Complete reduction of integrals in two-loop five-light-parton scattering amplitudes*

Xin Guan[1;1)] Xiao Liu[1;2)] Yan-Qing Ma(马滟青)[1,2,3;3)]

[1]School of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University, Beijing 100871, China
[2]Center for High Energy Physics, Peking University, Beijing 100871, China
[3]Collaborative Innovation Center of Quantum Matter, Beijing 100871, China

Abstract: We reduce all the most complicated Feynman integrals in two-loop five-light-parton scattering amplitudes to basic master integrals, while other integrals can be reduced even more easily. Our results are expressed as systems of linear relations in block-triangular form, which are very efficient for numerical calculation. Our results are crucial for complete next-to-next-to-leading order QCD calculation for three jets, photons, or hadrons production at hadron colliders. In order to find out the block-triangular relations, we develop a new method which is efficient and general. The method may provide a practical solution for the bottleneck problem of reducing multiloop multiscale integrals.

Keywords: Feynman integrals, reduction, five-light-parton scattering

DOI:

1 Introduction

Due to the good performance of the Large Hadron Collider (LHC), we now enter the era of precision high energy physics. Some of the most important observables are three light particles or jets production cross sections [1-3], which can both test the strong interaction at high energy and determine the QCD coupling constant. On the theoretical side, predictions with compatible precision are needed, which demands the perturbative QCD calculation up to next-to-next-to-leading order (NNLO). Although great progresses have been made in the past few years [4-24], a complete NNLO result is still unavailable. One of the main obstacles right now is the calculation of two-loop amplitudes.

To evaluate a two-loop five-light-parton scattering amplitude, one usually first generates integrand, then reduce all Feynman integrals to linear combinations of relatively simpler master integrals (MIs), and finally calculates these MIs. Because integrands can be obtained either by unitarity method [4-9] or by traditional Feynman diagram method and MIs have been calculated analytically [20-24], the bottleneck is the reduction of Feynman integrals. For example, non-planar contribution of two-loop three photons production at the LHC cannot be calculated for lack of reduction for nonplanar integrals [19].

Reduction is usually achieved by integration-by-parts (IBP) identities combined with Laporta's algorithm [25-35]. Even though many new ideas have been proposed to improve the IBP reduction [36-49] in recent years, the problem of reducing multiloop multiscale integrals has not been fully resolved yet. The difficulty is twofold. On the one hand, due to the number of scales, explicit solution of IBP system is usually too huge in size to be used for numerical calculation, besides it is very hard to obtain [47-51]. On the other hand, although solving IBP system numerically in a single run is tolerable, one usually needs to solve it for a huge number of times for the purpose of either phase space integration or fitting analytical expressions, which is very time-consuming and resource-consuming. For example, to reconstruct the fully analytical two-loop five-gluon all-plus helicity amplitude [17], one needs to run numerical IBP for about half a mil-

Received 7 May 2020

* The work is supported in part by the National Natural Science Foundation of China (Grants No. 11875071, No. 11975029) and the High-performance Computing Platform of Peking University

1) E-mail: guanxin0507@pku.edu.cn
2) E-mail: xiao6@pku.edu.cn
3) E-mail: yqma@pku.edu.cn

lion times[1)]. If one uses the same method to reconstruct analytical one-minus or maximal-helicity-violation amplitude, much more times of numerical IBP running may be needed, which is hard to achieve.

We note that reduction is effectively obtained if a system of block-triangular relations is found, which has small expression size and can be solved numerically with very high efficiency. Using our proposed series representation of Feynman integrals as input [52, 53], in Ref. [52] we constructed an algorithm to search for block-triangular relations and obtained some preliminary results. Although our method developed in Ref. [52] is good enough to reduce integrals with integrand having only denominators, we find it is very time-consuming for physical problems where integrands with numerators present.

In this paper, by further developing the method in Ref. [52], we propose a two-step search strategy along with a reduction scheme that is suitable for physical problems. Based on this, we successfully find out block-triangular relations to reduce integrals in two-loop five-light-parton scattering amplitudes. As expected, the relations are only 148MB in size, and can be numerically solved hundreds of times faster than other methods. Our work is an important step towards complete NNLO QCD calculation for three jets, photons, or hadrons production at the LHC. As our method is efficient and general, it can be straightforwardly applied to any other process, and thus provides a practical solution for the bottleneck problem of Feynman integrals reduction.

2 Feynman integrals in two-loop five-light-parton scattering amplitudes

To obtain the badly needed reduction of Feynman integrals in two-loop five-light-parton scattering amplitudes, we only need to consider integrals originated from the four topologies shown in Fig. 1. All other Feynman integrals are one-loop-like which can be dealt with much more easily.

