Investigations on the relationship between the mirror proton radii and neutron-skin thickness

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Abstract: Through the systematical investigations by using axially deformed solution of the Skyrme-Hartree-Fock-Bogoliubov equations with 132 sets of Skyrme interaction parameters, it is confirmed that the neutron-skin thickness \((S_n)\) of a neutron-rich nucleus is proportional to the difference between the proton radii of mirror nuclei \((R^\text{mir}_p)\). It indicates that \(S_n\) may be deduced from \(R^\text{mir}_p\). Compared with the results of the Skyrme-Hartree-Fock model, it is found that the pairing effects enhance the correlation for most mirror pairs while deformation effects may weaken the correlation. Furthermore, the correlation between \(S_n\) and \(R^\text{mir}_p\) have been studied for isotones with \(N=20\) and \(N=28\), which show a stronger linear correlation with the increasing of \([N-Z]\). This result demonstrates that it is possible to extract the neutron-skin thickness of the unstable nucleus from the proton radii difference of mirror nuclei of its isotones.

Keywords: neutron-skin thickness, mirror nuclei, Skyrme-Hartree-Fock model, the equation of state

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I. INTRODUCTION

Neutron skin is an interesting phenomenon in the domain of nuclear physics. In particular, for the neutron-rich systems, due to the large asymmetry between proton \((p)\) and neutron \((n)\) numbers (denoting as \(Z\) and \(N\), respectively), these two kinds of fermions tend to be decoupled around the surface region of a nucleus, providing valuable information to study nuclear force \([1]\) including symmetry energy regarding to the equation of state (EOS) of nuclear matter \([2−7]\). Moreover, the thickness of the neutron skin is a direct observation of nuclear structure, which is strongly associated with the difference in radii between mirror nuclei across the nuclear landscape. To this end, we aim to investigate the correlation between the neutron skin thickness and various physical quantities in this work by studying the region of light and mid-heavy nuclei, and gain more insight into the connection between nuclear structure and matter.

The neutron skin thickness is defined as the difference between the root-mean-square (rms) radius of the neutron and that of the proton: \(S_n \equiv (\langle r^2_n \rangle)^{1/2} - (\langle r^2_p \rangle)^{1/2} \approx R_n - R_p\). The proton radius of a nucleus can be extracted with relatively high accuracy by electromagnetic interaction \([8]\), but it is difficult to obtain precise neutron radii by using strong or weak interaction probes. Consequently, the extracted results of the neutron-skin thickness are usually model-dependent and vary greatly by different experimental methods \([9]\). To this end, high-precision data for neutron-skin thickness are anticipated and will play a significant role in nuclear physics and astrophysics.

According to EOS, the energy per nucleon of the nuclear matter can be approximately expressed as

\[
E(\rho, \delta) = E_0(\rho) + E_{\text{sym}}(\rho)\delta^2 + O(\delta^4),
\]

where \(\rho_n\), \(\rho_p\) and \(\rho = \rho_n + \rho_p\) are the neutron, proton, and total nucleon densities respectively, \(\delta = (\rho_n - \rho_p)/\rho\) is the isospin asymmetry, \(E_0(\rho) = E(\rho, \delta = 0)\) is the energy per nucleon of symmetric nuclear matter, and \(E_{\text{sym}}(\rho)\) is nuclear symmetry energy described as

\[
E_{\text{sym}}(\rho) \equiv \frac{1}{2\pi} \left. \frac{\partial^2 E(\rho, \delta)}{\partial \delta^2} \right|_{\delta=0}.
\]

The properties of symmetric nuclear matter are relat-
ively well-determined, while the isovector part remains largely uncertain and attracts a lot of attention, which could gain more knowledge of the dripline nuclei, astrophysics, and heavy-ion collisions [10–15]. Around the saturation density \( \rho_0 \), \( E_{\text{sym}}(\rho) \) can be expanded as

\[
E_{\text{sym}}(\rho) = E_{\text{sym}}(\rho_0) + \frac{L}{3} \left( \frac{\rho - \rho_0}{\rho_0} \right)
 + \frac{K_{\text{sym}}}{18} \left( \frac{\rho - \rho_0}{\rho_0} \right)^2 + \cdots, \tag{3}
\]

where \( L \) and \( K_{\text{sym}} \) are the slope and curvature of the symmetry energy at saturation density, respectively, defined as

\[
L \equiv 3\rho_0 \left. \frac{\partial E_{\text{sym}}(\rho)}{\partial \rho} \right|_{\rho=\rho_0}, \tag{4}
\]

\[
K_{\text{sym}} \equiv 9\rho_0^2 \left. \frac{\partial^2 E_{\text{sym}}(\rho)}{\partial \rho^2} \right|_{\rho=\rho_0}. \tag{5}
\]

