Neutrino oscillations in the Non-Kerr black hole with quantum phenomenon*

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Abstract: In this paper, we have investigated the mathematical components of the Dirac equation in curved spacetime and how it can be applied to the analysis of neutrino oscillations. More specifically, we have developed a method for calculating the phase shift in flavor neutrino oscillations by utilizing a Taylor series expansion of the action, taking into account Δm^4 orders. In addition, we have used this method to assess how the phase difference in neutrino mass eigenstates changes due to the gravitational field described by the Johannsen spacetime.

Keywords: Dirac equation, Neutrino flavour oscillation, Curved spacetime

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I. INTRODUCTION

The field of neutrino physics observes a phenomenon called neutrino oscillation, where neutrinos change from one flavor to another while moving through space. This oscillation happens because of the interaction between the three recognized neutrino flavors: electron neutrino ν_e, muon neutrino ν_μ, and tau neutrino ν_τ. Pontecorvo [1] first proposed the concept of neutrino oscillation. It suggests that neutrinos possess a mass that was previously thought to be nonexistent. The mathematical description of neutrino oscillation involves the use of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [2], which connects the flavor states of neutrinos to their mass eigenstates. This matrix includes four parameters: three mixing angles and one phase. The mixing angles determine the likelihood of a neutrino transitioning from one flavor to another, while the phase influences the relative probabilities of oscillation between different flavors. Neutrino oscillation has significant implications in astrophysics, particle physics, and cosmology, as it aids in our understanding of neutrino properties and their role in the universe [3, 4], including their contribution to dark matter. It is a captivating phenomenon that has opened up new avenues of research in particle physics and astrophysics [5]. The discovery of neutrino oscillation has challenged our existing knowledge of neutrinos and provided valuable insights into the nature of the universe The significant breakthrough of neutrino oscillation has received recognition through various awards, notably the 2015 Nobel Prize in Physics. This prestigious honor was bestowed upon Takaaki Kajita and Arthur B. McDonald for their remarkable contributions to the Super-Kamiokande [6] and SNO experiments.

Sudbury Neutrino Observatory (SNO) [7], The Super-Kamiokande [8], and MINOS experiments [9, 10] played a vital role in enabling the discovery of neutrino oscillation. These experiments observed the phenomenon of different types of neutrinos disappearing and reappearing as they passed through the Earth's atmosphere or matter.

The investigation aims to understand the phenomenon of neutrino lensing caused by gravitational sources, which reveals an intriguing connection between the probabilities of neutrino oscillations and their individual masses. This connection is illustrated through an analysis of the impact of weak lensing induced by a Schwarzschild mass [11]. The study explores the implications of gravitationally modified neutrino oscillations in realistic scenarios involving two or three flavors, such as the influence of a supernova's gravitational field on the travel of emitted neutrinos, which could have observable ef-

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fects on the neutrino signal [12]. Furthermore, the study examines the propagation of neutrinos in a strong gravitational field regime, incorporating electromagnetic interactions using the WKB approximation [13]. It also investigates the behavior of neutrino oscillations in the Schwarzschild spacetime, taking into account spin precession in the presence of a magnetic field [14, 15]. Notably, the study analyzes both the radial and nonradial propagation of neutrinos in the Schwarzschild spacetime [16]. Finally, the investigation explores the effects of universe expansion and torsion on neutrino oscillations [17]. The mass hierarchy of neutrinos refers to how the three types of neutrinos are arranged in terms of their relative sizes. Neutrinos consist of electron, muon, and tau neutrinos, each with their own antineutrinos. Despite being light compared to other particles, neutrinos do have small but nonzero masses according to current knowledge in neutrino physics. The mass hierarchy of neutrinos can be classified as normal or inverted. In the normal hierarchy, the masses are ordered as $m_1 < m_2 < m_3$, with m_1 being the lightest, m_2 the second lightest, and m_3 the heaviest neutrino [18]. Conversely, in the inverted hierarchy, the masses are arranged as $m_3 < m_1 < m_2$. Determining the neutrino mass hierarchy is an important topic in neutrino physics as it affects various astrophysical and cosmological phenomena. Current evidence for the neutrino mass hierarchy comes from the observation of neutrino oscillations. Neutrino oscillation refers to the ability of neutrinos to change their type as they travel in space due to quantum mechanical mixing between the three types of neutrinos. The probabilities of oscillation are influenced by the differences in the squared masses of the three neutrino types and the mixing angles between them. Experiments like Super-Kamiokande and Daya Bay have provided valuable information about the neutrino mass differences and mixing angles. Based on this data, it is highly probable that the neutrino mass hierarchy is normal [19]. However, future experiments such as the Deep Underground Neutrino Experiment (DUNE) will provide more accurate measurements, allowing for a definitive determination of the mass hierarchy.