Let us take the most complicated one, topology (a) in Fig. 1, as an example to explain what kind of Feynman integrals do we need to reduce. There are five external momenta $p_1,\cdots,p_5$ flowing into the diagram, satisfying on-shell conditions $p_i^2=0$ $(i=1,\cdots,5)$ and momentum conservation $\sum_{i=1}^5 p_i=0$. As a result, this problem contains five independent mass scales, which can be chosen as $\vec{s}=\{s_1,s_2,s_3,s_4,s_5\}$ with $s_i\equiv 2p_i\cdot p_{i+1}$ and $p_6\equiv p_1$. With two loop momenta ℓ_1 and ℓ_2, a complete set of Lorentz scalars can be chosen as

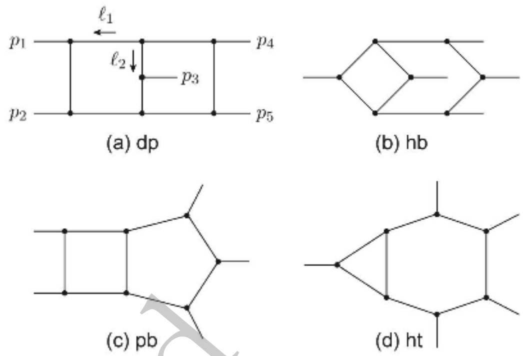

Fig. 1. All 8-propagator families: (a) double-pentagon; (b) hexa-box; (c) penta-box; (d) hexa-triangle.

$$\begin{aligned}
&D_1=\ell_1^2, D_2=(\ell_1+p_1)^2, D_3=(\ell_1+p_1+p_2)^2,\\
&D_4=\ell_2^2, D_5=(\ell_2+p_3)^2, D_6=(\ell_1+\ell_2)^2,\\
&D_7=(\ell_1+\ell_2-p_4)^2, D_8=(\ell_1+\ell_2-p_4-p_5)^2,\\
&D_9=(\ell_2+p_1)^2, D_{10}=(\ell_2+p_2)^2, D_{11}=(\ell_2+p_4)^2, \quad (1)
\end{aligned}$$

where the first eight are inverse propagators and the last three are introduced to make the set complete. Then the *family* of integrals defined by topology (a) can be expressed as

$$I_{\vec{\nu}}(\epsilon,\vec{s})=\int\frac{\mathrm{d}^{4-2\epsilon}\ell_1\,\mathrm{d}^{4-2\epsilon}\ell_2}{(\mathrm{i}\pi^{2-\epsilon})^2}\,\frac{D_9^{-\nu_9}D_{10}^{-\nu_{10}}D_{11}^{-\nu_{11}}}{D_1^{\nu_1}\ldots D_8^{\nu_8}}, \quad (2)$$

where the indexes $\nu_1,\cdots,\nu_8$ are integers, ν_9, ν_{10} and ν_{11} are nonpositive integers. Two integrals in this family are called in the same *sector* if positions of their positive indexes are the same. The *degree* of an integral is defined by the opposite value of the summation of all its negative indexes. Finally, we call a degree-m integral is $\frac{m}{n}$-*type* if it has n positive indexes and all these positive indexes are 1. For example, $I_{\{1,1,1,1,1,1,1,1,-4,0,-1\}}$ is a degree-5 integral in the top sector, and it is $\frac{5}{8}$-type.

For later convenience, we define operators $\hat{m}^\pm$ (for non-negative integer m), which generate a set of integrals in the same sector or its subsectors when acting on an integral. For any integral $I_{\vec{\nu}}$, $\hat{0}^\pm I_{\vec{\nu}}=I_{\vec{\nu}}$, $\widehat{m+1}^\pm I_{\vec{\nu}}=\hat{m}^\pm\hat{1}^\pm I_{\vec{\nu}}$, $\hat{1}^- I_{\vec{\nu}}$ generates a set of integrals with one index decreased by 1, and $\hat{1}^+ I_{\vec{\nu}}$ generates a set of integrals with one nonzero index increased by 1. For example, we have

$$\begin{aligned}
\hat{1}^+ I_{\{1,1,1,1,1,1,1,1,-4,0,-1\}}=\{&I_{\{2,1,1,1,1,1,1,1,-4,0,-1\}},\\
&I_{\{1,2,1,1,1,1,1,1,-4,0,-1\}}, I_{\{1,1,2,1,1,1,1,1,-4,0,-1\}},\\
&I_{\{1,1,1,2,1,1,1,1,-4,0,-1\}}, I_{\{1,1,1,1,2,1,1,1,-4,0,-1\}},\\
&I_{\{1,1,1,1,1,2,1,1,-4,0,-1\}}, I_{\{1,1,1,1,1,1,2,1,-4,0,-1\}},\\
&I_{\{1,1,1,1,1,1,1,2,-4,0,-1\}}, I_{\{1,1,1,1,1,1,1,1,-3,0,-1\}},\\
&I_{\{1,1,1,1,1,1,1,1,-4,0,0\}}\}, \quad (3)
\end{aligned}$$

1) We thank Y. Zhang for pointing out this. Here and in the rest of the paper, if not specified, "numerical" means rational numbers over a finite field of a big prime number.