These two characteristic parameters govern the behavior of the symmetry energy at subsaturation and oversaturation densities [16–23]. Similarly, the coefficient of the third-order term, named the incompressibility of symmetric nuclear matter, can be written as

\[
K_0 \equiv 9\rho_0^3 \left. \frac{\partial^2 E_{\text{sym}}(\rho)}{\partial \rho^3} \right|_{\rho=\rho_0}. \tag{6}
\]

Larger \( K_0 \) indicates that it is harder to compress nuclear matter, corresponding to a stiff EOS, while the contrary is regarded as a soft EOS [16,24–30]. According to the various studies of density functional theory [18,31–37], there is a strong linear correlation between \( S_n \) and \( L \) for the heavy nucleus \(^{208}\text{Pb}\). It has also been shown that these properties can be constrained by the scattering phase shift related to the nuclear force [38].

Supposing perfect charge symmetry, the neutron radius of a given nucleus \( (\frac{A}{2}) X \) is strictly equal to the proton radius of the corresponding mirror nucleus \( (\frac{A}{2}) Y \), with \( A = N + Z \) is the mass number. Therefore, the thickness of neutron skin can be evaluated through the difference of proton rms radius of the mirror nuclei [39–42], i.e.,

\[
S_n(\frac{A}{2} X) = R_p(\frac{A}{2} Y) - R_p(\frac{A}{2} X) \equiv R_{\text{mir}}^{\text{sym}}(\frac{A}{2} X). \tag{7}
\]

Although, in reality, the charge symmetry is slight broken mainly due to the presence of the Coulomb interaction between protons, it has been shown that there is a linear correlation between the difference of the rms charge radii of mirror nuclei \( \left(R_{\text{mir}}^{\text{sym}} \right) \) and \( |N-Z| \times L \) [39,40]. In addition, \( S_n \) is related to both \( |N-Z| \times L \) and \( E_{\text{sym}}(\rho = 0.10 \text{ fm}^{-3}) \) [39]. When \( |N-Z| \) is large, the \( L \) dependence in \( S_n \) dominates. It is worth noting that the above research results are about neutron-deficient nickel isotopes and their corresponding mirror nuclei. The lack of experimental data of charge radii for most of nickel isotopes makes it impossible to predict the \( S_n \) through their relationship with \( R_{\text{mir}}^{\text{sym}} \). Moreover, for the nuclei away from doubly magic nuclei, deformation and pairing effects play a significant role but they are not fully considered in the above studies. Refs [43,44] proposed that pairing effects weaken the correlation between \( R_{\text{mir}}^{\text{sym}} \) and \( L \). Therefore, further study on the dependence of \( R_{\text{mir}}^{\text{sym}} \) on isovector sensitive observables is still demanded.

In the present manuscript, we adopt the density-functional solver HFBTHO [45], in which the axially deformed solutions of the Skyrme-Hartree-Fock-Bogoliubov (HFB) equations are considered to study the neutron and proton rms radii for different nuclei. To investigate the pairing and deformation effect, the results of Skyrme-Hartree-Fock (SHF) method are also listed. By using various sets of Skyrme interaction parameters, different sizes of neutron skin is obtained and the correlations between \( S_n, L \) and \( R_{\text{mir}}^{\text{sym}} \) are investigated. By extracting the neutron-skin thickness of certain neutron-rich nuclei, the constraints for the characteristic parameters of the EOS – such as \( L \) – are studied.

II. METHOD

The SHF methodology [46] has been successfully applied to study the structure of finite nuclei across the nuclear landscape, including deformed nuclei near drip lines, superheavy elements, and heavy-ion collisions [47–51]. In the Hartree-Fock framework, a nucleon of the system is regarded as moving in the mean field of other nucleons. The total wave function of the system can be constructed by the Slater determinant of the single-particle ones, which is obtained through the single-particle Hamiltonian generated by the mean field [52–54]. Consequently, the total energy functional of the nucleus can be separated as

\[
E = E_{\text{Skyrme}} + E_{\text{Coulomb}} + E_{\text{pair}} - E_{\text{cm}}. \tag{8}
\]

