | 446 |
| $5^3S_1$ | 349.7 | | 255 | 368.91 | 432.437 | 419 |
| $6^3S_1$ | 335.3 | | 249 | 345.40 | 418.977 | 399 |

**Table 7.** Decay constants of $P$-wave states (in MeV).

| States | $f_{\chi_0}$ | States | $f_{\chi_1}$ |
|---|---|---|---|
| $1^3P_0$ | 227.8 | $1^3P_1$ | 262.4 |
| $2^3P_0$ | 248.7 | $2^3P_1$ | 286.6 |
| $3^3P_0$ | 257.3 | $3^3P_1$ | 296.6 |
| $4^3P_0$ | 262.0 | $4^3P_1$ | 302.1 |
| $5^3P_0$ | 264.8 | $5^3P_1$ | 305.4 |

**Table 8.** Di-leptonic decay widths (in keV for $S$ states and in eV for $D$ states).

| States | $\Gamma$ | $\Gamma_{cf}$ | Exp[66] | [60] | [45] | [53] | [56] | [58] |
|---|---|---|---|---|---|---|---|---|
| $1^3S_1$ | 1.268 | 0.883 | 1.34±0.018 | 1.370 | 1.65 | 1.44 | 0.7700 | 0.582 |
| $2^3S_1$ | 0.666 | 0.464 | 0.612±0.011 | 0.626 | 0.821 | 0.73 | 0.5442 | 0.197 |
| $3^3S_1$ | 0.501 | 0.349 | 0.443±0.008 | 0.468 | 0.569 | 0.53 | 0.4288 | 0.149 |
| $4^3S_1$ | 0.415 | 0.289 | 0.272±0.029 | 0.393 | 0.431 | 0.39 | 0.3549 | 0.129 |
| $5^3S_1$ | 0.360 | 0.251 | 0.31±0.07 | 0.346 | 0.348 | 0.33 | 0.3035 | 0.117 |
| $6^3S_1$ | 0.320 | 0.223 | 0.13±0.03 | 0.313 | 0.286 | 0.27 | 0.2586 | 0.109 |
| $1^3D_1$ | 1.149 | | | 2.0 | 1.88 | 1.38 | 5.0 | 1.65 |
| $2^3D_1$ | 2.166 | | | 3.0 | 2.81 | 1.99 | 5.8 | 2.42 |
| $3^3D_1$ | 3.059 | | | 5.0 | 3.00 | 2.38 | 5.9 | 3.19 |
| $4^3D_1$ | 4.573 | | | 6.0 | 3.00 | 2.18 | 5.8 | 3.97 |
| $5^3D_1$ | 5.219 | | | 8.0 | 3.02 | | 5.7 | |

**Table 9.** Di-photonic decay widths (in keV).

| States | $\Gamma$ | $\Gamma_{cf}$ | [45] | [53] | [56] | [77] | [58] |
|---|---|---|---|---|---|---|---|
| $1^1S_0$ | 0.426 | 0.344 | 1.05 | 0.94 | 0.3035 | 0.387 | 0.2361 |
| $2^1S_0$ | 0.223 | 0.180 | 0.489 | 0.41 | 0.2122 | 0.263 | 0.0896 |
| $3^1S_0$ | 0.168 | 0.135 | 0.323 | 0.29 | 0.1668 | 0.229 | 0.0726 |
| $4^1S_0$ | 0.139 | 0.112 | 0.237 | 0.20 | 0.1378 | 0.212 | 0.0666 |
| $5^1S_0$ | 0.120 | 0.097 | 0.192 | 0.17 | 0.1176 | 0.201 | 0.0636 |
| $6^1S_0$ | 0.107 | 0.086 | 0.152 | 0.14 | 0.1000 | 0.193 | 0.0619 |
| $1^3P_0$ | 0.042 | 0.042 | 0.199 | 0.15 | 0.1150 | 0.0196 | 0.0168 |
| $2^3P_0$ | 0.046 | 0.047 | 0.205 | 0.15 | 0.1014 | 0.0195 | 0.0172 |
| $3^3P_0$ | 0.046 | 0.047 | 0.180 | 0.13 | 0.0875 | 0.0194 | 0.0192 |
| $4^3P_0$ | 0.045 | 0.046 | 0.157 | 0.13 | 0.0768 | 0.0192 | |
| $5^3P_0$ | 0.044 | 0.046 | 0.146 | | 0.0686 | 0.0191 | |
| $1^3P_2$ | 0.011 | 0.007 | 0.0106 | 0.0093 | 0.0147 | 0.0052 | 0.0024 |
| $2^3P_2$ | 0.012 | 0.008 | 0.0133 | 0.012 | 0.0131 | 0.0052 | 0.0025 |
| $3^3P_2$ | 0.012 | 0.009 | 0.0141 | 0.013 | 0.0114 | 0.0051 | 0.0027 |
| $4^3P_2$ | 0.012 | 0.008 | 0.0142 | 0.015 | 0.0100 | 0.0051 | |
| $5^3P_2$ | 0.011 | 0.008 | 0.0143 | | 0.0090 | 0.0050 | |

Here, $\alpha = 1/137$ is the fine structure constant, $e_q$ is the quark charge, $E_f$ is the energy of the final state, $M_i$ is the mass of the initial state, and $E_\gamma = \left(M_i^2 - M_f^2\right)/2M_i$ is the emitted photon energy. $M_f$ is the mass of the final state. $E_f/M_i$ is the relativistic phase factor, and $C_{fi}$ is the statistical factor given by

$$C_{fi} = \max\left(L_i, L_f\right)\left(2J_f+1\right)\left\{\begin{array}{ccc} J_i & 1 & J_f \\ L_f & S & L_i \end{array}\right\}^2, \tag{29}$$

where $\{:::\}$ is the $6j$ symbol. In Eq. (28), $\epsilon_{fi}$ is the overlapping integral determined using the initial $R_{n_il_i}(r)$ and final state $R_{n_fl_f}(r)$ wavefunctions:

$$\epsilon_{fi} = \frac{3}{E_\gamma}\int_0^\infty \mathrm{d}r R_{n_il_i}(r)R_{n_fl_f}(r)\left[\frac{E_\gamma r}{2}j_0\left(\frac{E_\gamma r}{2}\right) - j_1\left(\frac{E_\gamma r}{2}\right)\right], \tag{30}$$

The $M1$ radiative partial widths between the states $\left(n_i^{2S_i+1}L_{J_i} \to \gamma + n_f^{2S_f+1}L_{J_f}\right)$ are given by [42, 57, 58]

$$\Gamma_{M1}(i \to \gamma + f) = \frac{4\alpha\mu_q^2}{3}\frac{2J_f+1}{2L+1}E_\gamma^3\frac{E_f}{M_i}|m_{fi}|^2\delta_{L_fL_i}\delta_{S_fS_i\pm1}, \tag{31}$$

**Table 10.** Tri-photonic decay widths (in $10^{-3}$ eV).

| States | $\Gamma$ | $\Gamma_{cf}$ | [45] | [53] | [80] | [58] |
|---|---|---|---|---|---|---|
| $1^3S_1$ | 42.16 | 11.92 | 19.4 | 17.0 | 3.44 | 30.67 |
| $2^3S_1$ | 22.17 | 6.27 | 10.9 | 9.8 | 2.00 | 11.58 |
| $3^3S_1$ | 16.66 | 4.71 | 8.04 | 7.6 | 1.55 | 9.376 |
| $4^3S_1$ | 13.81 | 3.91 | 6.36 | 6.0 | 1.29 | 8.590 |
| $5^3S_1$ | 11.98 | 3.39 | 5.43 | | 1.10 | 8.206 |
| $6^3S_1$ | 10.65 | 3.01 | 4.57 | | 0.96 | 7.982 |

