

Thermodynamic extremality relations in the massive gravity*

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Abstract: A universal relation between the leading correction to the entropy and extremality was gotten in the work of Goon and Penco. In this paper, we extend this work to the massive gravity and investigate thermodynamic extremality relations in a topologically higher-dimensional black hole. A rescaled cosmological constant is added to the action of the massive gravity as a perturbative correction. This correction modifies the extremality bound of the black hole and leads to the shifts of the mass, entropy, etc. Regarding the cosmological constant as a variable related to pressure, we get the thermodynamic extremality relations between the mass and entropy, pressure, charge, parameters c_i by accurate calculations, respectively. Finally, these relations are verified by a triple product identity, which shows that the universal relation exists in black holes.

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I. INTRODUCTION

The string landscapes formed by effective quantum field theories are broad and complex. However, there are some theories that look self-consistent but not compatible with the string theory. Thus, the swampland program was put forward [1-4]. Its aim is to find the subset of the infinite space in effective field theories arisen at low energies from quantum gravity theories by specific constraints. These constraints were first proposed in [1]. As one of the constraints, the weak gravity conjecture (WGC) has attracted people's attention. It asserts that for the lightest charged particle along the direction of some basis vectors in charge space, the charge-to-mass ratio is larger than for extremal black holes [2]. This conjecture shows that the extremal black holes are allowed to decay.

A proof to the WGC is that it is mathematically equivalent to a certain property of a black hole entropy. In [5], the authors introduced the higher-derivative operators to the action to compute the shift of the entropy. Due to these operators, the extremality condition of the black hole is modified and the mass and entropy are shifted. They derived the relation between the ratio of charge-to-mass and the entropy shift, $q/m - 1 \propto \Delta S$, where $\Delta S > 0$. The charge-to-mass ratio asymptotes to unity with the increase of the mass. Thus, the large extremal black hole is unstable and decays to a smaller extremal black hole with

the charge-to-mass ratios greater than unity. This phenomenon satisfies the requirement of the WGC. Subsequently, the WGC behavior was found in the four-dimensional rotating dyonic black hole and other spacetimes [6, 7]. Other researches on the WGC are referred to [8-23] and references therein.

In the recent work [24], Goon and Penco derived a universal extremality relation by the perturbative corrections to the free energy of generic thermodynamic systems. This relation takes the form

$$\frac{\partial M_{ext}(\vec{Q}, \epsilon)}{\partial \epsilon} = \lim_{M \rightarrow M_{ext}(\vec{Q}, \epsilon)} -T \left(\frac{\partial S(M, \vec{Q}, \epsilon)}{\partial \epsilon} \right)_{M, \vec{Q}}, \quad (1)$$

where $M_{ext}(\vec{Q}, \epsilon)$ and $S(M, \vec{Q}, \epsilon)$ are the extremality mass and entropy, respectively. Both of them are ϵ -dependent and ϵ is a control parameter in front of the free energy. \vec{Q} are additional quantities in the thermodynamic systems other than the mass. The above relation can be interpreted as a comparison between states in the classical and corrected theories. Meanwhile, an approximation relation $\Delta M_{ext}(\vec{Q}) \approx -T_0(M, \vec{Q}) \Delta S(M, \vec{Q})|_{M \approx M_{ext}^0(\vec{Q})}$ was gotten, where $\Delta M_{ext}(\vec{Q})$ and $\Delta S(M, \vec{Q})$ are the leading order corrections to the extremal bound and to the entropy of a state at the fixed mass and \vec{Q} , respectively. M_{ext}^0 is the

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mass in the extremal case without corrections. The result shows that the mass of the perturbed extremal black hole is less than that of the unperturbed one with the same quantum numbers, if $\Delta S > 0$, which implies that the perturbation decreases the mass of the extremal black hole. Therefore, the WGC-like behavior exists in the extremal black hole. In particular, the Goon-Penco relation (1) was verified in the AdS-Reissner-Nordström black hole by rescaling the cosmological constant as a perturbative correction. The approximation relation was also checked by the higher-derivative operators introduced in the action.