One of widely used Skyrme energy functionals can be written in the following form [46]

\[
E_{\text{Skyrme}} = 4\pi \int_0^\infty dr \int r^2 \left\{ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} - \frac{1}{2} f_0 (1 + 2x_0^2) + \frac{1}{2} f_0 (1 + x_0) \sum_q \rho_q^2 \right\} + \frac{1}{2} \rho_{\text{sym}} (1 + x_0) \sum_q \rho_q^2.
\]
\[ \frac{1}{12} t_3 (1 + \frac{1}{2} x_3) y^{\alpha+2} - \frac{1}{12} t_0 (1 + x_3) y^\alpha \sum_q \rho_q^2 \]
\[ + \frac{1}{4} \left[ t_1 (1 + \frac{1}{2} x_1) + t_2 (1 + \frac{1}{2} x_2) \right] \rho \tau \]
\[ - \frac{1}{4} \left[ t_1 (\frac{1}{2} x_1) - t_2 (\frac{1}{2} x_2) \right] \sum_q \rho_0 \rho_q \]
\[ - \frac{1}{16} \left[ 3 t_1 (1 + \frac{1}{2} x_1) - t_2 (1 + \frac{1}{2} x_2) \right] \rho \nabla^2 \rho \]
\[ + \frac{1}{16} \left[ 3 t_1 (1 + \frac{1}{2} x_1) + t_2 (1 + \frac{1}{2} x_2) \right] \sum_q \rho_0 \nabla^2 \rho_q \]
\[ - \frac{1}{2} W_0 \left[ \rho \nabla \cdot J + \sum_q \rho_q \nabla \cdot J_q \right], \] (9)

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \) is Laplacian operator, \( q \in \{n,p\} \) (\( n \)-neutron and \( p \)-proton) is the isospin label, and \( \rho = \rho_n + \rho_p \), \( \tau = \tau_n + \tau_p \), and \( J = J_n + J_p \) are the total density of nucleon, kinetic energy, and spin-orbit coupling, respectively. Eq. (9) contains 10 Skyrme interaction parameters, which are \( t_0, t_1, t_2, t_3, x_0, x_1, x_2, x_3, \alpha, \) and \( W_0 \). Additionally, \( \alpha \) is also known as the density-dependent coefficient of Skyrme interaction potential which commonly ranges from 1/6 to 1. When \( \alpha < 1 \), the potential is soft with a comparatively small \( K_0 \), while when \( \alpha = 1 \), the potential is stiff.

So far, a great deal of Skyrme parameter sets has been proposed by fitting experimental data on the properties of the ground state of finite nuclei and some observables of infinite nuclear matter near saturation density [55]. Each set of parameter has their own macroscopic quantities, such as \( \rho_0, L, E_{sym}, K_0, \) etc. Near the \( \beta \)-stability line, most Skyrme energy functionals provide similar results. However, the theoretical predictions for the properties of asymmetric nuclear matter or finite nuclei far from the stability line generally vary a lot, which strongly depends on the selected set of parameters [56]. To this end, we are trying to investigate the impact of density functionals on the neutron skin, in which the Coulomb energy \( E_{Coulomb} \) including the Coulomb-exchange part which is treated in the Slater approximation and a correction for the spurious center-of-mass (CoM) motion of the mean field \( E_{cm} \) [46,57] are taken into account. Except for SLy6 and SLy7 which use the simplified version of the two-body CoM correction [58], the one-body CoM motion is considered in the current work.

Moreover, nucleon-nucleon correlation widely exhibits in the nuclear landscape, and plays a significant role in the bulk properties. To investigate the pairing and deformation effects, the HFB solver is utilized, in which the axial transformed harmonic oscillator single-particle basis is used to expand quasi-particle wave functions. The pairing channel is parameterized by a density-dependent delta-pairing force with mixed volume and surface features:

\[ V_{pair,q}(r) = V_0,q(1 - \frac{1}{2} \frac{\rho(r)}{\rho_0}) \delta(r - r'). \] (10)

where \( V_0,q \) is the pairing strength, \( \rho(r) \) is the isoscalar local density, and \( \rho_0 \) is the saturation density fixed at 0.16 fm\(^{-3}\). A general review of the HFBTHO solver can be found in Ref. [45]. In the case of the UNEDF parameterizations, the pairing strengths should not be adjusted by the user since they were fitted together with the Skyrme coupling constants. For UNEDF0, UNEDF1, and UNEDF2 [59], recommended values of \( V_{0,n} \) are \(-170.374 \) MeV, \(-186.065 \) MeV, and \(-208.889 \) MeV, respectively, and recommended values of \( V_{0,p} \) are \(-199.202 \) MeV, \(-206.580 \) MeV, and \(-230.330 \) MeV, respectively. For other Skyrme interactions, \( V_{0,n} \) and \( V_{0,p} \) are chosen as \(-300 \) MeV, which have been checked by the current work that can reproduce the binding energy approximately consistent with the experimental data.