and

$$\begin{aligned}\hat{1}^- I_{\{1,1,1,1,1,1,1,1,-4,0,-1\}} = \{&I_{\{0,1,1,1,1,1,1,1,-4,0,-1\}},\\ &I_{\{1,0,1,1,1,1,1,1,-4,0,-1\}}, I_{\{1,1,0,1,1,1,1,1,-4,0,-1\}},\\ &I_{\{1,1,1,0,1,1,1,1,-4,0,-1\}}, I_{\{1,1,1,1,0,1,1,1,-4,0,-1\}},\\ &I_{\{1,1,1,1,1,0,1,1,-4,0,-1\}}, I_{\{1,1,1,1,1,1,0,1,-4,0,-1\}},\\ &I_{\{1,1,1,1,1,1,1,0,-4,0,-1\}}, I_{\{1,1,1,1,1,1,1,1,-5,0,-1\}},\\ &I_{\{1,1,1,1,1,1,1,1,-4,-1,-1\}}, I_{\{1,1,1,1,1,1,1,1,-4,0,-2\}}\}. \qquad (4)\end{aligned}$$

We also define operators $\hat{m}^{\ominus}$, which can generate a set of integrals as union of integrals generated by $\{\hat{m}^-, \widehat{m-1}^-, \cdots, \hat{0}^-\}$ when acting on an integral.

As is well-known, the *most complicated*[1)] integrals in the amplitudes are those with the highest number of propagators, i.e., $\nu_i = 1$ $(i = 1, \cdots, 8)$, and the highest numerator degree, i.e., $-(\nu_9 + \nu_{10} + \nu_{11})$. By studying the two-loop five-gluon scattering amplitude diagram by diagram, we find the highest numerator degree is 5 for all integrals. Therefore we define an integral set

$$S_{(a)} = \hat{5}^{\ominus} I_{\{1,1,1,1,1,1,1,1,0,0,0\}}, \qquad (5)$$

which contains 3 914 nonzero integrals with all the most complicated integrals in five-gluon scattering amplitude being included. Because the five-gluon scattering amplitude is general enough, all the most complicated integrals (if not all integrals) belonging to topology (a) appearing in five-light-parton scattering amplitudes are included in the set $S_{(a)}$. In fact, for two-loop five-gluon all-plus helicity amplitude, integrals in topology (a) form a subset of $S_{(a)}$ [5]. Therefore, for the purpose of reducing integrals in physical amplitudes, the main job for topology (a) is to reduce integrals in set $S_{(a)}$.

For topologies (b), (c) and (d) in Fig. 1, we define sets of target integrals $S_{(b)}$, $S_{(c)}$ and $S_{(d)}$, similar to $S_{(a)}$.

3 Search for block-triangular relations

Before presenting our method to reduce two-loop five-light-parton integrals, let us first point out that for multiscale problems expressing general integrals in terms of MIs explicitly is not preferred even at the one-loop level. Instead, one usually sets up a system of block-triangular relations that can numerically relate all integrals to MIs (see [54] and references therein).

The advantage of a system of block-triangular relations over the explicit solution can be understood from singularities of integrals. If we express a complicated integral as a linear combination of simpler MIs, powers of Gram determinants will present in denominators of coefficients of these MIs, which is necessary because only in this way the linear combination of MIs can generate correct singularities of the target integral. Then, numerators of these coefficients have high mass dimension and thus have very long expressions. This difficulty can be nicely resolved by a system of block-triangular relations. Relations in each block can be very simple, but the solution of them can naturally generate Gram determinants in the denominator. Furthermore, a good choice of block may enable that its solution only involves one Gram determinant.

Because reduction at multiloop level is much more complicated than one-loop case, the above discussion implies that constructing a system of block-triangular relations may be the best way to reduce multiloop multiscale integrals. Unlike one-loop case, where block-triangular systems can be achieved easily by analytically solving IBP relations, block-triangular systems at multiloop level are in general hard to obtain.

In Ref. [52], based on our proposed series representation of Feynman integrals [52, 53] as input information, we constructed an algorithm to search for block-triangular relations to reduce multiloop multiscale integrals. However, we find the method is very time-consuming for physical problems, although it is efficient to reduce integrals with integrand having only denominators. To deal with physical problems like two-loop five-light-parton integrals, we propose a two-step search strategy here.

In the first step, we set up a system of relations that can numerically express all target integrals in terms of MIs. It is fine if the system is less efficient in numerical calculation, and thus the system is not required to be block-triangular. This system can be obtained either by using our series representation of Feynman integrals [52], or simply by using the well-known IBP system.

In the second step, we search for a system of block-triangular relations, which needs to be very efficient for numerical use. The algorithm is the same as that proposed in Ref. [52] except that, instead of using our series representation of Feynman integrals, we use the numerical solution obtained in the first step as input information.

More details about the search strategy can be found in appendix.