The quantum field theory of neutrinos coupled to gravity serves as the theoretical framework for studying neutrino oscillation in curved spacetime [14]. In this framework, the probabilities of oscillation depend on various factors such as neutrino energy, mass-squared differences, and the curvature of spacetime. The metric tensor describes the curvature of spacetime, which is influenced by the gravitational field as well as the distribution of matter and energy. Numerous studies have explored the effects of curved spacetime on neutrino oscillation probabilities, considering aspects like the gravitational redshift and the curvature-induced potential. These investigations have demonstrated that the gravitational field can modify oscillation probabilities, leading to potentially observable consequences. Examining neutrino oscillation in curved spacetime is an active area of research with significant implications for astrophysics and cosmology [17, 20]. The development of theoretical models that describe the quantum field theory of neutrinos coupled to gravity and the investigation of curved spacetime's impact on oscillation probabilities have been pursued in several studies (see, for example, [21]).

The rotation of spacetime under weak gravity conditions has been extensively investigated in relation to neutrino oscillations, particularly when neutrinos travel along the equatorial plane. By using the asymptotic form of the Kerr metric, it has been demonstrated that the rotation of the gravitational source significantly changes the phase of neutrinos. Specifically, when neutrinos are generated near a black hole with angular momentum and detected on the same side without the influence of gravitational lensing, the probabilities of oscillation differ greatly compared to those observed in the Schwarzschild spacetime [18]. Another study explores the effects of gravitational lensing on neutrino oscillations within the framework of the γ-spacetime, employing a quantum-mechanical approach to relativistic neutrinos [22]. This investigation examines both radial and non-radial propagation, taking into consideration the phase of neutrino oscillations within this specific spacetime. Additionally, the presence of massive objects in the universe can impact the probabilities of neutrino oscillation, which has implications for predicting the cosmic neutrino background.

II. DIRAC EQUATION IN THE CURVED SPACETIME

It is crucial to first understand the fundamental methods used for solving the Dirac equation in curved spacetime before delving into the properties of neutrinos in the presence of gravity. The Dirac equation, which governs the behavior of a massive spinor field on a torsion-free pseudo-Riemannian manifold, can be easily extended as [23, 24]

$$i\hbar\gamma^{\mu}\,\mathcal{D}_{\mu}\,\psi(x) = mc\,\psi(x)\,, \tag{1}$$

Using the relation $\mathcal{D}_{\mu} = \left(\partial_{\mu} + \Gamma_{\mu}\right)$, which is the covariant derivative for a spinor field, the equation mentioned above can be expressed as [25, 26]

$$\left[i\hbar\gamma^{\mu}\,\left(\partial_{\mu} + \Gamma_{\mu}\right) - mc\right]\psi(x) = 0\,. \tag{2}$$

where Γ_{μ} represents the spin connection and γ^{μ} is associated with the covariant Dirac matrices, which are linked to space-time through the following relations [27, 28]

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \,. \tag{3}$$

In this context, the anti-commutation operation is denoted by curly brackets, and the spin connection Γ_μ is determined by the given condition [23]

$$\frac{\partial\gamma_\nu}{\partial x^\mu} - \Gamma^\lambda_{\nu\mu}\gamma_\lambda - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0 \tag{4}$$

We will now define the constant Dirac matrices $\gamma^{(a)}$ as follows

$$\gamma^{(a)} = e^{(a)}_\mu \gamma^\mu \tag{5}$$

where $e^{(a)}_\mu$ is the orthogonal tetrad that fulfills the relationship

$$g_{\mu\nu} = e^{(a)}_\mu e^{(b)}_\nu \eta_{ab} \tag{6}$$

In the convention for the flat metric, where $\eta_{ab} = diag(-c^2,1,1,1)$, the expression for the spin connection can be written using these constant Dirac matrices [25]

$$\Gamma_\mu = \frac{\{\gamma^{(a)},\gamma^{(b)}\}}{8}\, g_{\nu\lambda} e^\nu_{(a)}\, \nabla_\mu e^\lambda_{(b)} \tag{7}$$

The action corresponding to the Dirac Equation (1) is

$$\mathcal{S} = \int d^4x\, \sqrt{g}\, \mathcal{L}_D \tag{8}$$

where g is defined as $g = g^{\mu\nu} g_{\mu\nu}$, and we can express the Lagrangian by introducing it as [29]

$$\mathcal{L}_D = \frac{i}{2}\left[\bar\psi\gamma^\mu\, \mathcal{D}_\mu\, \psi - \left(\mathcal{D}_\mu\bar\psi\right)\gamma^\mu\psi\right] - m\bar\psi\psi \tag{9}$$