To further explore the WGC behavior and the Goon-Penco relation, people studied the thermodynamic corrections in the specific spacetimes by introducing the higher-derivative operators or perturbative parameters [25, 26]. The Goon-Penco relation was confirmed and other extremality relations were gotten. In [25], Cremonini et al. computed the four-derivative corrections to thermodynamic quantities in the higher-dimensional AdS-Reissner-Nordström black hole and found the extremality relation between the mass and charge,

$$\lim_{T \rightarrow 0} \left(\frac{\partial M}{\partial \epsilon} \right)_{Q,T} = \lim_{T \rightarrow 0} -\Phi \left(\frac{\partial Q}{\partial \epsilon} \right)_{M,T}. \quad (2)$$

Extended this work to rotating anti-de Sitter spacetimes, Liu et al. derived the extremality relation between the mass and angular momentum in the BTZ and Kerr anti-de Sitter spacetimes [26],

$$\left(\frac{\partial M_{ext}}{\partial \epsilon} \right)_{J,I} = \lim_{M \rightarrow M_{ext}} -\Omega \left(\frac{\partial J}{\partial \epsilon} \right)_{M,S,I}. \quad (3)$$

The relations (2) and (3) are the extensions of the Goon-Penco relation (1). These relations will shed light in theories of quantum gravity.

In this paper, we extend the work of [24] to the massive gravity, and investigate the extremality relations between the mass and pressure, entropy, charge, parameters c_i of a charged topological black hole in the higher-dimensional spacetime, respectively. Einstein's general relativity (GR) is a low energy effective theory. The UV completeness requires that GR be modified to meet physical descriptions in the high energy region. The massive gravity is a straightforward modification to GR. We introduce a perturbative correction by adding a rescaled cosmological constant to the action of the massive gravity. This scenario is different from that in [24] where the cosmological constant was directly rescaled in the action and consistent with that in [26]. In our investigation, the cosmological constant is regarded as a variable related to pressure [27-31]. Its conjugate quantity is a thermodynamic volume. The black hole mass is naturally interpreted as an enthalpy. The first reason for this is that the cosmological constant, as a variable, can reconcile the inconsistency between the first law of thermodynamics of

black holes and the Smarr relation derived from the scaling method. The second reason is that physical constants, such as the gauge coupling constants, Newtonian constant or cosmological constant arisen as vacuum expectation values are not fixed and vary in the more fundamental theories [32].

The rest of this paper is organized as follows. In the next section, the solution of the higher-dimensional black hole in the massive gravity is given and its thermodynamic properties are discussed. In section 3, we introduce a perturbative correction into the action and derive the extremality relations between the mass and pressure, entropy, charge, parameters c_i , respectively. Section 4 is devoted to our discussion and conclusion.

II. THE BLACK HOLE SOLUTION IN THE MASSIVE GRAVITY

The action for an $(n+2)$ -dimensional massive gravity is [33]

$$S = \frac{1}{16\pi} \int dx^{n+2} \sqrt{-g} \left[R + \frac{n(n+1)}{l^2} - \frac{F^2}{4} + m^2 \sum_{i=1}^4 c_i u_i(g, f) \right], \quad (4)$$

where the terms including m^2 are the massive potential associate with graviton mass, f is a fixed symmetric tensor called as the reference metric, c_i are constants, and u_i are symmetric polynomials of the eigenvalues of the $(n+2) \times (n+2)$ matrix $\mathcal{K}_\nu^\mu = \sqrt{f^{\mu\alpha} g_{\alpha\nu}}$:

$$\begin{aligned} u_1 &= [\mathcal{K}], & u_2 &= [\mathcal{K}]^2 - [\mathcal{K}^2], \\ u_3 &= [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ u_4 &= [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]. \end{aligned} \quad (5)$$

The square root in \mathcal{K} denotes $(\sqrt{A})_\nu^\mu (\sqrt{A})_\lambda^\nu = A_\lambda^\mu$ and $[\mathcal{K}] = \mathcal{K}_\mu^\mu$.