4 Reduction scheme and results

To apply the above proposed search strategy on physical problems, we still need to fix a reduction scheme, which means the choice of target integrals and other integrals allowed to appear in each block. In this paper, integrals in each block are defined by the operator $\hat{m}^{\ominus}$ acting on a proper integral. For example, to reduce the integrals in $S_{(a)}$, all integrals are allowed to appear in the first

1) The definition of complexity is a consequence of a convention to order integrals. In our convention, integrals are thought to be more complicated if they have more propagators, integrals in the same sector are more complicated if they have higher total denominator powers or if they have higher degree, and so on.

block, and the target integrals in this block are all the 21 most complicated integrals in the top-sector with degree 5. The first block enables us to express all the 21 most complicated integrals in terms of simpler integrals. Then in the second block, we choose the most complicated integrals among the rest of the integrals as target integrals, and use the operator $\hat{m}^{\ominus}$ acting on a proper integral to generate a set of integrals that covers all the target integrals. And so on. Eventually, we can express any integral in terms of simpler integrals.

Based on the above method, we successfully find out systems of block-triangular relations for integrals in the four topologies in Fig. 1. The file size of all these relations is acceptable, about 148 MB. To obtain them, it costs about 200 CPU-core hours to search for relations in the second step of the two-step search strategy, in addition to hundreds of CPU-core hours to generate input information by numerically solving the system obtained in the first step. Basic information to obtain these relations are collected in Table 1.

Table 1. Main information of the obtained reduction relations. $t_{\rm search}$ represents CPU time to search for these relations in the unit of CPU-core hour. $t_{\rm solve}$ represents the time spent to solve these relations numerically using one CPU.

top.	#int.	#MIs	$t_{\rm search}$ (h)	$t_{\rm solve}$ (s)	size(MB)
(a)	3914	108	112	0.17	66
(b)	3584	73	31	0.090	40
(c)	3458	61	56	0.075	31
(d)	2634	28	8	0.035	11

To be more intuitive, we show a matrix density plot for the block-triangular system of topology (a) in Fig. 2. This system contains 3 914 integrals and 108 MIs, which means we need 3 806 linear relations to reduce all target integrals. In this plot, each line represents a relation, each column corresponds to an integral, and black points represent nonzero elements in the matrix. Integrals are ordered from the most complicated one to the simplest one, with MIs being put at the end of each line. It can be found that the matrix is in an exact block-triangular form and the largest block contains only tens of relations.

Analytic expressions of all these relations are available from the website in [57]. Technical details of our reduction scheme can be found in appendix.

5 Checking our results and comparison with other methods

Our final reduction relations have been verified numerically by an independent code `FIRE6` [29] for randomly chosen phase space points, and the results agree

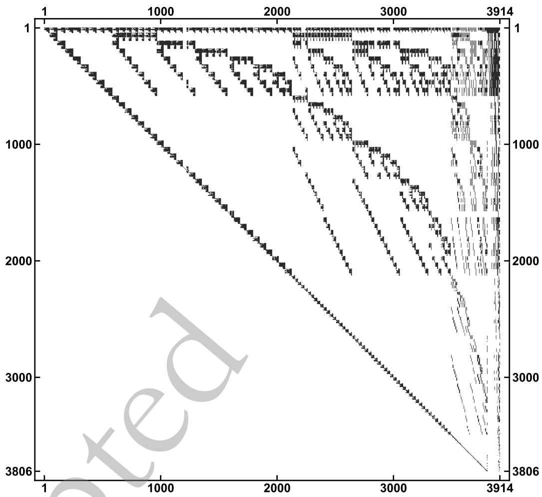

Fig. 2. Matrix density plot for relations of topology (a). Each row represents a relation and each column represents an integral. Black points represent nonzero elements.

with each other.

For each given numerical point ϵ and $\vec{s}$, solving our reduction relations of the four families totally costs 0.4 second using one CPU, as can be found in Table 1. The time spent can be divided into two parts: assignment (substituting numerical ϵ and $\vec{s}$ into the system), which is proportional to the file size; and solving the system, which depends on both the number of relations and how these relations are coupled with each other. Because our systems are in nice block-triangular form, the time spent in the latter part is shorter than the former one. Therefore, the efficiency of numerical calculation of our reduction relations can be simply estimated by the file size.

Comparing with explicit solutions, the file size of our reduction relations is much smaller. The file size of explicit solutions of eight-propagator integrals with degree up to 4 in topology (a), 26 integrals in total, is about 2GB [48]; that of eight-propagator integrals with degree up to 4 in topology (b), 32 integrals in total, is about 0.8GB [47]; and that of all integrals in topology (c) is in excess of 20GB for compressed format [49]. It can be expected that our relations should be hundreds of times smaller than the complete explicit solution in file size, which results in at least hundreds of times faster in numerical calculation, even if there is no problem for memory to store the huge expression of explicit solutions.