In this work, we have considered 128 sets of Skyrme interaction parameters out of 240 available in the literature [55] to obtain the density distributions of neutron/proton, and calculated the corresponding rms radii as well as \( S_n \). These 128 Skyrme interactions are selected by taking into account two aspects:

1. The range of \( L \) (the slope of the symmetry energy at saturation density) is limited to 0 \(-130 \) MeV, as previous researches suggested [12,60\textendash}62];

2. The calculated binding energy per nucleon should approximately match the experimental data for all nuclei involved in the current work.

Also, we want to emphasize that the value of \( L \) is still not determined which requires further studies, and our paper is mainly focused on the correlations between the neutron skin thickness and various physical quantities. Removing a small number of Skyrme interactions has little effect on the overall results and conclusions.

Both SHF and HFB frameworks are used to make comparisons and investigate the influence of pairing and deformation effects. The 132 parameter sets give a relatively wide range of nuclear-matter quantities regarding the EOS, including \( 0.145 \leq \rho_0 \leq 0.175 \) fm\(^{-3}\), \( 0.13 \leq L \leq 129.33 \) MeV, \( 22.83 \leq E_{sym}(\rho_0) \leq 37.40 \) MeV, and \( 200.97 \leq K_0 \leq 370.38 \) MeV. The correlations between \( S_n \), \( R_{pair} \) and \( L \) were studied. Based on the experimental data [41,42,63], we investigated 12 pairs of mirror nuclei that have the charge radii data. By taking into account the electromagnetic spin-orbit effects, the conversion formula between charge and point-proton radius can be ex-
pressed approximately as [64]

\[ R_p = \sqrt{R_{ch}^2 - \frac{3}{4M^2} \left( \langle r^2 \rangle_{so} \right) + \frac{\langle R_p^2 \rangle}{Z} - \frac{\langle R_n^2 \rangle}{N}}. \]  

(11)

where \( \langle r^2 \rangle_{so} \) is the spin-orbit contribution [65]. Here \( \langle R_p^2 \rangle = 0.769(12) \) fm\(^2\) and \( \langle R_n^2 \rangle = -0.1161(22) \) fm\(^2\) are the mean-square charge radii of the proton and neutron, respectively, and the term \( \frac{3}{4M^2} \) is known as the Darwin-Foldy term [66]. Consequently, the experimental value of the difference in proton radii and the corresponding error by means of the error propagation formula can be obtained. In this way, we can not only forecast the \( S_n \) of some neutron-rich nuclei, but also find out which sets of Skyrme parameters are more reasonable to constrain the range of \( L \). Besides, we also explore correlation strength based on the linearity between \( S_n \) and \( R_{p \text{mir}}^n \) for \( N = 20 \) and \( N = 28 \) isotope chains [40].

III. RESULTS AND DISCUSSION

As a starting point, we took \(^{48}\text{Ca}\) as an example, and calculated the correlations of \( S_n - L \), \( S_n - R_{p \text{mir}}^n \), and \( R_{p \text{mir}}^n - L \) (see Fig. 1). Although either \( R_{p \text{mir}}^n \) or \( R_{n \text{mir}}^n \) can be used in these analysis, the results will be nearly identical. For the sake of consistency, we will be using \( R_{p \text{mir}}^n \) in our discussions. Each point in Fig. 1 corresponds to the results of individual Skyrme energy functional. Especially, we label the results of HFB9, UNEDF0, UNEDF1 and UNEDF2 parameterizations by black crosses.

As shown in Fig. 1(a), a perfect linear relation between the neutron-skin thickness and the difference in proton radii of mirror nuclei exhibits when the Coulomb interaction is not considered due to Eq. (7). In reality, the Coulomb interaction should be taken into account as shown in Fig. 1(b-d). It is interesting to note that a roughly linear correlation remains even in the presence of the Coulomb interaction, especially in Fig. 1(b). Therefore, we carry out the statistical analysis approach as follows. If two physical quantities (\( x \) and \( y \)) have a linear statistical correlation, we can establish the linear regression equation as

\[ \hat{y} = C_0 + C_1x, \]

(12)

where constant \( C_0 \) and \( C_1 \) are linear regression intercept and slope, respectively. To signify the degree of linear relation, the coefficient of determination (\( \varepsilon \)) is defined as

\[ \varepsilon = 1 - \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}, \]

(13)

where

\[ \hat{y}_i = C_0 + C_1x_i, \quad \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i. \]

(14)

In the above, \( (x_i, y_i) \) is the \( i \)th observed value of \( (x, y) \) which corresponds to each calculation result and \( n \) is the total number of samples. The \( \varepsilon \) ranges from zero to one, and the closer it gets to one, the stronger the linear correlation is.