**Table 11.** Di-gluonic decay widths of $S$, $P$ (in MeV) and $D$ (in keV) states.

| States | $\Gamma$ | $\Gamma_{cf}$ | [45] | [53] | [80] | [58] | [56] |
|---|---|---|---|---|---|---|---|
| $1^1S_0$ | 4.608 | 5.763 | 17.9 | 16.6 | 20.18 | 11.326 | 6.8520 |
| $2^1S_0$ | 2.412 | 3.016 | 8.33 | 7.2 | 10.64 | 4.301 | 5.2374 |
| $3^1S_0$ | 1.811 | 2.264 | 5.51 | 4.9 | 7.94 | 3.485 | 4.3182 |
| $4^1S_0$ | 1.500 | 1.876 | 4.03 | 3.4 | | 3.193 | 3.6829 |
| $5^1S_0$ | 1.301 | 1.627 | 3.26 | | | 3.051 | 3.2196 |
| $6^1S_0$ | 1.156 | 1.446 | 2.59 | | | 2.968 | 2.8519 |
| $1^3P_0$ | 0.454 | 0.713 | 3.37 | 2.6 | 2.00 | 1.34 | 1.4297 |
| $2^3P_0$ | 0.499 | 0.783 | 3.52 | 2.6 | 2.37 | 1.39 | 1.2358 |
| $3^3P_0$ | 0.503 | 0.789 | 3.10 | 2.2 | 2.46 | 1.54 | 1.0539 |
| $4^3P_0$ | 0.496 | 0.779 | 2.73 | 2.1 | | | 0.9175 |
| $5^3P_0$ | 0.486 | 0.763 | 2.54 | | | | 0.8127 |
| $1^3P_2$ | 0.119 | 0.118 | 0.165 | 0.147 | 0.837 | 0.209 | 0.2370 |
| $2^3P_2$ | 0.131 | 0.130 | 0.220 | 0.207 | 0.104 | 0.215 | 0.2064 |
| $3^3P_2$ | 0.132 | 0.132 | 0.243 | 0.227 | 0.111 | 0.240 | 0.1767 |
| $4^3P_2$ | 0.131 | 0.130 | 0.251 | 0.248 | | | 0.1543 |
| $5^3P_2$ | 0.128 | 0.127 | 0.258 | | | | 0.1370 |
| $1^1D_2$ | 0.321 | 0.281 | 0.657 | 1.8 | 0.37 | 0.489 | |
| $2^1D_2$ | 0.534 | 0.468 | 1.22 | 3.3 | 0.67 | 0.764 | |
| $3^1D_2$ | 0.679 | 0.595 | 1.59 | 4.7 | | 1.06 | |
| $4^1D_2$ | 0.785 | 0.686 | 1.86 | | | 1.38 | |
| $5^1D_2$ | 0.861 | 0.754 | 2.13 | | | | |

**Table 12.** Di-gluonic decay widths of $F$ (in keV) and $G$ (in eV) states.

| States | Γ | [45] | [53] | States | Γ | [45] | [53] |
|---|---|---|---|---|---|---|---|
| $1^3F_2$ | 0.282 | 0.834 | 0.70 | $1^3F_4$ | 0.031 | 0.05 | 0.048 |
| $2^3F_2$ | 0.618 | 2.04 | 1.77 | $2^3F_4$ | 0.067 | 0.126 | 0.13 |
| $3^3F_2$ | 0.946 | 3.17 | | $3^3F_4$ | 0.102 | 0.210 | |
| $4^3F_2$ | 1.248 | | | $4^3F_4$ | 0.135 | | |
| $5^3F_2$ | 1.517 | | | $5^3F_4$ | 0.164 | | |
| $1^3F_3$ | 0.031 | 0.0672 | 0.060 | $1^1G_4$ | 0.289 | 0.661 | 2.3 |
| $2^3F_3$ | 0.067 | 0.167 | 0.16 | $2^1G_4$ | 0.778 | | |
| $3^3F_3$ | 0.103 | 0.270 | | $3^1G_4$ | 1.383 | | |
| $4^3F_3$ | 0.135 | | | $4^1G_4$ | 2.044 | | |
| $5^3F_3$ | 0.165 | | | $5^1G_4$ | 2.723 | | |

**Table 13.** Tri-gluonic decay widths (in keV).

| States | Γ | $\Gamma_{cf}$ | Exp[66] | [45] | [53] | [56] | [80] |
|---|---|---|---|---|---|---|---|
| $1^3S_1$ | 41.85 | 30.18 | 44.13±1.09 | 50.8 | 47.6 | 28.5 | 41.63 |
| $2^3S_1$ | 22.00 | 15.86 | 18.8±1.59 | 28.4 | 26.3 | 19.3 | 24.25 |
| $3^3S_1$ | 16.54 | 11.92 | 7.25±0.85 | 21.0 | 19.8 | 14.8 | 18.76 |
| $4^3S_1$ | 13.71 | 9.89 | | 16.7 | 15.1 | 12.1 | 15.58 |
| $5^3S_1$ | 11.89 | 8.58 | | 14.2 | 13.1 | 10.2 | 13.33 |
| $6^3S_1$ | 10.57 | 7.62 | | 12.0 | 11.0 | 8.5 | 11.57 |
| $1^1P_1$ | 20.10 | | | 44.7 | 37.0 | 35.7 | 35.26 |
| $2^1P_1$ | 26.99 | | | 64.6 | 54.0 | 34.6 | 52.70 |
| $3^1P_1$ | 30.23 | | | 71.1 | 59.0 | 33.1 | 62.16 |
| $4^1P_1$ | 31.99 | | | 73.2 | 64.0 | 32.7 | |
| $5^1P_1$ | 32.97 | | | 76.2 | | 30.9 | |
| $1^3D_1$ | 3.11 | | | 10.4 | 8.11 | 10.6 | 9.97 |
| $2^3D_1$ | 5.61 | | | 20.1 | 14.8 | 11.9 | 9.69 |
| $3^3D_1$ | 7.54 | | | 26.0 | 21.2 | 11.8 | |
| $4^3D_1$ | 9.05 | | | 30.4 | | 11.3 | |
| $5^3D_1$ | 1.02 | | | 34.7 | | 10.8 | |
| $1^3D_2$ | 0.37 | | | 0.821 | 0.69 | | 0.62 |
| $2^3D_2$ | 0.66 | | | 1.65 | 1.4 | | 0.61 |
| $3^3D_2$ | 0.89 | | | 2.27 | 2.0 | | |
| $4^3D_2$ | 0.11 | | | 2.75 | | | |
| $5^3D_2$ | 0.12 | | | 3.23 | | | |
| $1^3D_3$ | 1.46 | | | 2.19 | 2.07 | 6.0 | 0.22 |
| $2^3D_3$ | 2.64 | | | 4.56 | 4.3 | 5.6 | 1.25 |
| $3^3D_3$ | 3.54 | | | 6.65 | 6.6 | 5.5 | |
| $4^3D_3$ | 4.26 | | | 8.38 | | 5.3 | |
| $5^3D_3$ | 4.82 | | | 10.1 | | 5.1 | |

**Table 14.** Photo-gluon decay widths of $S$ states and quark-gluon decay widths of $P$ states (in keV).

| States | Γ | $\Gamma_{cf}$ | Exp[66] | [45] | [80] | [58] | [56] |
|---|---|---|---|---|---|---|---|
| $1^3S_1$ | 1.37 | 0.79 | 1.19±0.33 | 1.32 | 0.79 | 0.903 | 0.7220 |
| $2^3S_1$ | 0.72 | 0.42 | 0.60±0.10 | 0.739 | 0.46 | 0.341 | 0.4982 |
| $3^3S_1$ | 0.54 | 0.31 | 0.20±0.04 | 0.547 | 0.36 | 0.276 | 0.3874 |
| $4^3S_1$ | 0.45 | 0.26 | | 0.433 | 0.30 | 0.253 | 0.3176 |
| $5^3S_1$ | 0.39 | 0.22 | | 0.370 | 0.25 | 0.242 | 0.2698 |
| $6^3S_1$ | 0.34 | 0.19 | | 0.311 | 0.22 | 0.235 | 0.2272 |
| $1^3P_1$ | 32.25 | | | 81.7 | 71.53 | 45.55 | 57.9585 |
| $2^3P_1$ | 43.28 | | | 117.0 | 106.14 | 56.16 | 55.3966 |
| $3^3P_1$ | 48.46 | | | 126.0 | 124.53 | 68.97 | 52.9585 |
| $4^3P_1$ | 51.27 | | | 128.0 | | | 52.4466 |
| $5^3P_1$ | 52.82 | | | 132.0 | | | 49.5181 |

where $m_{fi}$ is given by

$$m_{fi} = \int_0^\infty \mathrm{d}r R_{n_i l_i}(r) R_{n_f l_f}(r) \left[ j_0\left(\frac{E_\gamma r}{2}\right)\right], \tag{32}$$

and $\mu_q$ is the magnetic dipole moment given by [55]

$$\mu_q = \frac{m_{\bar{q}} e_q - m_q e_{\bar{q}}}{2 m_q m_{\bar{q}}}. \tag{33}$$

The $E1$ transitions widths for $S, P, D, F$ and $G$ wave states are presented in Tables 15−19, respectively, and the $M1$ transitions widths for $S$ and $P$ wave states are presented in Table 20.