We note that the file size of trimmed IBP relations to reduce all integrals considered in this work is a few GB, which is also much larger than that of our reduction relations. The reason is that, although each IBP relation is simpler than ours, the IBP system involves hundreds of times more equations. Furthermore, the time spent for numerical IBP is dominated by the latter part because IBP relations are coupled in a complicated way. As a result, numerical IBP should be much more inefficient than our method. Through our test, numerical IBP via FiniteFlow [35] combined with LiteRed [34] costs about 2 minutes for each phase space point, which is slower than our method by hundreds of times.

The above comparison reveals the advantage of our method. Numerical evaluation of explicit solutions spends too much time on assignment; while numerical IBP spends too much time to solve linear equations. Our method has improved both parts, and therefore it is much more efficient. Similar to numerical evaluation over field of a prime number, our reduction relations should also be much more efficient for numerical evaluation with floating numbers, which enables phase space integration to obtain physical cross sections.

6 Summary and outlook

In this paper, we achieve the reduction of a set of integrals which covers all the most complicated integrals in two-loop five-light-parton scattering amplitudes. Our results are expressed as systems of linear relations in block-triangular form, which are very efficient for numerical calculation. The remaining integrals involved in amplitudes can be easily reduced using the same method upon demanding. Therefore, complete reduction of integrals in two-loop five-light-parton scattering amplitudes, which challenges all other methods, is available now. As MIs are already known [20-24], our results make the complete calculation of two-loop five-light-parton scattering amplitudes, and thus complete NNLO calculation of three light particles or jets production at the LHC on the horizon.

To obtain the block-triangular relations, we develop the method in Ref. [52] by proposing a two-step search strategy along with a reduction scheme. As our newly developed method is general and efficient, other more complicated problems, like two-loop integrals for $t\bar{t}+\text{jet}$, $t\bar{t}H$, or 4jets hadproduction, are also within reach. Our work opens the door for complete NNLO QCD calculation for three or more particles production at the LHC.

In the current application of our method, most CPU time is cost to numerically solve the system obtained in the first step. Although the time spent is tolerable for the current problem, improvement may be needed for more complicated applications. There are different possible ways. Within the method of [52], one needs to explore better integral sets. Another possible choice is to use trimmed IBP systems obtained by solving syzygy equations [44-48]. We leave this for future study.

We thank K.T. Chao, F. Feng, Q.J. Jin, Z. Li, X.H. Liu, H. Luo, C. Meng, J. Usovitsch and Y. Zhang for many useful communications and discussions.

Appendix: reduction method

A.1 Search strategy: Step one

We take integrals originated from topology (a) in the main text as an example to explain our technique details.

We want to set up a set of relations, with which we can express all integrals in $S_{(a)}$ in terms of MIs for any given phase space point (rational numbers for both $\vec{s}$ and ϵ), with coefficients calculated in the finite field of a 63-bit prime number. Although IBP method [25-35] can do this job, we would like to explain in the following that our method proposed in Ref. [52] may provide a better choice.

For each given integral $I_{\vec{\nu}}$, called a *seed*, there are 12 IBP relations among the integral set

$$G_{\vec{\nu}}^{\rm IBP}=\{\hat{1}^+,\hat{1}^-\hat{1}^+\}I_{\vec{\nu}}. \tag{A1}$$

Besides, there are additional 6 relations due to Lorentz invariance [55], which can be interpreted as linear combinations of IBP relations from other seeds [56].

The above IBP relations can also be found out easily using the method proposed in Ref. [52]. To this end, we introduce a parameter η for all integrals in $G_{\vec{\nu}}^{\rm IBP}$, and then search relations among them using input information from series representation [52, 53]. Up to $d_{\rm max}=1$, where $d_{\rm max}$ is half of the maximal value of mass dimension for coefficients of relations, we can find at least 12 relations; while up to $d_{\rm max}=2$ we can find at least 12+6 relations. Because these relations are analytical in η, we can take $\eta\to 0$ directly and recover the aforementioned $12+6$ IBP relations.

The advantage of our method in Ref. [52] is that it has freedom to search relations among any set of integrals. As the simplest generalization of $G_{\vec{\nu}}^{\rm IBP}$, we can define an integral set

$$G_{\vec{\nu}}=\{\hat{1}^+,\hat{1}^-\hat{1}^+,\hat{1}^-\}I_{\vec{\nu}}, \tag{A2}$$

and search relations among them. Up to $d_{\rm max}=2$, there are typically 2 more relations besides 12+6 IBP relations for each seed. With more relations in hand, it is possible to select better relations to achieve a more efficient reduction. For example, our relations from all $\frac{4}{8}$-type seeds can already reduce 15 out of all $\frac{5}{8}$-type integrals to integrals with lower degree (these relations are available at [57]). IBP relations from these seeds cannot do this job because $\frac{5}{8}$-type integrals do not show up.

One can certainly explore other integral sets for each seed to find even better reduction efficiency. We did not do that because efficiency of either the IBP set (6) or the generalized set (7) is sufficient for us to deal with the problem in this work.