According to our calculation (see Fig. 1), the difference of radii between mirror nuclei has a much better correlation with the neutron-skin thickness than the slope of the symmetry energy. It is due to the fact that the correlation between the neutron-skin thickness and the slope of the symmetry energy becomes weaker with decreasing nucleus mass. A strong correlation only exists for the heavier nuclei such as \(^{208}\text{Pb}, \; ^{132}\text{Sn}, \; \) and \(^{124}\text{Sn}, \) which has been found in Ref. [31] by SHF model with 21 sets of Skyrme interaction parameters. This indicates that, as one of characteristic parameters of the EOS of nuclear matter, the slope of the symmetry energy only has an obvious linear correlation with \( S_n \) or \( R_{p \text{mir}}^n \) in heavy nuclei near nuclear matter. However, the thickness of neutron-skin is a direct observation of nuclear structure, so it would strongly associate with the mirror proton radii due to Eq. (7) for both light and heavy nuclei. The mirror nuclei

![Fig. 1](color online) The correlation between two physics quantities of \(^{48}\text{Ca}\) calculated by the HFBTHO using 132 sets of Skyrme interaction parameters. \( S_n \) and \( R_{p \text{mir}}^n \) (a), \( S_n \) and \( R_{n \text{mir}}^n \) (b), \( R_{p \text{mir}}^n \) and \( L \) (c), \( S_n \) and \( L \) (d). (a) and (b-d) represent the results obtained without and with the Coulomb interaction, respectively. Each point corresponds to a set of parameters and the black crosses denote the results of HFB9, UNEDF0, UNEDF1 and UNEDF2 parameterizations. \( \varepsilon \) is the coefficient of determination of linear fit.
pairs studied in this paper are the systems with relatively light masses. Therefore, we focus on the correlation between the neutron-skin thickness and difference of proton radii in the presence of Coulomb interaction in our subsequent study. By making use of the good linear correlations, $S_n$ or $R_{\text{ch}}^\text{mir}$ can be deduced through the other's experimental value.

Nuclear charge radii can be extracted through different experimental methods, such as elastic electron scattering, muonic atom X-rays, $K_e$ isotope shifts, and optical isotope shifts. Up to now, there are 12 pairs of mirror nuclei having the known information on charge radii, i.e., $^{18}\text{O}$–$^{18}\text{Ne}$, $^{19}\text{F}$–$^{19}\text{Ne}$, $^{21}\text{Ne}$–$^{21}\text{Na}$, $^{22}\text{Ne}$–$^{22}\text{Mg}$, $^{23}\text{Na}$–$^{23}\text{Mg}$, $^{34}\text{S}$–$^{34}\text{Ar}$, $^{36}\text{S}$–$^{36}\text{Ca}$, $^{35}\text{Cl}$–$^{35}\text{Ar}$, $^{37}\text{Cl}$–$^{37}\text{Ca}$, $^{37}\text{Ar}$–$^{37}\text{K}$, $^{38}\text{Ar}$–$^{38}\text{Ca}$, $^{54}\text{Fe}$–$^{54}\text{Ni}$. It is noted that $R_{\text{ch}}$ of $^{54}\text{Fe}$–$^{54}\text{Ni}$ and $^{36}\text{Ca}$ were evaluated in Ref. [63] and Ref. [41], respectively, and $R_{\text{ch}}$ of other nuclei were listed in Ref. [42]. Consequently, one can deduce the information of the difference in proton radii by Eq. (11) and the corresponding error by means of the error propagation formula. The estimated experimental proton radii are listed in Table 1. Based on HFB framework with 132 sets of Skyrme parameters, the predicted correlations between the neutron-skin thickness and difference in proton radii from oxygen to iron isotopes are shown in Fig. 2(a)–(l) respectively. The solid blue lines are the results of the linear fit, which can be expressed as $S_n = C_0 + C_1 R_{\text{ch}}^\text{mir}$. It indicates that the resulting correlations vary for different nuclei. Roughly, the linear correlation is stronger when the neutron-rich nucleus has a larger neutron-proton ratio ($N/Z$). Moreover, except for $^{37}\text{Ar}$–$^{37}\text{K}$ and $^{54}\text{Fe}$–$^{54}\text{Ni}$ with small $N/Z$, the coefficients of determination of other pairs are larger than 0.7, making it possible to estimate $S_n$ from experimental $R_{\text{ch}}^\text{mir}$. Especially for $^{18}\text{O}$–$^{18}\text{Ne}$ mirror pair as shown in Fig. 2(a), its coefficient of determination is extremely close to 1, making it possible to predict the neutron-skin thickness of $^{18}\text{O}$ with a moderate neutron-proton ratio ($N/Z = 1.25$). As a result, the linear expression as