The solution of the charged black hole with the spacetime metric and reference metric is given by [34]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij} dx^i dx^j, \quad (6)$$

$$f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_{ij}), \quad (7)$$

where

$$\begin{aligned} f(r) &= k + \frac{r^2}{l^2} - \frac{16\pi M}{n\Omega_n r^{n-1}} + \frac{(16\pi Q)^2}{2n(n-1)\Omega_n^2 r^{2(n-1)}} + \frac{c_0 c_1 m^2 r}{n} \\ &\quad + c_0^2 c_2 m^2 + \frac{(n-1)c_0^3 c_3 m^2}{r} + \frac{(n-1)(n-2)c_0^4 c_4 m^2}{r^2}, \end{aligned} \quad (8)$$

l^2 is related to the cosmological constant Λ as $l^2 = -\frac{n(n+1)}{2\Lambda}$. M and Q are the mass and charge of the black hole, respectively. Ω_n is the volume spanned by coordinates x^i and c_0 is a positive integral constant. $h_{ij}dx^i dx^j$ is the line element for an Einstein space with the constant curvature $n(n-1)k$. $k = 1, 0$ or -1 denote a spherical, Ricci flat or hyperbolic topology black hole horizons, respectively. The thermodynamics in the extended phase space of the massive gravity were studied in [35-41]. The event horizon r_+ is determined by $f(r) = 0$. A general formula for the Hawking temperature can be defined as $T = \frac{\kappa}{2\pi}$, where $\kappa = -\frac{1}{2} \lim_{r \rightarrow r_+} \sqrt{\frac{-g^{11}}{g^{00}}} \frac{\partial \ln(-g^{00})}{\partial r}$ is the surface gravity. For this black hole, the Hawking temperature is

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+} \left[\frac{(n+1)r_+^2}{l^2} + \frac{(16\pi Q)^2}{2n\Omega_n^2 r_+^{2(n-1)}} + c_0 c_1 m^2 r_+ + (n-1)c_0^2 c_2 m^2 + (n-1)k + \frac{(n-1)(n-2)c_0^3 c_3 m^2}{r_+} + \frac{(n-1)(n-2)(n-3)c_0^4 c_4 m^2}{r_+^2} \right]. \quad (9)$$

The mass expressed by the horizon radius and charge is

$$M = \frac{n\Omega_n r_+^{n-1}}{16\pi} \left[k + \frac{r_+^2}{l^2} + \frac{(16\pi Q)^2}{2n(n-1)\Omega_n^2 r_+^{2(n-1)}} + \frac{c_0 c_1 m^2 r_+}{n} + c_0^2 c_2 m^2 + \frac{(n-1)c_0^3 c_3 m^2}{r_+} + \frac{(n-1)(n-2)c_0^4 c_4 m^2}{r_+^2} \right]. \quad (10)$$

The cosmological constant was seen as a fixed constant in the past. In this paper, it is regarded as a variable related to pressure, $P = -\frac{\Lambda}{8\pi} = \frac{n(n+1)}{16\pi l^2}$, and its conjugate quantity is a thermodynamic volume V . The entropy, volume and electric potential at the event horizon are given by

$$S = \frac{\Omega_n r_+^n}{4}, \quad V = \frac{\Omega_n r_+^{n+1}}{n+1}, \quad \Phi_e = \frac{16\pi Q}{(n-1)\Omega_n r_+^{n-1}}, \quad (11)$$

respectively. Due to the appearance of the pressure, the mass is no longer interpreted as the internal energy, but as an enthalpy. c_1, c_2, c_3 and c_4 are seen as extensive parameters for the mass. Their conjugate quantities are

$$\begin{aligned}
 \Phi_1 &= \frac{\Omega_n c_0 m^2 r_+^n}{16\pi}, \\
 \Phi_2 &= \frac{n\Omega_n c_0^2 m^2 r_+^{n-1}}{16\pi}, \\
 \Phi_3 &= \frac{n(n-1)\Omega_n c_0^3 m^2 r_+^{n-2}}{16\pi}, \\
 \Phi_4 &= \frac{n(n-1)(n-2)\Omega_n c_0^4 m^2 r_+^{n-3}}{16\pi}, \quad (12)
 \end{aligned}$$

respectively. It is easy to verify that these thermodynamic quantities obeys the first law of thermodynamics

$$dM = TdS + VdP + \Phi_e dQ + \sum_{i=1}^4 \Phi_i dc_i. \quad (13)$$

When the cosmological constant is fixed, the term VdP disappears and the mass is interpreted as the internal energy. When a perturbative correction is introduced, the related thermodynamic quantities are shifted, which is discussed in the next section.