## IV. *S-D* MIXING

In bottomonium, the proximity of energy levels of higher excited states with the same $J^{PC}$ can result in a mixing of states. The mixing is caused by the tensor potential term, but it is not sufficiently strong to induce significant mixing [59, 60]. However, for the states above the open flavor threshold, the mixing can be caused by coupled-channel dynamics, threshold effects, meson exchange, and multi-gluon exchange interactions [61−64]. These effects can modifiy the wavefunctions, causing a mass shift and mixing between states and affecting decay properties such as open channel strong and leptonic decays. Consequently, the conventional representation of bottomonium states as pure $S$ and $D$ wavefunctions breaks down, and the states are instead identified as admixtures of both components. The mixed states can be represented in terms of pure $|nS\rangle$ and $|n'D\rangle$ states as [60]

**Table 15.** $E1$ transition widths (in keV) and photon energies (in MeV) of $S$ wave states.

| Initial State | Final State | Ours $E_\gamma$ | Ours $\Gamma_{E1}$ | Exp[66] | [45] | [53] | [80] | [85] |
|---|---|---|---|---|---|---|---|---|
| $2^1S_0$ | $1^1P_1$ | 125.3 | 4.769 | | 2.467 | 2.48 | 2.85 | 3.41 |
| $2^3S_1$ | $1^3P_0$ | 183.4 | 1.632 | 1.22±0.11 | 0.907 | 0.91 | 1.09 | 1.09 |
| | $1^3P_1$ | 156.9 | 3.092 | 2.21±0.22 | 1.60 | 1.63 | 1.84 | 2.17 |
| | $1^3P_2$ | 137.7 | 3.472 | 2.29±0.22 | 1.86 | 1.88 | 2.08 | 2.62 |
| $3^1S_0$ | $2^1P_1$ | 102.7 | 6.596 | | 2.88 | 2.96 | 2.60 | 4.25 |
| | $1^1P_1$ | 489.9 | 0.226 | | 1.12 | 1.3 | 0.0084 | 0.67 |
| $3^3S_1$ | $2^3P_0$ | 148.2 | 2.157 | 1.20±0.12 | 1.06 | 1.03 | 1.21 | 1.21 |
| | $2^3P_1$ | 127.2 | 4.132 | 2.56±0.26 | 1.96 | 1.91 | 2.13 | 2.61 |
| | $2^3P_2$ | 111.3 | 4.651 | 2.66±0.27 | 2.37 | 2.30 | 2.56 | 3.16 |
| | $1^3P_0$ | 540.6 | 0.057 | 0.055±0.01 | 0.0099 | 0.01 | 0.15 | 0.097 |
| | $1^3P_1$ | 515.1 | 0.115 | 0.018±0.01 | 0.0363 | 0.05 | 0.16 | 0.0005 |
| | $1^3P_2$ | 496.1 | 0.139 | 0.2±0.03 | 0.359 | 0.45 | 0.0827 | 0.14 |
| $4^1S_0$ | $3^1P_1$ | 88.6 | 7.329 | | 1.50 | 1.24 | | |
| | $2^1P_1$ | 392.5 | 0.718 | | | 0.732 | | |
| | $1^1P_1$ | 768.9 | 0.022 | | 0.688 | | | |
| $4^3S_1$ | $3^3P_0$ | 127.9 | 2.384 | | 0.587 | 0.48 | 0.61 | |
| | $3^3P_1$ | 109.5 | 4.544 | | 1.14 | 0.84 | 1.17 | |
| | $3^3P_2$ | 95.4 | 5.057 | | 1.16 | 0.82 | 1.45 | |
| | $2^3P_0$ | 434.0 | 0.160 | | 0.0137 | | 0.17 | |
| | $2^3P_1$ | 413.6 | 0.344 | | 0.0138 | | 0.18 | |
| | $2^3P_2$ | 398.1 | 0.440 | | 0.226 | | 0.11 | |
| | $1^3P_0$ | 815.7 | 0.007 | | $5.12\times10^{-4}$ | | 0.0588 | |
| | $1^3P_1$ | 790.8 | 0.012 | | 0.0507 | | 0.0474 | |
| | $1^3P_2$ | 772.4 | 0.013 | | 0.219 | | 0.012 | |

$$|\phi\rangle = \cos\theta|nS\rangle + \sin\theta|n'D\rangle,$$
$$|\phi'\rangle = -\sin\theta|nS\rangle + \cos\theta|n'D\rangle, \tag{34}$$

where $|\phi\rangle$ and $|\phi'\rangle$ are the mixed states, and $\theta$ is the mixing angle. The masses of the mixed states can be calculated using [60]

$$M_\phi = \left[\left(\frac{M_{nS}+M_{n'D}}{2}\right)+\left(\frac{M_{nS}-M_{n'D}}{2\cos 2\theta}\right)\right],$$
$$M_{\phi'} = \left[\left(\frac{M_{nS}+M_{n'D}}{2}\right)+\left(\frac{M_{n'D}-M_{nS}}{2\cos 2\theta}\right)\right]. \tag{35}$$

Here, $M_\phi$ and $M_{\phi'}$ are the masses of the mixed states, and $M_{nS}$ and $M_{n'D}$ are the masses of the corresponding pure $S$ and $D$ states, respectively.

The leptonic decay widths of the mixed states are given by [60, 65]

**Table 16.** $E1$ transition widths (in keV) and photon energies (in MeV) of $1P$ and $2P$ wave states.

| Initial State | Final State | Ours $E_\gamma$ | Ours $\Gamma_{E1}$ | Exp[66] | [45] | [53] | [80] | [85] |
|---|---|---|---|---|---|---|---|---|
| $1^1P_1$ | $1^1S_0$ | 455.4 | 38.692 | 35.77 | 34.4 | 35.7 | 43.66 | 35.8 |
| $1^3P_0$ | $1^3S_1$ | 379.9 | 23.099 | | 22.8 | 23.8 | 28.07 | 27.5 |
| $1^3P_1$ | $1^3S_1$ | 405.8 | 27.901 | 32.544 | 28.3 | 29.5 | 35.66 | 31.9 |
| $1^3P_2$ | $1^3S_1$ | 424.9 | 31.805 | 34.38 | 31.4 | 32.8 | 39.15 | 31.8 |
| $2^1P_1$ | $1^1D_2$ | 121.8 | 3.604 | | 1.81 | 1.7 | 5.36 | 2.24 |
| | $2^1S_0$ | 269.1 | 21.962 | 40.32 | 15.0 | 14.1 | 17.60 | 16.2 |
| | $1^1S_0$ | 828.8 | 11.071 | | 10.8 | 13.0 | 14.90 | 16.1 |
| $2^3P_0$ | $1^3D_1$ | 104.9 | 2.316 | | 1.05 | 1.0 | 0.74 | 1.77 |
| | $2^3S_1$ | 218.7 | 12.165 | $1.2\times10^{-4}$ | 11.1 | 10.9 | 12.80 | 14.4 |
| | $1^3S_1$ | 763.0 | 8.198 | | 2.31 | 2.5 | 5.44 | 5.54 |
| $2^3P_1$ | $1^3D_1$ | 126.0 | 0.995 | | 0.511 | 0.5 | 0.41 | 0.56 |
| | $1^3D_2$ | 117.6 | 2.436 | | 1.25 | 1.2 | 1.26 | 0.50 |
| | $2^3S_1$ | 239.6 | 15.790 | 19.4±5 | 13.7 | 13.3 | 15.89 | 15.3 |
| | $1^3S_1$ | 782.7 | 8.991 | 8.9±2.2 | 5.09 | 5.5 | 9.13 | 10.8 |
| $2^3P_2$ | $1^3D_1$ | 141.9 | 0.056 | | 0.0267 | 0.03 | 0.0209 | 0.026 |
| | $1^3D_2$ | 133.5 | 0.708 | | 0.339 | 0.3 | 0.35 | 0.42 |
| | $1^3D_3$ | 127.3 | 3.442 | | 1.61 | 1.5 | 2.06 | 2.51 |
| | $2^3S_1$ | 255.2 | 18.908 | 15.1±5.6 | 14.6 | 14.3 | 17.50 | 15.3 |
| | $1^3S_1$ | 797.6 | 9.626 | 9.8±2.3 | 7.86 | 8.4 | 11.38 | 12.5 |

$$\Gamma_\phi = \left[\frac{2\alpha e_q}{M_{nS}}|R_{nS}(0)|\cos\theta + \frac{5\alpha e_q}{\sqrt{2}m_q^2 M_{n'D}}|R''_{n'D}(0)|\sin\theta\right]^2,$$
$$\Gamma_{\phi'} = \left[\frac{5\alpha e_q}{\sqrt{2}m_q^2 M_{n'D}}|R''_{n'D}(0)|\cos\theta - \frac{2\alpha e_q}{M_{nS}}|R_{nS}(0)|\sin\theta\right]^2. \tag{36}$$

The leptonic decay of the mixed states is fitted to the experimental data to obtain the mixing angle, which is then used to calculate the masses of the mixed states. Our results of $S-D$ mixing are presented in Table 21.