$$S_n^{(18}\text{O}) = -0.043 + 1.071 R_{\text{ch}}^\text{mir}(18\text{O}).$$

According to $R_{\text{ch}}^\text{mir}(18\text{O}) = 0.183 \pm 0.009$ fm in Table 1, we can deduce $S_n^{(18}\text{O})$ to be $0.153 \pm 0.010$ fm, which is close to 0.17 fm measured by $\pi^-$ and $\pi^+$ elastic scattering at 180 MeV in Ref. [67] and 0.179 fm performed at the Swiss Institute of Nuclear Research (SIN) using 163 MeV pions in Ref. [68]. Additionally, there are six sets of Skyrme parameters that give the predicted $R_{\text{ch}}^\text{mir}$ within experimental uncertainty, namely SK13 [69], SKT4 [69], SK15 [69], SK16 [70], SK255 [71], SK272 [71]. The corresponding $L$’s are 100.53 MeV, 60.39 MeV, 129.33 MeV, 59.24 MeV, 95.05 MeV and 91.67 MeV, respectively, which limits the value of $L$ to the range of 59–101 MeV.

Furthermore, Fig. 3 displays the range of the slope of the symmetry energy at saturation density restricted by experimental $R_{\text{ch}}^\text{mir}$ for each mirror pair. The horizontal gray band shows the results from Ref. [62]. It is based on 24 new analyses of neutron star observables since GW170817, which gives the range for $L$ of 38.7–76.7 MeV at a 68 confidence level. Also, it is consistent with its fiducial value from surveys of over 50 earlier analyses of both terrestrial and astrophysical data within error bars. We can see that except for mirror pairs $^{19}\text{F}$–$^{19}\text{Ne}$ and $^{37}\text{Cl}$–$^{37}\text{Ca}$, the constraints on $L$ of the other 10 pairs can overlap with the gray band to some extent. As for the $^{19}\text{F}$–$^{19}\text{Ne}$ pair, none of 132 Skyrme parameters sets provides predictions that match the experimental data. In terms of the $^{37}\text{Cl}$–$^{37}\text{Ca}$ pair, its constraint on $L$ is relatively lower than the deductions in Refs. [62,72]. Because Ref. [41] has pointed out that its experimental uncertainty is large and the combined analysis cannot be used for the evaluation of $R_{\text{ch}}(37\text{Cl})$ due to the lack of muonic atom data and no reliable value of $R_{\text{ch}}$ is available. That could be why the data of $^{37}\text{Cl}$–$^{37}\text{Ca}$ far from the other data.

In order to investigate the effects of pairing correlations and deformation, we also compared the results of the SHF model about the correlation between $R_{\text{ch}}^\text{mir}$ and $S_n$ with 128 sets of Skyrme interaction parameters (without HFB9, UNEDF0, UNEDF1 and UNEDF2) as shown in Fig. 2. We summarize the values of $C_0$, $C_1$ and $\varepsilon$ with the HFBTHO and SHF solver in Table 2 to make the comparison easy. It is found that for the mirror pairs

<table>
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<th>Table 1. The experimental proton radii for mirror pairs. $R_{\text{ch}}^\text{mir}(\text{Exp})$ is the experimental value of the difference in proton rms radii with the uncertainty including both experimental and systematic errors (all in fm).</th>
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<tr>
<td>Mirror nuclei</td>
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<tr>
<td>$^{18}\text{O}$, $^{18}\text{Ne}$</td>
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<tr>
<td>$^{19}\text{F}$, $^{19}\text{Ne}$</td>
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<td>$^{21}\text{Ne}$, $^{21}\text{Na}$</td>
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<td>$^{38}\text{Ar}$, $^{38}\text{Ca}$</td>
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<td>$^{54}\text{Fe}$, $^{54}\text{Ni}$</td>
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</table>
The correlation between $R_{\text{mir}}^p$ and $S_n$ for neutron-rich nuclei $^{18}\text{O}$ (a), $^{19}\text{F}$ (b), $^{21}\text{Ne}$ (c), $^{22}\text{Ne}$ (d), $^{23}\text{Na}$ (e), $^{34}\text{S}$ (f), $^{36}\text{S}$ (g), $^{35}\text{Cl}$ (h), $^{37}\text{Cl}$ (i), $^{37}\text{Ar}$ (j), $^{38}\text{Ar}$ (k), and $^{54}\text{Fe}$ (l) calculated by the HFBTHO (including Coulomb interaction) using 132 sets of Skyrme interaction parameters. Each point represents the result of individual set and the black crosses denote the results of HFB9, UNEDF0, UNEDF1 and UNEDF2 parameterizations. The solid blue lines are the results of the linear fit. The coefficients of determination ($\epsilon$), linear regression intercepts ($C_0$) and slopes ($C_1$) are presented below and the corresponding errors are given in parentheses. The blue and red shadows are the 95% confidence bands and 95% prediction bands, respectively. The vertical gray shadows represent the range of experimental $R_{\text{mir}}^p$.