III. EXTREMALITY RELATIONS IN THE MASSIVE GRAVITY

In this section, we derive the extremality relations between the mass and entropy, charge, pressure, parameters c_i by adding a rescaled cosmological constant to the action as the perturbative correction. The rescaled parameter is ϵ . Here, the black hole is designated as an extremal one.

We first introduce the correction

$$\Delta S = \frac{1}{16\pi} \int dx^{n+2} \sqrt{-g} \frac{n(n+1)\epsilon}{l^2}, \quad (14)$$

to the action (4). The corrected action is $S + \Delta S$. The action (4) is recovered when $\epsilon = 0$. A black hole solution is gotten from the corrected action and takes the form as Eqs. (6) and (8), but there is a shift. Due to the correction, the Hawking temperature is also shifted and given by

$$\begin{aligned}
 T &= \frac{1}{4\pi r_+} \left[\frac{(n+1)r_+^2 \epsilon}{l^2} + \frac{(n+1)r_+^2}{l^2} + \frac{(16\pi Q)^2}{2n\Omega_n^2 r_+^{2(n-1)}} + c_0 c_1 m^2 r_+ + (n-1)c_0^2 c_2 m^2 + (n-1)k + \frac{(n-1)(n-2)c_0^3 c_3 m^2}{r_+} + \frac{(n-1)(n-2)(n-3)c_0^4 c_4 m^2}{r_+^2} \right]. \quad (15)
 \end{aligned}$$

The corrected mass is

$$M = \frac{n\Omega_n r_+^{n-1}}{16\pi} \left[\frac{r_+^2 \epsilon}{l^2} + k + \frac{r_+^2}{l^2} + \frac{(16\pi Q)^2}{2n(n-1)\Omega_n^2 r_+^{2(n-1)}} + \frac{c_0 c_1 m^2 r_+}{n} + \frac{(n-1)c_0^3 c_3 m^2}{r_+} + \frac{(n-1)(n-2)c_0^4 c_4 m^2}{r_+^2} \right], \quad (16)$$

which is a function of parameters $r_+, \epsilon, Q, l, c_1, c_2, c_3$ and c_4 . Our interest is focused on the thermodynamic extremality relation. The Hawking temperature (15) in the extremal case is zero, which leads to a solution $r_+ = r_+(\epsilon)$. Inserting this solution into the above equation yields an expression about the mass, $M_{ext} = M_{ext}(\epsilon)$. Carrying out the differential on $M_{ext}(\epsilon)$, we have

$$\left(\frac{\partial M_{ext}}{\partial \epsilon} \right)_{Q,l,c_1,c_2,c_3,c_4} = \frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \quad (17)$$

Since the expression of the differential expressed by ϵ is very complex, we adopted the expression of r_+ in the above derivation. In fact, this relation can also be derived by the following calculation. For convenience, we use c to denote all parameters Q, l, c_1, c_2, c_3, c_4 except for r_+ and ϵ . From Eq. (16), the differential of M to ϵ is gotten as follows

$$\begin{aligned} \left(\frac{\partial M}{\partial \epsilon} \right)_c &= \left(\frac{\partial M}{\partial r_+} \right)_{c,\epsilon} \left(\frac{\partial r_+}{\partial \epsilon} \right)_c + \left(\frac{\partial M}{\partial \epsilon} \right)_{c,r_+} \\ &= \left(\frac{\partial M}{\partial S} \right)_{c,\epsilon} \left(\frac{\partial S}{\partial r_+} \right)_{c,\epsilon} \left(\frac{\partial r_+}{\partial \epsilon} \right)_c + \left(\frac{\partial M}{\partial \epsilon} \right)_{c,r_+} \end{aligned}$$

$$\left(\frac{\partial \epsilon}{\partial S} \right)_{M,Q,l,c_1,c_2,c_3,c_4} = \frac{4l^2}{n\Omega_n r_+^{n-1}} \left[-\frac{(n+1)16\pi M}{n\Omega_n r_+^{n+2}} + \frac{2k}{r_+^3} + \frac{(16\pi Q)^2}{(n-1)\Omega_n^2 r_+^{2n+1}} + \frac{c_0 c_1 m^2}{nr_+^2} + \frac{2c_0^2 c_2 m^2}{r_+^3} + \frac{3(n-1)c_0^3 c_3 m^2}{r_+^4} + \frac{4(n-1)(n-2)c_0^4 c_4 m^2}{r_+^5} \right]. \quad (20)$$