## V. RESULTS AND DISCUSSION

In this study, a screened potential model within a relativistic framework is employed to compute the mass spectrum and decay widths of $b\bar{b}$ bound system. The masses of $S$-wave states are presented in Table 2 and are compared with experimental data and other theoretical models. The well-established $1S$ and $2S$ states serve as benchmarks, with our model predicting $\eta_b(1S) = 9406.4$ MeV, $\Upsilon(1S) = 9451.1$ MeV, $\eta_b(2S) = 9998.9$ MeV, and $\Upsilon(2S) = 10023.8$ MeV. The hyperfine mass splittings, given by $\Delta m(nS) = m[\Upsilon(nS)] - m[\eta_b(nS)]$, are evaluated to be $\Delta m(1S) = 44.7$ MeV and $\Delta m(2S) = 24.9$ MeV. These

**Table 17.** $E1$ transition widths (in keV) and photon energies (in MeV) of $3P$ states.

| Initial State | Final State | Ours $E_\gamma$ | Ours $\Gamma_{E1}$ | [45] | [53] | [80] | [85] |
|---|---|---|---|---|---|---|---|
| $3^1P_1$ | $2^1D_2$ | 105.9 | 5.482 | 1.44 | 1.6 | 4.72 | 4.21 |
| | $1^1D_2$ | 424.7 | 0.208 | 0.0585 | 0.081 | 0.35 | 0.17 |
| | $3^1S_0$ | 205.8 | 18.156 | 9.94 | 8.9 | 12.27 | 14.1 |
| | $2^1S_0$ | 567.7 | 7.175 | 4.60 | 8.2 | 6.86 | 7.63 |
| | $1^1S_0$ | 1110.9 | 5.592 | 3.91 | 3.6 | 7.96 | 10.7 |
| $3^3P_0$ | $2^3D_1$ | 91.8 | 3.593 | 0.966 | 1.0 | 3.50 | 2.20 |
| | $1^3D_1$ | 411.7 | 0.163 | 0.189 | 0.20 | $3.59\times10^{-2}$ | 0.15 |
| | $3^3S_1$ | 163.9 | 9.527 | 7.15 | 6.9 | 8.50 | 7.95 |
| | $2^3S_1$ | 522.0 | 5.156 | 1.26 | 1.7 | 2.99 | 2.55 |
| | $1^3S_1$ | 1050.1 | 4.462 | 0.427 | 0.3 | 1.99 | 1.87 |
| $3^3P_1$ | $2^3D_1$ | 110.2 | 1.541 | 0.425 | 0.47 | 1.26 | 1.07 |
| | $2^3D_2$ | 102.5 | 3.738 | 0.950 | 1.1 | 3.34 | 0.94 |
| | $1^3D_1$ | 429.6 | 0.056 | 0.00418 | $7.0\times10^{-3}$ | $4.80\times10^{-2}$ | 0.010 |
| | $1^3D_2$ | 421.4 | 0.147 | 0.0615 | 0.080 | 0.11 | 0.015 |
| | $3^3S_1$ | 182.3 | 12.897 | 8.36 | 8.4 | 9.62 | 10.3 |
| | $2^3S_1$ | 539.8 | 5.876 | 2.49 | 3.1 | 4.58 | 5.63 |
| | $1^3S_1$ | 1066.9 | 4.754 | 1.62 | 1.3 | 4.17 | 6.41 |
| $3^3P_2$ | $2^3D_1$ | 124.3 | 0.088 | 0.0248 | 0.027 | 0.18 | 0.049 |
| | $2^3D_2$ | 116.6 | 1.090 | 0.295 | 0.32 | 0.79 | 0.78 |
| | $2^3D_3$ | 110.6 | 5.229 | 1.37 | 1.5 | 4.16 | 4.60 |
| | $1^3D_1$ | 443.2 | 0.003 | $1.15\times10^{-4}$ | | $3.38\times10^{-3}$ | 0.047 |
| | $1^3D_2$ | 435.1 | 0.037 | $3.11\times10^{-4}$ | | $4.41\times10^{-2}$ | 0.068 |
| | $1^3D_3$ | 429.0 | 0.188 | 0.0288 | 0.046 | 0.21 | 0.12 |
| | $3^3S_1$ | 196.2 | 15.889 | 9.30 | 9.3 | 10.38 | 10.8 |
| | $2^3S_1$ | 553.2 | 6.478 | 3.66 | 4.5 | 5.62 | 6.72 |
| | $1^3S_1$ | 1079.7 | 4.986 | 3.17 | 2.8 | 5.65 | 8.17 |

values are consistent with the experimental results of $\Delta m(1S) = 62.3 \pm 3.2$ MeV and $\Delta m(2S) = 24.3 \pm 3.5^{+2.8}_{-1.9}$ MeV [66]. The $\Upsilon(10355)$ is well established as the $\Upsilon(3S)$ in the literature. In our model, its mass is evaluated to be 10394.2 MeV. Our model predicts the mass difference $m[\Upsilon(3S)] - m[\Upsilon(2S)] = 370.4$ MeV compared with the experimental value of $331.50 \pm 0.02 \pm 0.13$ MeV [66]. The $\Upsilon(10580)$ is traditionally identified as the $\Upsilon(4S)$ state [45, 56, 58]. Our model calculates the mass of $\Upsilon(4S)$ as 10688.1 MeV, which is overestimated by 108.7 MeV compared with the experimental value. This overestimation is a consistent trend observed across all potential models [45, 58]. A $^3P_0$ model analysis suggests that $\Upsilon(10580)$ exhibits a significant meson-meson component owing its proximity with the $B^*\bar{B}^*$ channel [67]. Ref. [68] suggests that the state discovered by the CLEO Collaboration at $10684 \pm 10 \pm 8$ MeV, identified as a $b\bar{b}g$ hybrid [12], is a more suitable assignment for the $\Upsilon(4S)$ state, which is also corroborated by our model. The intermediate $B^*\bar{B}^*$ channel may induce observable $S-D$ mixing within the $\Upsilon(10580)$ state [67], and Ref. [60] predicts it to be the $\Upsilon(4S)-\Upsilon(3D)$ mixture state with a substantial mixing angle. The $\Upsilon(10860)$ and $\Upsilon(11020)$ states are associated with the $\Upsilon(5S)$ and $\Upsilon(6S)$ states, respectively [45]. Our model predicts their masses as 10938.9 MeV and 11160.9 MeV, which are overestimated by 53.7 MeV and 160.9 MeV, respectively. Theoretical models commonly show discrepancies in $\Upsilon(5S)$ and $\Upsilon(6S)$ mass predictions, either overestimating or underestimating their values. Various interpretations have been explored in the