The correlations between $R_{\text{mir}}^p$ and $S_n$ are weakened to some extent after considering pairing and deformation effects. More specifically, the $\epsilon$’s of $^{19}\text{F}$--$^{19}\text{Ne}$ and $^{21}\text{Ne}$--$^{21}\text{Na}$ decrease a little bit, namely about 0.05. While for $^{23}\text{Na}$--$^{23}\text{Mg}$ and $^{54}\text{Fe}$--$^{54}\text{Ni}$, the dependence of $S_n$ on $R_{\text{mir}}^p$ become much weaker, that the $\epsilon$ obtained by HFBTHO is about 0.15 less than by SHF. For the case of $^{19}\text{F}$--$^{19}\text{Ne}$ pair, compared to the HFBTHO calculations, some SHF results have better agreement with the experimental $R_{\text{mir}}^p$ as shown in Fig 4(b). However, for most of the other mirror pairs, the HFBTHO code with proper consideration of pairing and deformation effects is able to provide a more appropriate description, which is evidently reflected in the mirror pairs of $^{21}\text{Ne}$--$^{22}\text{Ne}$, $^{23}\text{Na}$, and $^{35}\text{Cl}$. In general, both the SHF and HFBTHO solver can give reliable descriptions for the nuclei with closed shells. Nevertheless, for nuclei away from doubly magic nuclei, the HFBTHO framework, which properly considers the pairing and deformation effects through the Bogoliubov transformation, is more reasonable. Furthermore, we set the values of $V_{\text{odd}}$ in Eq. (10) to zero in order to eliminate the pairing energy and explore the influence of deformation separately. Most nuclei studied in the current work are around the $\beta$-stability line near the closed shell, which are spherical or
As shown in Fig. 5, deformation effects may weaken the correlation between $R_{p}^{\text{mir}}$ and $S_n$. One possible reason is because deformation enhances the coupling of the orbitals with high angular-momenta, which makes the nuclear structure more complicated.

The discussion above is based on the nuclei with experimental $R_{p}^{\text{mir}}$. Nevertheless, for a neutron-rich nucleus with a large value of $|N-Z|$, it is more unstable and difficult to measure its radii in experiments. Consequently, for lack of experimental data, it is a challenge to extract its $S_n$ from the linear correlation between the neutron-skin thickness and difference of mirror proton radii. In order to resolve this situation, we now investigate the dependence of $S_n$ and $R_{p}^{\text{mir}}$ for $N = 20$ and $N = 28$ isotonic chains (see Fig. 6). From left to right, the mass number decreases but $|N-Z|$ increases. It indicates that $S_n$ of a

Table 2. Comparisons of the values of linear regression intercepts ($C_0$), slopes ($C_1$) and coefficients of determination ($\varepsilon$) between the HFBTHO and SHF solver. The corresponding uncertainties are given in parentheses.

<table>
<thead>
<tr>
<th></th>
<th>HFBTHO</th>
<th>SHF</th>
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<tbody>
<tr>
<td></td>
<td>$C_0$</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$^{18}$O</td>
<td>-0.043(1)</td>
<td>1.07(19)</td>
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<tr>
<td>$^{19}$F</td>
<td>-0.035(1)</td>
<td>1.10(28)</td>
</tr>
<tr>
<td>$^{21}$Ne</td>
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<td>1.13(19)</td>
</tr>
<tr>
<td>$^{23}$Na</td>
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<td>0.98(37)</td>
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<td>$^{34}$S</td>
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<td>0.96(37)</td>
</tr>
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<td>$^{35}$Ar</td>
<td>-0.058(3)</td>
<td>1.03(21)</td>
</tr>
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<td>0.99(22)</td>
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<td>$^{37}$Ar</td>
<td>-0.054(2)</td>
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</tr>
<tr>
<td>$^{38}$Ar</td>
<td>-0.061(3)</td>
<td>1.05(29)</td>
</tr>
</tbody>
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Fig. 4. (color online) The same as Fig. 2, but calculated by the SHF model using 128 sets of Skyrme interaction parameters.
neutron-rich nucleus is also roughly proportional to $R_{\text{mir}}$ of its isotopes. In Fig. 6, we label the values of $\varepsilon$ next to fitting lines and exhibit the dependence of linear regression slopes $C_1$ on $\ln(|N-Z|)$ for isotope chains in inserts. It can be seen that as $|N-Z|$ decreasing, the linear correlation becomes weaker and the value of linear regression slope becomes larger. This indicates that the experimental data of charge radii need to be more accurate to obtain the neutron-skin thickness within a certain precision. Furthermore, the linear regression slope $C_1$ is proportional to $-\ln(|N-Z|)$ as shown in inserts of Fig. 6, which can be quantitatively expressed as $C_1(N = 20) = 3.324 - 1.041\ln(|N-Z|)$ and $C_1(N = 28) = 5.289 - 2.065\ln(|N-Z|)$. The above conclusion offers the possibility of estimating $S_n$ of a neutron-rich nucleus by $R_{\text{mir}}$ of another nucleus with the same $N$ but larger $Z$. For example, as shown in Fig. 6(a), even if there are no experimental data on charge radii of $^{32}\text{Mg}$ and $^{32}\text{Ca}$, the neutron-skin thickness of $^{32}\text{Mg}$ can be evaluated by the mirror nuclei pair $^{36}\text{S}$-$^{36}\text{Ca}$ with the relatively good linear relation. By using $R_{\text{mir}}^{(36}\text{S}) = 0.156 \pm 0.005$ fm in Table 1 and the result of linear fit,