To evaluate the value, we insert the expression of the mass into the above equation and get

$$\left(\frac{\partial \epsilon}{\partial S} \right)_{M,Q,l,c_1,c_2,c_3,c_4} = \frac{4l^2}{n\Omega_n r_+^{n-1}} \left[-\frac{(n-1)k}{r_+^3} - \frac{(n+1)(1+\epsilon)}{r_+ l^2} + \frac{(16\pi Q)^2}{2n\Omega_n^2 r_+^{2(n+1)}} - \frac{c_0 c_1 m^2}{r_+^2} - \frac{(n-1)c_0^2 c_2 m^2}{r_+^3} - \frac{(n-1)(n-2)c_0^3 c_3 m^2}{r_+^4} - \frac{(n-1)(n-2)(n-3)c_0^4 c_4 m^2}{r_+^5} \right]. \quad (21)$$

Combining the inverse of the above differential with the expression of the temperature given in Eq. (15), we have

$$\frac{1}{4} T \Omega_n r_+^{n-1} \left(\frac{\partial r_+}{\partial \epsilon} \right)_c + \left(\frac{\partial M}{\partial \epsilon} \right)_{c,r_+}. \quad (18)$$

In the extremal case, the first term in the last line of the above equation disappears, and the mass is rewritten as $M = M_{ext}$. Therefore, Eq. (17) is readily recovered.

The entropy S , pressure P , charge Q , c_1 , c_2 , c_3 and c_4 are usually regarded as a complete set of extensive parameters for the mass. Their conjugate quantities can be derived from the mass and take the same form as those given in section 2 except for the temperature and volume. We first verify the extremality relation between the mass and entropy.

From Eq. (16), the expression of ϵ is gotten and takes the form

$$\epsilon = \left[\frac{16\pi M}{n\Omega_n r_+^{n+1}} - \frac{k}{r_+^2} - \frac{(16\pi Q)^2}{2n(n-1)\Omega_n^2 r_+^{2n}} - \frac{c_0 c_1 m^2}{nr_+} - \frac{c_0^2 c_2 m^2}{r_+^2} - \frac{(n-1)c_0^3 c_3 m^2}{r_+^3} - \frac{(n-1)(n-2)c_0^4 c_4 m^2}{r_+^4} \right] l^2 - 1. \quad (19)$$

Due to the relation between the entropy and horizon radius given in Eq. (11), the above equation is a function $\epsilon(S)$ and $\frac{\partial r_+}{\partial S} = \frac{4}{n\Omega_n r_+^{n-1}}$. Carrying out the differential calculation on this function yields

$$T \left(\frac{\partial S}{\partial \epsilon} \right)_{M,Q,l,c_1,c_2,c_3,c_4} = -\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \quad (22)$$

Compared it with the relation (17), it is easy to get

$$\left(\frac{\partial M_{ext}}{\partial \epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4} = \lim_{M \rightarrow M_{ext}} -T \left(\frac{\partial S}{\partial \epsilon}\right)_{M,Q,l,c_1,c_2,c_3,c_4}, \quad (23)$$

where S is a function of M , Q , l , c_1 , c_2 , c_3 , c_4 and ϵ . Therefore, the Goon-Penco relation is verified in the higher-dimensional black hole.

In this paper, the cosmological constant is regarded as a variable related to pressure. The entropy, pressure, charge, c_1 , c_2 , c_3 and c_4 are usually regarded as the extensive parameters for the mass. Since the entropy satisfies the thermodynamic extremality relation, it is natural to ask whether other extensive quantities also satisfy corresponding relations. What we need to do in the following investigation is to find out these relations. Let's first derive the extremality relation between the mass and pressure. The pressure is expressed by the constant l^2 as $P = \frac{n(n+1)}{16\pi l^2}$. Then, $\frac{\partial P}{\partial l^2} = -\frac{16\pi l^4}{n(n+1)}$. Using Eqs. (16) and (19), we get the differential of ϵ to the pressure,

$$\left(\frac{\partial \epsilon}{\partial P}\right)_{M,r_+,Q,c_1,c_2,c_3,c_4} = \frac{-16\pi l^2(1+\epsilon)}{n(n+1)}. \quad (24)$$