With integral sets in hand, we generate a system of linear equations from all seeds belonging to $\frac{m}{n}$-type with $3\leqslant n\leqslant 8$ and $0\leqslant m\leqslant 5$, and use the package FiniteFlow [35] to trim the system by removing redundant relations and solve the trimmed system numerically, which expresses all integrals in $S_{(a)}$ as linear combinations of 108 MIs (after exploring symmetries between MIs).

A.2 Search strategy: Step two

In this step, we search linear relations to reduce the given target integrals in $G_1 \subseteq S_{(a)}$ to simpler integrals in $G_2 \subseteq S_{(a)}$ (the reducibility can be tested numerically easily). Combining the reduction scheme which will be described in the next section, a block-triangular system can be finally obtained.

We first describe how to search linear relations among the integral set $G := \{I_1, \ldots, I_N\} \subseteq S_{(a)}$ of the form

$$\sum_{i=1}^{N} Q_i(\epsilon, \vec{s}) I_i(\epsilon, \vec{s}) = 0, \tag{A3}$$

where $Q_i(\epsilon, \vec{s})$ can be decomposed as

$$Q_i(\epsilon, \vec{s}) = \sum_{\kappa=0}^{\epsilon_{\max}} \sum_{\vec{\lambda} \in \Omega_{d_i}} \tilde{Q}_i^{\kappa\lambda_1 \ldots \lambda_5} \epsilon^{\kappa} s_1^{\lambda_1} \cdots s_5^{\lambda_5}, \tag{A4}$$

where $\epsilon_{\max}$ is the maximal power of ϵ allowed to appear in the relation, $\Omega_{d_i} = \{\vec{\lambda} \in \mathbb{N}^r | \lambda_1 + \cdots + \lambda_r = d_i\}$, d_i is half of the mass dimension of Q_i which can be fixed by $d_{\max} \equiv \max\{d_1, \cdots, d_N\}$, and $\tilde{Q}_i^{\kappa\lambda_1 \ldots \lambda_5}$ are unknown rational numbers to be determined. It is crucial to point out that, for given $\epsilon_{\max}$ and $d_{\max}$, the number of unknowns are finite. Therefore, they can be determined by finite number of constraints. As will be explained in the following, these unknowns can be determined by the result obtained in the first step.

Based on the system of equations in the first step, for a given numerical point ϵ and $\vec{s}$ every integral in G can be represented as an 108-dimensional vector, with elements being the projection onto MIs,

$$I_i = \{C_{i,1}, \cdots, C_{i,108}\}, \quad i = 1, \cdots, N. \tag{A5}$$

By inserting these numerical vectors into Eq. (8), we obtain a vector equation, which results in at most 108 independent constraints over the unknowns. By repeating the above procedure for many times (at most several thousand in this work), sufficient number of constraints can be obtained to determine all the unknowns. As the above values are actually calculated in the finite field of a given prime number, we still need to repeat the procedure for several different prime numbers (at most 15 in this work) and use the Chinese remainder theorem to reconstruct the real results of the unknowns. Finally, linear relations with given $d_{\max}$ and $\epsilon_{\max}$ are obtained.

To reduce G_1 to G_2, we just set $G := G_1 \cup G_2$ and search relations among G with different values of $d_{\max}$ and $\epsilon_{\max}$. For the purpose of the current work, we find it is sufficient to fix $\epsilon_{\max} = 3$. In order to find out simple relations, we follow the algorithm proposed in Ref. [52] by starting the search procedure with $d_{\max} = 0$ and increasing $d_{\max}$ by 1 each time, until enough relations are obtained to reduce G_1 to G_2.

A.3 Reduction scheme

Reduction scheme determines which integrals should be involved in each block. We generate the integrals through previously defined operator $\hat{m}^{\ominus}$ acting on properly chosen integrals.

For example, in the first block for topology (a), we need to reduce the most complicated $\frac{5}{8}$-type integrals. To this end, we set $G := S_{(a)} = \hat{5}^{\ominus} I_{\{1,1,1,1,1,1,1,1,0,0,0\}}$ with G_1 chosen as all 21 $\frac{5}{8}$-type integrals. We indeed find out 21 independent relations, which can reduce all $\frac{5}{8}$-type integrals to simpler integrals. The most complicated relation corresponds to $d_{\max} = 7$, which means that the coefficients of $\frac{5}{8}$-type integrals are degree-2 polynomials in $\vec{s}$. We then reduce $\frac{4}{8}$-type integrals, which can be realized by setting $G = \hat{4}^{\ominus} I_{\{1,1,1,1,1,1,1,1,0,0,0\}}$ with G_1 chosen as all 15 $\frac{4}{8}$-type integrals. To reduce the rest of top-sector integrals, we set $G = \hat{3}^{\ominus} I_{\{1,1,1,1,1,1,1,1,0,0,0\}}$ with G_1 chosen as 11 top-sector integrals that are not MIs.