$$S_{n}(^{32}\text{Mg}) = 0.012 + 1.708R_{\text{mir}}^{(36}\text{S}),$$

one can obtain that the value of $S_{n}(^{32}\text{Mg})$ is $0.278 \pm 0.009$ fm. Similarly, using $R_{\text{mir}}$ of $^{54}\text{Fe}-^{54}\text{Ni}$, $S_{n}(^{48}\text{Ca}) = 0.119 \pm 0.014$ fm, which agrees with the latest measurement of the calcium radius experiments CREX giving $S_{n}(^{48}\text{Ca}) = 0.121 \pm 0.026(\text{exp}) \pm 0.024(\text{model})$ fm [73]. It also coincides with the results both from Ref. [74] by $104$ MeV $\alpha$ particles scattering ($S_{n}(^{48}\text{Ca}) = 0.17 \pm 0.05$ fm) and from Ref. [75] by pion scattering analyzed through using model densities in which the neutron matter distributions are considered to have two components corresponding to core and valence neutrons ($S_{n}(^{48}\text{Ca}) = 0.11 \pm 0.04$ fm).

**IV. SUMMARY**

Based on the framework of axially deformed solution of the Skyrme-Hartree-Fock-Bogoliubov equations with 132 sets of interaction parameters, we have investigated systematically the neutron and proton rms radii of different nuclei to investigate the correlation between $S_{n}$, $L$ and $R_{\text{mir}}$. It has been confirmed that the neutron-skin thickness is proportional to the difference in the proton...
radii of mirror nuclei, especially without considering the Coulomb interaction. In order to explore the effects of pairing and deformation, we compare the results with the calculations by the SHF model with 128 sets of Skyrme interaction parameters (without HFB9, UNEDF0, UNEDF1 and UNEDF2). It can be seen that with the inclusion of pairing effects, the correlation between $R_{\text{mir}}$ and $S_n$ becomes stronger for most mirror pairs, while the deformation effect seems to weaken this correlation.

By studying the 12 pairs of mirror nuclei with available experimental data on charge radii, the $^{18}\text{O}^{18}\text{Ne}$ mirror pair shows an almost ideal linear correlation between these two quantities. The neutron-skin thickness of $^{18}\text{O}$ is deduced to be $0.155 \pm 0.010$ fm, which is consistent with the experimental data. The constraints for the characteristic parameters of the EOS are also studied.

Furthermore, the correlations between $R_{\text{mir}}$ of isotones with $N = 20$ and $N = 28$ and $S_n$ of the neutron-rich nucleus with the smallest $Z$ are studied. With the increasing of $|N - Z|$, the linear correlation becomes stronger and the value of linear regression slope becomes smaller, which offers a possible way to determine $S_n$ of the unstable nucleus without experimental data of $R_{p}$. Based on this relation, $S_n^{(32}\text{Mg})$ is deduced to be $0.278 \pm 0.009$ fm through the $R_{\text{mir}}$ of $^{36}\text{S}^{16}\text{O}$ pair and $S_n^{(48}\text{Ca})$ is deduced to be $0.119 \pm 0.014$ fm through the $R_{\text{mir}}$ of $^{54}\text{Fe}^{54}\text{Ni}$ pair. This work has shown that the proton radii of mirror nuclei maybe a good observable to extract the neutron-skin thickness for some neutron-rich nuclei. To get more precise $S_n$ for unstable nuclei, further theoretical study and more experimental data of proton radii of mirror nuclei are necessary.

References