**Table 18.** $E1$ transition widths (in keV) and photon energies (in MeV) of $D$ states.

| Initial State | Final State | Ours $E_\gamma$ | Ours $\Gamma_{E1}$ | [45] | [53] | [80] | [85] |
|---|---|---|---|---|---|---|---|
| $1^1D_2$ | $1^1P_1$ | 272.5 | 28.719 | 24.3 | 24.9 | 17.23 | 30.3 |
| $1^3D_1$ | $1^3P_0$ | 296.3 | 20.292 | 16.3 | 16.5 | 20.98 | 19.8 |
| | $1^3P_1$ | 270.0 | 11.661 | 9.51 | 9.7 | 12.29 | 13.3 |
| | $1^3P_2$ | 250.6 | 0.626 | 0.550 | 0.56 | 0.65 | 1.02 |
| $1^3D_2$ | $1^3P_1$ | 278.3 | 22.890 | 18.8 | 19.2 | 21.95 | 21.8 |
| | $1^3P_2$ | 258.9 | 6.194 | 5.49 | 5.6 | 6.23 | 7.23 |
| $1^3D_3$ | $1^3P_2$ | 265.1 | 26.518 | 23.9 | 24.3 | 24.74 | 32.1 |
| $2^1D_2$ | $1^1F_3$ | 108.9 | 2.640 | 1.35 | 1.8 | 2.20 | |
| | $2^1P_1$ | 202.6 | 21.128 | 16.8 | 16.5 | 11.66 | 15.6 |
| | $1^1P_1$ | 586.0 | 6.313 | 3.36 | 3.0 | 4.15 | 5.66 |
| $2^3D_1$ | $1^3F_2$ | 103.1 | 2.246 | 1.18 | 1.6 | 2.05 | |
| | $2^3P_0$ | 220.0 | 14.846 | 11.0 | 10.6 | 8.35 | 9.58 |
| | $2^3P_1$ | 199.1 | 8.380 | 6.71 | 6.5 | 4.84 | 6.74 |
| | $2^3P_2$ | 183.3 | 0.441 | 0.40 | 0.4 | 0.24 | 0.47 |
| | $1^3P_0$ | 609.7 | 4.118 | 2.99 | 2.9 | 3.52 | 5.56 |
| | $1^3P_1$ | 584.3 | 2.599 | 1.03 | 0.9 | 1.58 | 2.17 |
| | $1^3P_2$ | 565.5 | 0.152 | 0.030 | 0.02 | 0.061 | 0.44 |
| $2^3D_2$ | $1^3F_2$ | 110.8 | 0.308 | 0.164 | 0.21 | 0.24 | |
| | $1^3F_3$ | 107.6 | 2.265 | 1.21 | 1.5 | 1.93 | |
| | $2^3P_1$ | 206.7 | 16.793 | 13.1 | 12.7 | 9.10 | 11.4 |
| | $2^3P_2$ | 190.9 | 4.462 | 3.96 | 3.8 | 2.55 | 3.75 |
| | $1^3P_1$ | 591.6 | 4.921 | 2.81 | 2.6 | 3.43 | 4.00 |
| | $1^3P_2$ | 572.9 | 1.441 | 0.489 | 0.4 | 0.80 | 1.11 |
| $2^3D_3$ | $1^3F_2$ | 116.8 | 0.007 | 0.004 | 0.005 | 0.005 | |
| | $1^3F_3$ | 113.6 | 0.237 | 0.125 | 0.16 | 0.19 | |
| | $1^3F_4$ | 112.0 | 2.632 | 1.37 | 1.7 | | |
| | $2^3P_2$ | 196.9 | 19.488 | 16.8 | 16.4 | 10.70 | 17.0 |
| | $1^3P_2$ | 578.6 | 5.997 | 2.99 | 2.6 | 3.80 | 5.22 |

**Table 19.** $E1$ transition widths (in keV) and photon energies (in MeV) of $F$ and $G$ states.

| Initial | Final | Ours | Ours | [45] | [53] |
|---|---|---|---|---|---|
| State | State | $E_\gamma$ | $\Gamma_{E1}$ | | |
| $1^1F_3$ | $1^1D_2$ | 215.3 | 27.505 | 22.0 | 18.8 |
| $1^3F_2$ | $1^3D_1$ | 221.8 | 25.149 | 19.4 | 16.4 |
| | $1^3D_2$ | 213.4 | 4.171 | 3.26 | 2.7 |
| | $1^3D_3$ | 207.3 | 0.109 | 0.0852 | 0.070 |
| $1^3F_3$ | $1^3D_2$ | 216.6 | 24.873 | 19.7 | 16.7 |
| | $1^3D_3$ | 210.4 | 28.592 | 2.26 | 1.9 |
| $1^3F_4$ | $1^3D_3$ | 212.0 | 26.299 | 21.2 | 18.0 |
| $2^1F_3$ | $1^1G_4$ | 98.8 | 2.069 | 1.06 | 1.5 |
| | $2^1D_2$ | 174.5 | 22.171 | 17.4 | 19.9 |
| | $1^1D_2$ | 491.2 | 4.759 | 1.99 | 1.6 |
| $2^3F_2$ | $1^3G_3$ | 95.6 | 1.877 | 0.946 | 1.4 |
| | $2^3D_1$ | 180.0 | 20.371 | 15.1 | 17.5 |
| | $2^3D_2$ | 172.4 | 3.332 | 2.55 | 3.0 |
| | $2^3D_3$ | 166.4 | 0.086 | 0.0681 | 0.080 |
| | $1^3D_1$ | 497.2 | 4.214 | 1.95 | 1.6 |
| | $1^3D_2$ | 489.1 | 0.727 | 0.224 | 0.16 |
| | $1^3D_3$ | 483.1 | 0.019 | 0.00367 | 0.002 |
| $2^3F_3$ | $1^3G_3$ | 98.9 | 0.129 | 0.0664 | 0.10 |
| | $1^3G_4$ | 98.2 | 1.910 | 0.957 | 1.4 |
| | $2^3D_2$ | 175.6 | 20.091 | 15.4 | 17.9 |
| | $2^3D_3$ | 169.7 | 2.275 | 1.80 | 2.1 |
| | $1^3D_2$ | 492.3 | 4.273 | 1.83 | 1.4 |
| | $1^3D_3$ | 486.3 | 0.506 | 0.145 | 0.1 |
| $2^3F_4$ | $1^3G_3$ | 100.8 | 0.002 | $8.80\times10^{-4}$ | 0.001 |
| | $1^3G_4$ | 100.2 | 0.104 | 0.0535 | 0.080 |
| | $1^3G_5$ | 100.9 | 2.098 | 1.05 | 1.5 |
| | $2^3D_3$ | 171.6 | 21.148 | 16.9 | 19.6 |
| | $1^3D_3$ | 488.2 | 4.635 | 0.126 | 1.4 |
| $1^1G_4$ | $1^1F_3$ | 184.5 | 27.129 | 21.1 | 23.1 |
| $1^3G_3$ | $1^3F_2$ | 187.5 | 26.098 | 20.1 | 22.3 |
| | $1^3F_3$ | 184.3 | 2.174 | 1.67 | 1.8 |
| | $1^3F_4$ | 182.7 | 0.034 | 0.0256 | 0.028 |
| $1^3G_4$ | $1^3F_3$ | 184.9 | 25.586 | 20.1 | 22.0 |
| | $1^3F_4$ | 183.3 | 0.034 | 0.0256 | 0.028 |
| $1^3G_5$ | $1^3F_4$ | 182.6 | 26.304 | 21.1 | 23.1 |