After reducing top-sector integrals, we still need to reduce integrals in subsectors. For example, for seven-propagator sector $I_{\{1,1,1,1,1,1,1,0,0,0,0\}}$, whose most complicated integrals in $S_{(a)}$ are of $\frac{4}{7}$-type, we set $G = \hat{4}^{\ominus} I_{\{1,1,1,1,1,1,1,0,0,0,0\}}$ with G_1 chosen as all 35 $\frac{4}{7}$-type integrals in this sector.

Based on the above scheme, we obtain 3 801 reduction relations. By introducing additional 5 symmetry relations among MIs, we have 3 806 relations in total that can express 3 914 integrals in $S_{(a)}$ as linear combinations of 108 MIs.

We note that there is a way to further reduce the block size that has not been applied in this work. For example, by setting $G = \hat{3}^{\ominus} I_{\{0,1,1,1,1,1,1,1,-1,0,0\}}$, we can generate a smaller size of block to reduce part of $\frac{4}{7}$-type integrals.

References

1 ATLAS, M. Aaboud *et al.*., Phys. Lett., **B781**(55): 76 (2018), arXiv:1712.07291
2 ATLAS, M. Aaboud *et al.*., Eur. Phys. J., **C77**: 872 (2017), arXiv:1707.02562
3 CMS, A. M. Sirunyan *et al.*., JHEP, **12**: 117 (2018), arXiv:1811.00588
4 S. Badger, H. Frellesvig, and Y. Zhang, JHEP, **12**: 045 (2013), arXiv:1310.1051
5 S. Badger, G. Mogull, A. Ochirov, and D. O'Connell, JHEP, **10**: 064 (2015), arXiv:1507.08797
6 S. Badger, C. Brønnum-Hansen, H. B. Hartanto, and T. Peraro, Phys. Rev. Lett., **120**: 092001 (2018), arXiv:1712.02229
7 S. Abreu, F. Febres Cordero, H. Ita, B. Page, and M. Zeng, Phys. Rev., **D97**: 116014 (2018), arXiv:1712.03946
8 S. Badger, C. Brønnum-Hansen, H. B. Hartanto, and T. Peraro, JHEP, **01**: 186 (2019), arXiv:1811.11699
9 S. Abreu, F. Febres Cordero, H. Ita, B. Page, and V. Sotnikov, JHEP, **11**: 116 (2018), arXiv:1809.09067
10 S. Abreu, L. J. Dixon, E. Herrmann, B. Page, and M. Zeng, Phys. Rev. Lett., **122**: 121603 (2019), arXiv:1812.08941
11 R. H. Boels, Q. Jin, and H. Luo, *Efficient integrand reduction for particles with spin*, [arXiv: 1802.06761] [InSPIRE].
12 S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, and B. Page, Phys. Rev. Lett., **122**: 082002 (2019), arXiv:1812.04586
13 D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang, and S. Zoia, Phys. Rev. Lett., **122**: 121602 (2019), arXiv:1812.11057
14 D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang, and S. Zoia, JHEP, **03**: 115 (2019), arXiv:1901.05932
15 S. Abreu, L. J. Dixon, E. Herrmann, B. Page, and M. Zeng,