The perturbation parameter ϵ exists in the above differential relation as an explicit function. The reason is that the perturbation correction is introduced by adding the rescaled cosmological constant to the action and this constant is related to the pressure. Due to the shift of the mass, the thermodynamic volume is also shifted and its expression is different from that given given in Eq. (11). The volume is

$$V = \frac{\epsilon+1}{n+1} \Omega_n r_+^{n+1}. \quad (25)$$

Using Eq. (25) and the inverse of the differential of ϵ to P yields

$$V \left(\frac{\partial P}{\partial \epsilon}\right)_{M,r_+,Q,c_1,c_2,c_3,c_4} = -\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \quad (26)$$

Comparing the above equation with Eq. (17), we get the extremality relation between the mass and pressure,

$$\left(\frac{\partial M_{ext}}{\partial \epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4} = \lim_{M \rightarrow M_{ext}} -V \left(\frac{\partial P}{\partial \epsilon}\right)_{M,r_+,Q,c_1,c_2,c_3,c_4}, \quad (27)$$

where P is a function of M , r_+ , Q , c_1 , c_2 , c_3 , c_4 and ϵ . This relation is an extension of the Goon-Penco relation.

We continue to investigate the extremality relation

between the mass and charge. The calculation process is similar. From Eq. (19), the differential of ϵ to Q takes the form

$$\left(\frac{\partial \epsilon}{\partial Q}\right)_{M,r_+,l,c_1,c_2,c_3,c_4} = -\frac{(16\pi)^2 Q l^2}{n(n-1)\Omega_n^2 r_+^{2n}}. \quad (28)$$

Multiplying the electric potential $\Phi_e = \frac{16\pi Q}{(n-1)\Omega_n r_+^{n-1}}$ by the inverse of the above differential yields

$$\Phi_e \left(\frac{\partial Q}{\partial \epsilon}\right)_{M,r_+,l,c_1,c_2,c_3,c_4} = -\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \quad (29)$$

Obviously, there is a minus sign difference between Eqs. (17) and (29). Therefore,

$$\left(\frac{\partial M_{ext}}{\partial \epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4} = \lim_{M \rightarrow M_{ext}} -\Phi_e \left(\frac{\partial Q}{\partial \epsilon}\right)_{M,r_+,l,c_1,c_2,c_3,c_4}, \quad (30)$$

which is the extremality relation between the mass and charge. Now, Q is a function of M , r_+ , l , c_1 , c_2 , c_3 , c_4 and ϵ . This relation is also an extension of the Goon-Penco relation.

For the extremality relations between the mass and parameters c_1 , c_2 , c_3 , c_4 , the calculations are parallel. Their differential relations are

$$\left(\frac{\partial \epsilon}{\partial c_1}\right)_{M,r_+,Q,l,c_2,c_3,c_4} = -\frac{c_0 m^2 l^2}{nr_+}, \quad (31)$$

$$\left(\frac{\partial \epsilon}{\partial c_2}\right)_{M,r_+,Q,l,c_1,c_3,c_4} = -\frac{c_0^2 m^2 l^2}{r_+^2}, \quad (32)$$

$$\left(\frac{\partial \epsilon}{\partial c_3}\right)_{M,r_+,Q,l,c_1,c_2,c_4} = -\frac{(n-1)c_0^3 m^2 l^2}{r_+^3}, \quad (33)$$

$$\left(\frac{\partial \epsilon}{\partial c_4}\right)_{M,r_+,Q,l,c_1,c_2,c_3} = -\frac{(n-1)(n-2)c_0 m^2 l^2}{r_+^4}. \quad (34)$$