**Table 20.** $M1$ transition widths (in keV) and photon energies (in MeV) of $S$ and $P$ states.

| Initial | Final | Ours | Ours | Exp[66] | [45] | [53] | [80] |
|---|---|---|---|---|---|---|---|
| State | State | $E_\gamma$ | $\Gamma_{M1}$ | | | | |
| $1^3S_1$ | $1^1S_0$ | 44.5 | 4.228 | | 9.52 | 10.0 | 9.34 |
| $2^1S_0$ | $1^3S_1$ | 532.8 | 4.848 | | 70.6 | 68.0 | 45.0 |
| $2^3S_1$ | $2^1S_0$ | 24.8 | 0.732 | | 0.582 | 0.590 | 0.580 |
| | $1^1S_0$ | 598.3 | 3.569 | 12.5±4.9 | 68.8 | 81.0 | 56.50 |
| $3^1S_0$ | $2^3S_1$ | 345.1 | 2.015 | | 11.1 | 9.10 | 9.20 |
| | $1^3S_1$ | 882.6 | 5.342 | | 73.2 | 74.0 | 5.10 |
| $3^3S_1$ | $3^1S_0$ | 19.3 | 0.343 | | 0.337 | 0.250 | 0.658 |
| | $2^1S_0$ | 387.7 | 1.488 | <13 | 11.8 | 19.0 | 11.0 |
| | $1^1S_0$ | 940.8 | 2.836 | 10±2 | 60.4 | 60.0 | 57.0 |
| $2^1P_1$ | $1^3P_0$ | 423.9 | 0.439 | | 5.56 | 0.320 | 36.40 |
| | $1^3P_1$ | 398.0 | 0.857 | | 1.30 | 1.10 | 1.280 |
| | $1^3P_2$ | 378.9 | 1.018 | | 0.992 | 2.20 | 0.007 |
| $2^3P_0$ | $1^1P_1$ | 365.2 | 0.475 | | 5.21 | 9.70 | 2.390 |
| $2^3P_1$ | $1^1P_1$ | 385.8 | 0.692 | | $3.90\times10^{-6}$ | 2.20 | 0.167 |
| $2^3P_2$ | $1^1P_1$ | 401.2 | 0.905 | | 3.86 | 0.240 | 1.780 |
| $3^1P_1$ | $2^3P_0$ | 332.5 | 0.366 | | 2.16 | | 1.710 |
| | $2^3P_1$ | 311.8 | 0.709 | | 0.559 | | 0.597 |
| | $2^3P_2$ | 269.2 | 0.831 | | 0.407 | | 0.007 |
| | $1^3P_0$ | 717.9 | 0.279 | | 5.10 | 0.980 | 3.770 |
| | $1^3P_1$ | 692.8 | 0.639 | | 1.01 | 0.930 | 1.230 |
| | $1^3P_2$ | 674.2 | 0.867 | | 1.48 | 0.140 | 0.051 |
| $3^3P_0$ | $2^1P_1$ | 283.4 | 0.372 | | 2.05 | | |
| | $1^1P_1$ | 664.2 | 0.465 | | 6.23 | | |
| $3^3P_1$ | $2^1P_1$ | 301.9 | 0.569 | | $9.80\times10^{-4}$ | | |
| | $1^1P_1$ | 681.7 | 0.566 | | 0.032 | | |
| $3^3P_2$ | $2^1P_1$ | 315.7 | 0.772 | | 1.74 | | |
| | $1^1P_1$ | 694.9 | 0.654 | | 3.53 | | |

literature, where $\Upsilon(10860)$ is considered as mixture of $\Upsilon(5S)-P$ wave hybrid [69], whereas lattice QCD studies remain inconclusive on whether $\Upsilon(11020)$ corresponds to $\Upsilon(S)$ or $\Upsilon(D)$ state [70]. The $^3P_0$ model of Ref. [67] indicated that $\Upsilon(10860)$ and $\Upsilon(11020)$ are structures are primarily $b\bar{b}$ states with small $S-D$ mixing components. This was analyzed in Ref. [71], proposing $\Upsilon(10860)$ as a $\Upsilon(5S)-\Upsilon(4D)$ mixture, and Ref. [60] suggests that both $\Upsilon(10860)$ and $\Upsilon(11020)$ are $\Upsilon(5S)-\Upsilon(4D)$ mixtures. More experimental data are required to understand their nature. We discuss the possibility of $S-D$ mixing in $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$ later in this section.

The $P$-wave masess are presented in Table 3 and our evaluated masses for $1P$ and $2P$ states correspond with the experimental values. The experimentally determined mass difference are $m[\chi_{b2}(1P)]-m[\chi_{b1}(1P)] = 19.10\pm0.25$ MeV, $m[\chi_{b1}(1P)]-m[\chi_{b0}(1P)] = 32.49\pm0.93$ MeV, $m[\chi_{b2}(2P)]-m[\chi_{b1}(2P)] = 13.10\pm0.24$ MeV, and $m[\chi_{b1}(2P)]-m[\chi_{b0}(2P)] = 23.8\pm1.7$ MeV [66]. Our model calculates these values as 19.9 MeV, 27 MeV, 16.1 MeV, and 21.3

**Table 21.** $S-D$ mixed states with the masses of mixed states $M_\phi$ and $M_{\phi'}$ (in MeV) and their di-leptonic decay widths $\Gamma_\phi$ and $\Gamma_{\phi'}$ (in keV).

| $S-D$ States | $M_S$ $M_D$ | $\theta$ | $\theta$ [60] | $M_\phi$ $M_{\phi'}$ | $M_{exp}$ [66] | $\Gamma_\phi$ $\Gamma_{\phi'}$ | $\Gamma_{exp}$ [66] |
|---|---|---|---|---|---|---|---|
| 3$S$ | 10394.2 | 19.28 | −9.0 | 10374.9 | 10355.1 | 0.440 | 0.443±0.008 |
| 2$D$ | 10467.3 | | | 10486.5 | | 0.036 | |
| 4$S$ | 10688.1 | −28.82 | −12.5 | 10656.4 | 10579.4 | 0.272 | 0.272±0.029 |
| 3$D$ | 10741.3 | | | 10772.9 | 10752.7 | 0.129 | |
| 5$S$ | 10938.9 | 44.55 | −38.0 | 10909.8 | 10885.2 | 0.291 | 0.31±0.07 |
| 4$D$ | 10981.0 | | | 11010.1 | 11000.0 | 0.142 | 0.13±0.03 |

MeV, respectively, exhibiting good agreement with the experimental data. Among $3P$ bottomonium states, only $\chi_{b1}(3P)$ and $\chi_{b2}(3P)$ have been identified. In our model, their masses are obtained as 10578.1 MeV and 10592.3 MeV, which are higher by 64.7 MeV and 68.9 MeV, respectively. This discrepancy can be due to proximity to the open-flavor $B\bar{B}^*$ threshold, potentially causing mixing effects [72, 73]. The experimentally measured mass difference $m[\chi_{b2}(3P)]-m[\chi_{b1}(3P)] = 10.60\pm0.64\pm0.17$ MeV [66] is calculated as 14.2 MeV in our model. Masses of $D$-wave states are presented in Table 4. The mass of $\Upsilon_1(1D)$ state in our model is evaluated to be 10147.9 MeV, deviating by 15.8 MeV from the experimental value [66]. The $\Upsilon_2(1D)$ and $\Upsilon_3(1D)$ states are estimated to have mass values of 10.13 GeV and 10.18 GeV, respectively [15], whereas our model calculates them as 10139.4 MeV and 10154.2 MeV, respectively. The recently observed $\Upsilon(10753)$ is generally associated with $\Upsilon_1(3D)$ [45, 73], although alternative interpretations suggest a tetraquark [74, 75], hybrid meson [3], etc. The mass of $\Upsilon_1(3D)$ state in our model is evaluated to be 10741.3 MeV, aligning with the experimental value [66]. A reanalysis of BABAR data estimated the mass of $\Upsilon_1(2D)$ to be $10495\pm5$ MeV with a $10.7\sigma$ significance [76], whereas our model calculates it as 10467.3 MeV, showing consistency with experimental result. The masses of $F$- and $G$-wave states are presented in Table 5. Our model shows consistency with other models for lower states, but deviations occur for higher excitations. The masses for different $J$ states in Table 5 are very close to each other, which could make differentiating these states experimentally more difficult.

Decay constants of pseudoscalar ($f_P$), vector ($f_V$), and tensor ($f_{\chi_0}, f_{\chi_1}$) states are presented in Tables 6 and 7, respectively. Our calculated values for the vector decay constants ($f_V$) correspond with experimental values and