JHEP, **03**: 123 (2019), arXiv:1901.08563
16 S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, B. Page, and V. Sotnikov, JHEP, **05**: 084 (2019), arXiv:1904.00945
17 S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J. M. Henn, T. Peraro, P. Wasser, Y. Zhang, and S. Zoia, Phys. Rev. Lett., **123**: 071601 (2019), arXiv:1905.03733
18 H. B. Hartanto, S. Badger, C. Brønnum-Hansen, and T. Peraro, JHEP, **09**: 119 (2019), arXiv:1906.11862
19 H. A. Chawdhry, M. L. Czakon, A. Mitov, and R. Poncelet, *NNLO QCD corrections to three-photon production at the LHC*, [arXiv: 1911.00479][InSPIRE].
20 T. Gehrmann, J. M. Henn, and N. A. Lo Presti, *Analytic form of the two-loop planar five-gluon all-plus-helicity amplitude in QCD, Phys. Rev. Lett.* **116** (2016) 062001[arXiv: 1511.05409][InSPIRE]. [Erratum: Phys. Rev. Lett.116, no.18, 189903(2016)].
21 C. G. Papadopoulos, D. Tommasini, and C. Wever, JHEP, **04**: 078 (2016), arXiv:1511.09404
22 T. Gehrmann, J. M. Henn, and N. A. Lo Presti, JHEP, **10**: 103 (2018), arXiv:1807.09812
23 D. Chicherin, T. Gehrmann, J. M. Henn, N. A. Lo Presti, V. Mitev, and P. Wasser, JHEP, **03**: 042 (2019), arXiv:1809.06240
24 D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang, and S. Zoia, Phys. Rev. Lett., **123**: 041603 (2019), arXiv:1812.11160
25 K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys., **B192**: 159-204 (1981)
26 S. Laporta, Int. J. Mod. Phys., **A15**: 5087-5159 (2000), arXiv:hep-ph/0102033
27 A. V. Smirnov, JHEP, **10**: 107 (2008), arXiv:0807.3243
28 A. V. Smirnov, Comput. Phys. Commun., **189**: 182-191 (2015), arXiv:1408.2372
29 A. V. Smirnov and F. S. Chuharev, *FIRE6: Feynman Integral REduction with Modular Arithmetic*, [arXiv: 1901.07808][InSPIRE].
30 P. Maierhoefer, J. Usovitsch, and P. Uwer, *Kira - A Feynman Integral Reduction Program*, [arXiv: 1705.05610][InSPIRE].
31 P. Maierh ofer and J. Usovitsch, *Kira 1.2 Release Notes*, [arXiv: 1812.01491][InSPIRE].
32 C. Studerus, Comput. Phys. Commun., **181**: 1293-1300 (2010), arXiv:0912.2546
33 A. von Manteuffel and C. Studerus, *Reduze 2 - Distributed Feynman Integral Reduction*, [arXiv: 1201.4330][InSPIRE].
34 R. N. Lee, *Presenting LiteRed: a tool for the Loop InTEgrals REDuction*, [arXiv: 1212.2685][InSPIRE].
35 T. Peraro, JHEP, **07**: 031 (2019), arXiv:1905.08019
36 A. von Manteuffel and R. M. Schabinger, Phys. Lett., **B744**: 101-104 (2015), arXiv:1406.4513
37 T. Peraro, JHEP, **12**: 030 (2016), arXiv:1608.01902
38 D. A. Kosower, Phys. Rev., **D98**: 025008 (2018), arXiv:1804.00131
39 Y. Wang, Z. Li, and N. Ul Basat, *Direct Reduction of Amplitude*, [arXiv: 1901.09390][InSPIRE].
40 P. Mastrolia and S. Mizera, JHEP, **02**: 139 (2019), arXiv:1810.03818
41 H. Frellesvig, F. Gasparotto, S. Laporta, M. K. Mandal, P. Mastrolia, L. Mattiazzi, and S. Mizera, JHEP, **05**: 153 (2019), arXiv:1901.11510
42 H. Frellesvig, F. Gasparotto, M. K. Mandal, P. Mastrolia, L. Mattiazzi, and S. Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*, [arXiv: 1907.02000][InSPIRE].
43 J. Klappert and F. Lange, Comput. Phys. Commun., **247**: 106951 (2020), arXiv:1904.00009
44 J. Gluza, K. Kajda, and D. A. Kosower, Phys. Rev., **D83**: 045012 (2011), arXiv:1009.0472
45 R. M. Schabinger, JHEP, **01**: 077 (2012), arXiv:1111.4220
46 K. J. Larsen and Y. Zhang, Phys. Rev., **D93**: 041701 (2016), arXiv:1511.01071
47 J. Böhm, A. Georgoudis, K. J. Larsen, H. Schönemann, and Y. Zhang, JHEP, **09**: 024 (2018), arXiv:1805.01873
48 D. Bendle, J. Boehm, W. Decker, A. Georgoudis, F.-J. Pfreundt, M. Rahn, P. Wasser, and Y. Zhang, *Integration-by-parts reductions of Feynman integrals using Singular and GPI-Space*, [arXiv: 1908.04301] [InSPIRE].
49 H. A. Chawdhry, M. A. Lim, and A. Mitov, Phys. Rev., **D99**: 076011 (2019), arXiv:1805.09182
50 S. Borowka, N. Greiner, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, U. Schubert, and T. Zirke, *Higgs Boson Pair Production in Gluon Fusion at Next-to-Leading Order with Full Top-Quark Mass Dependence, Phys. Rev. Lett.* **117** (2016) 012001 [arXiv: 1604.06447][InSPIRE]. [Erratum: Phys. Rev. Lett.117, no.7, 079901(2016)].
51 S. P. Jones, M. Kerner, and G. Luisoni, Phys. Rev. Lett., **120**: 162001 (2018), arXiv:1802.00349
52 X. Liu and Y.-Q. Ma, *Determine Arbitrary Feynman Integrals by Vacuum Integrals*, [arXiv: 1801.10523] [InSPIRE].
53 X. Liu, Y.-Q. Ma, and C.-Y. Wang, Phys. Lett., **B779**: 353-357 (2018), arXiv:1711.09572
54 Denner, Ansgar and Dittmaier, S., Nucl. Phys., **B734**: 62-115 (2006), arXiv:hep-ph/0509141
55 T. Gehrmann and E. Remiddi, Nucl. Phys., **B580**: 485-518 (2000), arXiv:hep-ph/9912329
56 R. N. Lee, JHEP, **07**: 031 (2008), arXiv:0804.3008
57 http://www.phy.pku.edu.cn/~yqma/work/Packages/Multiloop/reduction.html, .