The conjugate quantities of c_1 , c_2 , c_3 and c_4 are

$$\Phi_1 = \frac{\Omega_n c_0 m^2 r_+^n}{16\pi}$$

$$\Phi_2 = \frac{n\Omega_n c_0^2 m^2 r_+^{n-1}}{16\pi}$$

$$\Phi_3 = \frac{n(n-1)\Omega_n c_0^3 m^2 r_+^{n-2}}{16\pi}$$

$$\Phi_4 = \frac{n(n-1)(n-2)\Omega_n c_0^4 m^2 r_+^{n-3}}{16\pi}$$

respectively. Using these quantities, it is not difficult for us to get

$$\begin{aligned} \left(\Phi_1 \frac{\partial c_1}{\partial \epsilon}\right)_{M,r_+,Q,l,c_2,c_3,c_4} &= \left(\Phi_2 \frac{\partial c_2}{\partial \epsilon}\right)_{M,r_+,Q,l,c_1,c_3,c_4} = \left(\Phi_3 \frac{\partial c_3}{\partial \epsilon}\right)_{M,r_+,Q,l,c_1,c_2,c_4} \\ &= \left(\Phi_4 \frac{\partial c_4}{\partial \epsilon}\right)_{M,r_+,Q,l,c_1,c_2,c_3} = -\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \end{aligned} \quad (35)$$

Thus, the extremality relations between the mass and extensive parameters c_i are

$$\left(\frac{\partial M_{ext}}{\partial \epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4} = \lim_{M \rightarrow M_{ext}} -\Phi_i \left(\frac{\partial c_i}{\partial \epsilon}\right)_{M,r_+,Q,l,c_j,c_k,c_u}, \quad (36)$$

where $i, j, k, u = 1, 2, 3, 4$ and $i \neq j \neq k \neq u$. Therefore, the Goon-Penco relation is extended to the case of the extensive parameters c_i of the higher-dimensional black hole.

In the above investigation, the thermodynamic extremality relations between the mass and entropy, pressure, charge, parameters c_i were gotten by the accurate calculations. They are expressed by Eqs. (22), (27), (30) and (36), respectively. The values of these relations are equal. In fact, these relations can be derived uniformly by using the triple product identity

$$\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T} \left(\frac{\partial X^i}{\partial \epsilon}\right)_{M,T} \left(\frac{\partial \epsilon}{\partial M}\right)_{T,X^i} = -1, \quad (37)$$

which yields

$$\left(\frac{\partial M}{\partial \epsilon}\right)_{T,X^i} = -\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T} \left(\frac{\partial X^i}{\partial \epsilon}\right)_{M,T} = -\Phi_i \left(\frac{\partial X^i}{\partial \epsilon}\right)_{M,T}. \quad (38)$$

In the above derivation, $\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T}$ were identified to Φ_i which are the conjugate quantities to X^i . Here, X^i are

chosen as S, Q, P, c_1, c_2, c_3 and c_4 . M and T are the corrected mass and temperature given in (16) and (15), respectively. In the extremal case, $T \rightarrow 0$ and $M \rightarrow M_{ext}$. The above relation becomes

$$\left(\frac{\partial M_{ext}}{\partial \epsilon}\right)_{M,X^i} = \lim_{M \rightarrow M_{ext}} -\Phi_i \left(\frac{\partial X^i}{\partial \epsilon}\right)_{M,X^i}, \quad (39)$$

where $X^i \neq X^j$, and X^j are parameters S, Q, P, c_1, c_2, c_3 or c_4 , except for X^i . This relation implies that the universal extremality relation exist in black holes. The relation (39) is easily reduced to (22), (27), (30) and (36) when X^i is the entropy, charge, parameters c_i and pressure. In the calculation, due to the shift of the mass, the expression of the volume $V = \frac{\epsilon+1}{n+1} \Omega_n r_+^{n+1}$ is different from that given in Eq. (11). In [26], the authors derived the extremality relation between the mass and angular momentum in the BTZ and Kerr anti-de Sitter spacetimes, and made a conjecture that a general formula of the extremality relation existed in black holes. Our result gives a verification to this conjecture.

IV. CONCLUSION

In this paper, we extended the work of Goon and Penco to the massive gravity and investigated the thermodynamic extremality relations in the higher-dimensional black hole. The extremality relations between the mass and pressure, entropy, charge, parameters c_i were derived by the accurate calculations, respectively. The values of these extremality relations are equal, which may be due to the first law of thermodynamics. In the calculation, the cosmological constant was seen as a variable related to pressure. The perturbative correction was introduced by adding the rescaled cosmological constant to the action, but this addition does not affect the form of the extremality relation between the mass and pressure.

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