**Table 22.** Our assignments for $b\bar{b}$ states with masses (in MeV) and di-leptonic decay widths (in keV).

| States | Assignment | $M_{\rm exp}$ [66] | $M_{\rm cal}$ | $\Gamma^{ee}_{\rm exp}$ [66] | $\Gamma^{ee}_{\rm cal}$ |
|---|---|---|---|---|---|
| $\eta_b(1S)$ | $\eta_b(1S)$ | 9398.7±2 | 9406.4 | | |
| $\Upsilon(1S)$ | $\Upsilon(1S)$ | 9460.4±0.09±0.04 | 9451.1 | 1.34±0.018 | 1.268 |
| $\chi_{b0}(1P)$ | $\chi_{b0}(1P)$ | 9859.44±0.42±0.31 | 9838.7 | | |
| $\chi_{b1}(1P)$ | $\chi_{b1}(1P)$ | 9892.78±0.26±0.31 | 9865.7 | | |
| $h_b(1P)$ | $h_b(1P)$ | 9899.3±0.8 | 9872.9 | | |
| $\chi_{b2}(1P)$ | $\chi_{b2}(1P)$ | 9912.21±0.26±0.31 | 9885.6 | | |
| $\eta_b(2S)$ | $\eta_b(2S)$ | $9999.0\pm3.5^{+2.8}_{-1.9}$ | 9998.9 | | |
| $\Upsilon(2S)$ | $\Upsilon(2S)$ | 10023.4±0.5 | 10023.3 | 0.612±0.011 | 0.666 |
| $\Upsilon_2(1D)$ | $\Upsilon_2(1D)$ | 10163.7±1.4 | 10147.9 | | |
| $\chi_{b0}(2P)$ | $\chi_{b0}(2P)$ | 10232.5±0.4±0.5 | 10244.9 | | |
| $\chi_{b1}(2P)$ | $\chi_{b1}(2P)$ | 10255.46±0.22±0.5 | 10266.2 | | |
| $h_b(2P)$ | $h_b(2P)$ | 10259.8±0.5±1.1 | 10271.7 | | |
| $\chi_{b2}(2P)$ | $\chi_{b2}(2P)$ | 10268.65±0.22±0.5 | 10282.3 | | |
| $\Upsilon(10355)$ | $\Upsilon(3S)-\Upsilon(2D)$ | 10355.1±0.5 | 10374.9 | 0.443±0.008 | 0.440 |
| $\chi_{b1}(3P)$ | $\chi_{b1}(3P)$ | 10513.42±0.41±0.53 | 10578.1 | | |
| $\chi_{b2}(3P)$ | $\chi_{b2}(3P)$ | 10524.02±0.57±0.53 | 10592.3 | | |
| $\Upsilon(10580)$ | $\Upsilon(4S)-\Upsilon(3D)$ | 10579.4±1.2 | 10656.4 | 0.272±0.029 | 0.272 |
| $\Upsilon(10753)$ | $\Upsilon(4S)-\Upsilon(3D)$ | $10752.7\pm5.9^{+0.7}_{-1.1}$ | 10772.9 | | 0.129 |
| $\Upsilon(10860)$ | $\Upsilon(5S)-\Upsilon(4D)$ | $10885.2^{+2.6}_{-1.6}$ | 10909.8 | 0.31±0.07 | 0.291 |
| $\Upsilon(11020)$ | $\Upsilon(5S)-\Upsilon(4D)$ | 11000±4 | 11010.1 | 0.13±0.03 | 0.142 |

exhibit more consistency over other theoretical models. The di-leptonic decay widths $\Gamma(l^+l^-)$ of $\Upsilon(nS)$ and $\Upsilon(nD)$ states, without ($\Gamma$) and with the correction factor ($\Gamma_{cf}$) are presented in Table 8. The di-leptonic decay widths of $\Upsilon(nS)$ states evaluated without the correction term are more in agreement with the experimental values, whereas the correction factor significantly suppresses them. The di-leptonic decay widths of $\Upsilon(nD)$ are smaller than $\Upsilon(nS)$ by a factor of 1000, serving as a key distinguishing feature in most models [45, 56]. The di-leptonic decay width difference between $\Upsilon(nS)$ and $\Upsilon(nD)$ states is used as a justification for assigning $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$ states to $\Upsilon(4S)$, $\Upsilon(5S)$, and $\Upsilon(6S)$, respectively, in potential models. Because $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$ exhibit $S$ state characteristics rather than being purely $D$ state, their widths are often overestimated, suggesting a potential for $S-D$ mixing [60]. For $n\geq 3$, the probability of $S-D$ mixing increases, and even a small mixing angle can increase the di-leptonic decay widths of $\Upsilon(nD)$ by an order of 2 [79]. To study $S-D$ mixing in our model, we use the di-leptonic decay widths without the correction factor to obtain the mixing angle. The di-photonic decay widths $\Gamma(\gamma\gamma)$ of bottomonium states without ($\Gamma$) and with ($\Gamma_{cf}$) are listed in Table 9. Our results are comparable to those of Ref. [56, 77] in magnitude but are lower than those of Ref. [45, 53]. The di-photonic decay width of $\eta_b(1S)$ is not observed experimentally and we predict it to be 0.344 keV. The tri-photonic decay widths $\Gamma(\gamma\gamma\gamma)$ without ($\Gamma$) and with ($\Gamma_{cf}$) are listed in Table 10. The values of tri-photonic decay widths vary significantly among models, highlighting the need for experimental validation. The di-gluonic decay widths $\Gamma(gg)$ without ($\Gamma$) and with ($\Gamma_{cf}$) are calculated in Table 11 and Table 12. Our di-gluonic decay widths of $S$, $P$, and $D$ states are comparable to those in Ref. [56] but are 2-4 times smaller than those in Ref. [45, 53, 58, 80]. For lower-lying $\eta_b(nS)$ states, the di-gluonic decay widths constitute approximately $\sim 100\%$ of their total width owing to the suppression of OZI-allowed two-body strong decays [45]. Our evaluated di-gluonic width for $\eta_b(1S)$ is 5.763, close to the lower limit of the total width estimate of $10.0^{+5}_{-4}$ MeV [66]. The tri-gluonic decay widths $\Gamma(ggg)$ without ($\Gamma$) and with ($\Gamma_{cf}$) are presented in Table 13. The tri-gluonic decay width for $\Upsilon(1S)$ is lower by 13.95 MeV, whereas those for $\Upsilon(2S)$ and $\Upsilon(3S)$ agree well with experimental results. For $P$ and $D$ states, our predicted widths are lower than those of other models. The photo-gluonic decay widths $\Gamma(\gamma gg)$ and quark-gluonic decay width $\Gamma(q\bar{q}g)$ without ($\Gamma$) and with ($\Gamma_{cf}$) are evaluated in Table 14. The photo-gluonic decay widths of $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ agree with the experimental data. The multi-gluon or hybrid $q\bar{q}g$ decays are dominant channels for the $\chi_{b1}(1P)$ state [45]. Our predictions for quark-gluonic decay width for $\chi_{b1}(1P)$ are observed to be lower than those of other models.

The $S$-wave $E1$ transitions widths are calculated in Table 15. The transition widths $\Gamma(2S\to\gamma\chi_b(1P))$ in our model align well with experimental data. The transition widths for $\Gamma(\Upsilon(3S)\to\gamma\chi_b(P))$ presents a complex scenario owing to discrepancies in $\Gamma(\Upsilon(3S)\to\gamma\chi_b(2P))$ and $\Gamma(\Upsilon(3S)\to\gamma\chi_b(1P))$ predictions across models [45, 80]. Our model estimates $\Gamma(\Upsilon(3S)\to\gamma\chi_b(2P))$ slightly higher than the experimental values, whereas $\Gamma(\Upsilon(3S)\to\gamma\chi_b(1P))$ are highly suppressed, a typical feature of $E1$ transitions among states separated by two radial nodes, making them susceptible to relativistic corrections [81, 82]. This suppression is evident in our evaluation, where $\Gamma(\Upsilon(3S)\to\gamma\chi_{b0}(1P))=0.057$ keV and $\Gamma(\Upsilon(3S)\to\gamma\chi_{b2}(1P))=0.139$ align with experimental results, whereas $\Gamma(\Upsilon(3S)\to\gamma\chi_{b1}(1P))=0.115$ keV exceeds the experimental value. This atypical hierarchy of $\Gamma(\Upsilon(3S)\to\gamma\chi_{b2}(1P))>\Gamma(\Upsilon(3S)\to\gamma\chi_{b0}(1P))>\Gamma(\Upsilon(3S)\to\gamma\chi_{b1}(1P))$ mentioned in Ref. [83] is also observed in our model. This is attributed to $\chi_{b1}(1P)$ mixing with $\chi_b(2P)$ and $\chi_b(3P)$, further suppressing $\Gamma(\Upsilon(3S)\to\gamma\chi_{b1}(1P))$. In Ref. [60], the $S-D$ mixing in $\Upsilon(3S)$ is considered to explain the $E1$ transitions widths of $\Gamma(\Upsilon(3S)\to\gamma\chi_b(2P))$, which enable them to reproduce the experimental widths. This explanation may also be extended for analysis of $\Gamma(\Upsilon(3S)\to\gamma\chi_b(1P))$ transition widths. We also evaluate $\Gamma(4S\to\gamma P)$. The $P$ wave $E1$ transitions widths are presented in Table 16 and 17. The transition width $\Gamma(1P\to\gamma S)$ in our model agrees with the experimental and theoretical results. Using the measured branching ratios $B[\chi_{b0}(1P)\to\gamma\Upsilon(1S)]=1.94\pm0.27\%$, $B[\chi_{b1}(1P)\to\gamma\Upsilon(1S)]=35.2\pm2.0\%$, and $B[\chi_{b2}(1P)\to\gamma\Upsilon(1S)]=18.0\pm1.0\%$ [66], we calculate the total decay width as 1.19 MeV for $\chi_{b0}(1P)$, 79.0 keV for $\chi_{b1}(1P)$, and 177.0 keV for $\chi_{b2}(1P)$. Our total width for $\chi_{b0}(1P)$ is consistent with the $1.3\pm0.9$ MeV and $\Gamma_{\text{total}}<2.4$ MeV condition predicted by the Belle Collaboration [84]. The $h_b(1P)$ has the primary transition $h_b(1P)\to\gamma\eta_b(1S)$ with a measured branching ratio of $52^{+6}_{-5}\%$. Using this, we estimate the total decay width of $h_b(1P)$ as 74.0 keV, consistent with Ref. [85]. The evaluated transition width $h_b(2P)\to\gamma\eta_b(2S)$ is lower than the experimental value, a trend observed in most of the potential models. Using measured branching ratios $B[h_b(2P)\to\gamma\eta_b(2S)]=48\pm13\%$ and $B[h_b(2P)\to\gamma\eta_b(1S)]=22\pm5\%$ [66], we estimate the total decay widths of $h_b(2P)$ as 46.0 keV and 50.0 keV, respectively, with an average of 48.0 keV, which is smaller than the estimate in Ref. [85]. The transition width $\Gamma(\chi_{b0}(1P)\to\gamma\Upsilon(2S))$ is overestimated in most models. From the measured branching ratios $B[\chi_{b0}(2P)\to\gamma\Upsilon(2S)]=1.38\pm0.30\%$ and $B[\chi_{b0}(2P)\to\gamma\Upsilon(1S)]=(3.8\pm1.7)\times10^{-3}$ [66], we determine the total decay widths of $\chi_{b0}(2P)$ as 0.88 MeV and 2.20 MeV, respectively. While these values vary significantly, the latter aligns with the ($\sim 2.5$ MeV) prediction of Ref. [53]. Our model predicts $\Gamma(\chi_{b1}(2P)\to\gamma\Upsilon(S))$ and $\Gamma(\chi_{b2}(2P)\to\gamma\Upsilon(S))$ in excellent agreement with experimental results. Using measured branching ratios $B[\chi_{b1}(2P) \to \gamma\Upsilon(2S)] = 18.1 \pm 1.9\%$ and $B[\chi_{b1}(2P) \to \gamma\Upsilon(1S)] = 9.9 \pm 1.0\%$ [66], we determine the total decay widths of $\chi_{b1}(2P)$ as 87.0 keV and 91.0 keV, respectively, with an average of 89.0 keV, which corresponds with the CLEO Collaboration value of $96 \pm 16$ keV [86]. Using the branching ratio $B[\chi_{b2}(2P) \to \gamma\Upsilon(1S)] = 6.6 \pm 0.8\%$ [66], we calculate the decay width as 146 keV for $\chi_{b2}(2P)$, which agrees with $138 \pm 19$ keV obtained by the CLEO Collaboration [86]. No experimental data exist for $\Gamma(3P \to \gamma S)$ and $\Gamma(3P \to \gamma D)$ transitions, although detections of $\Gamma(\chi_{b1}(3P) \to \gamma\Upsilon(1S))$, $\Gamma(\chi_{b1}(3P) \to \gamma\Upsilon(2S))$, $\Gamma(\chi_{b1}(3P) \to \gamma\Upsilon(3S))$, and $\Gamma(\chi_{b2}(3P) \to \gamma\Upsilon(3S))$ have been reported. We estimate these transition widths as 4.754 keV, 5.876 keV, 12.895 keV, and 15.899 keV, respectively. $E1$ transitions widths for $D$-wave states are presented in Table 18. The transition $\Gamma(\Upsilon_2(1D) \to \gamma\chi_b(1P))$ has been observed [66] and our model estimates $\Gamma(\Upsilon_2(1D) \to \gamma\chi_{b2}(1P)) = 6.194$ keV and $\Gamma(\Upsilon_2(1D) \to \gamma\chi_{b1}(1P)) = 22.890$ keV are consistent with other theoretical models. Ref. [45, 53, 80] suggest the total decay widths of $\eta_b(1D)$ and $\Upsilon_3(1D)$ are equivalent to their transition widths $\Gamma(\eta_b(1D) \to \gamma h_b(1P))$ and $\Gamma(\Upsilon_3(1D) \to \gamma h_b(1P))$, respectively. We estimate these transition widths to be 28.719 keV and 26.518 keV, respectively, agreeing with other models. The $\Upsilon_1(1D)$ state is predicted to be detected in $\Gamma(\Upsilon_1(1D) \to \gamma\chi_{b0}(1P))$ and $\Gamma(\Upsilon_1(1D) \to \gamma\chi_{b1}(1P))$ owing to their large branching ratios [45, 53, 80]. Our model estimates these transition widths as 20.292 keV and 11.661 keV, respectively. The large branching ratio of $\Gamma(\eta_b(2D) \to \gamma h_b(2P))$ suggests that the unobserved $\eta_b(2D)$ state can be detected [45, 53]. Our model estimates $\Gamma(\eta_b(2D) \to \gamma h_b(2P))$ to be 21.128 keV. The transition widths of $2D$ states in our model correspond with other theoretical predictions. $E1$ transitions widths for $F$ and $G$ wave states are listed in Table 19, which are slightly higher than those in other potential models. The $M1$ transition widths are presented in Table 20. Our $M1$ transition widths have noticeable differences from Refs. [45, 53, 56]. While our $\Gamma(\Upsilon(nS) \to \gamma\eta_b(nS))$ estimates are lower than other models, they agree more closely with experimental results. Because decay widths are highly dependent on the wavefunction, estimates vary significantly across models.

Table 21 presents the masses and leptonic decay widths of $S-D$ mixed states, which are assigned to experimentally observed states. The $\Upsilon(10355)$ state is considered as the $3S-2D$ mixed state with a small mixing component, having a mass of 10374.9 MeV and leptonic width of 0.440 keV. The $\Upsilon(10580)$ and $\Upsilon(10753)$ are considered to be the $4S-3D$ mixed state with a significant mixing component, with a mass of 10656.4 MeV and leptonic width of 0.272 keV for $\Upsilon(10580)$ and mass of 10772.9 MeV and leptonic width of 0.129 keV for $\Upsilon(10753)$. The $\Upsilon(10860)$ and $\Upsilon(11020)$ are assigned to be $5S-4D$ mixed states, which is also supported by Ref. [60]. The mass and leptonic decay width values of the $\Upsilon(10860)$ and $\Upsilon(11020)$ are consistent with the experimental results. When $S-D$ mixing is considered, our final assignments are presented in Table 22.

## VI. CONCLUSION

In this study, we explore the bottomonium system using a screened potential model within a relativistic framework to compute the mass spectrum of $S$, $P$, $D$, $F$, and $G$ waves, decay widths, and $E1$ and $M1$ transition widths, along with mass and leptonic decay widths of $S-D$ mixed states. This study emphasizes the relevance of relativistic dynamics, screening, and state mixing, offering a framework that bridges gaps between theory and experiment. The computed mass values exhibit strong agreement with experimental data, particularly for lower states, whereas our predictions for higher excited states demonstrate notable improvements compared with previous potential models. A recurring challenge in bottomonium spectroscopy has been reconciling theoretical predictions with experimental measurements, particularly for the masses and leptonic decay widths of higher states such as $\Upsilon(10355)$, $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$. Our study addresses this by incorporating $S-D$ mixing, yielding results that align closely with experimental values and providing a more refined interpretation of these states and emphasizing the need for beyond-static-potential effects in quarkonium spectroscopy. Our calculations of decay constants, particularly for vector states, show improvements over prior models along with annihilation decay widths. Beyond mass spectra, our evaluation of $E1$ and $M1$ transition widths offers valuable insights into radiative decays, supporting experimental searches for unobserved bottomonium states. We also utilize $E1$ transition widths to estimate the total decay widths of higher bottomonium states, achieving reasonable agreement with experimental values and reinforcing the validity of our model. Additionally, the calculated transition widths for higher excited states serve as references. This research paves the way for future investigations, particularly in the exploration of higher excited states and the effects of $S-D$ mixing.

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