III. WKB APPROXIMATION FOR DIRAC EQUATION

We are searching for an approximate solution to the Dirac Equation (1) by employing the Wentzel–Kramers–Brillouin (WKB) approximation method [30]. Several different forms of the WKB approximation have been found up to now. The investigation of the complex spinor $\psi(x)$ revealed that it can be decomposed into two components: an amplitude $\xi = \xi(x)$ and a semi-classical phase $\mathcal{S} = \mathcal{S}(x)$ as [31]

$$\psi(x) = e^{-\frac{i}{\hbar}\mathcal{S}(x)}\xi(x) \,. \tag{10}$$

After conducting certain investigations, the wave function expression in the presence of spin connection has been formulated as [32, 33, 29]

$$\psi(x) = e^{-\frac{i}{\hbar}\mathcal{S}(x)} e^{-\Gamma_\mu x^\mu} \sum_{n=0}^{\infty}\left(\frac{\hbar}{i}\right)^n \xi_n(x) \,. \tag{11}$$

and the Dirac matrix product in the spin connection term is given as [34, 27]

$$\gamma^a\{\gamma^b,\gamma^c\} = 2\eta^{ab}\gamma^c - 2\eta^{ac}\gamma^b - 2i\epsilon^{dabc}\gamma_5\gamma_d \tag{12}$$

where η^{ab} represents the metric in a flat space, while ϵ^{abcd} denotes the totally antisymmetric tensor in the same flat space. The spin connection can be expressed in terms of the matrix that violates parity

$$\Gamma_\mu = \frac{\gamma_5}{2i}\sqrt{-g}\,\mathcal{A}_\mu \tag{13}$$

where

$$\mathcal{A}_\mu = \frac{\sqrt{-g}}{4} e^\mu_a \epsilon^{abcd}\left(e_{b\nu,\sigma} - e_{b\sigma,\nu}\right) e^\nu_c e^\sigma_d \tag{14}$$

One can infer that the additional phase factor in Equation (11) effectively acknowledges the interaction between the metric and the spin orientation of the spinor.

By substituting Eq. (10) into Eq. (1) and equating terms with equal powers of $\hbar$, we obtain a series of recursive equations for the amplitudes ξ_n as

$$\left[\gamma^\nu\partial_\nu\mathcal{S}(x) + mc\right]\xi_0(x) = 0 \,, \tag{15}$$

$$\left[\gamma^\nu\partial_\nu\mathcal{S}(x) + mc\right]\xi_n(x) = \left[\gamma^\nu\partial_\nu\mathcal{S}(x) + mc\right]\xi_{n-1}(x) \tag{16}$$

Multiplying $\left[\gamma^\nu\partial_\nu\mathcal{S}(x) - m\right]$ from the left-hand side of Eq.(15), The Hamilton-Jacobi equation for a massive particle in a curved space- time can be written as [35, 36, 37]

$$g^{\mu\nu}\partial_\mu\mathcal{S}(x)\partial_\nu\mathcal{S}(x) - m^2c^2 = 0 \tag{17}$$

As long as the four-momentum of the particle is known, we can link the expression to the classical action of a particle with mass m on a torsion-free pseudo-Riemannian manifold, allowing us to associate the phase $\mathcal{S}(x)$ with this action

$$p_\mu = mg_{\mu\nu}\frac{dx^\nu}{d\tau} \tag{18}$$

If we recognize Eq. (15) and establish the identification, it becomes equivalent to the mass-shell condition

$$p_\mu = \partial_\mu \mathcal{S}(x) \tag{19}$$

and solution to Eq. (15) can be written as

$$\mathcal{S}(x) = \int p_\mu dx^\mu \tag{20}$$

The Lagrangian expression, which characterizes the geodesic motion, can be stated as $\mathcal{L} = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$, and geodesic equation can be expressed as

$$\ddot{x}^\mu + \Gamma^\nu_{\mu\sigma}\dot{x}^\mu\dot{x}^\sigma = 0\,. \tag{21}$$

A. The dynamics of neutrino spin in external fields within a curved spacetime.

The study of spin oscillations of massive Dirac neutrinos in the presence of background matter, electromagnetic fields, and gravitational fields is a complex and ongoing research area in theoretical physics. Neutrinos are fundamental particles with extremely weak interactions with matter, making their study challenging and intriguing. Let's delve deeper into some of the key aspects of studying spin oscillations of massive Dirac neutrinos in background matter, electromagnetic fields, and gravitational fields:

● Matter Effects and Neutrino Oscillations: When neutrinos propagate through a medium, such as the dense matter found in the core of a star or during the early universe, their interactions with the medium can modify their oscillation behavior. This is known as the matter effect or the MSW effect, named after the physicists who first studied it (Mikheyev, Smirnov, and Wolfenstein).

The matter effect arises due to the presence of charged particles in the medium. Neutrinos can experience forward scattering interactions with these charged particles, leading to an effective potential that depends on the neutrino flavor. As a result, the flavor oscillation probabilities of neutrinos can be significantly altered compared to their behavior in a vacuum. The matter effect can induce resonances, where the oscillation probabilities are maximally modified, leading to interesting phenomena in neutrino oscillation experiments.

● Electromagnetic Fields and Spin Precession: Neutrinos, being electrically neutral particles, do not directly interact with electromagnetic fields. However, they possess a magnetic dipole moment, which allows for an indirect interaction with magnetic fields. When neutrinos propagate through regions with magnetic fields, such as in astrophysical environments or laboratory experiments, they can experience spin precession.

Spin precession refers to the rotation of the neutrino's spin about the direction of the magnetic field. This precession can modify the flavor oscillation probabilities of neutrinos and introduce new effects that depend on the relative orientation between the neutrino's momentum, magnetic field, and direction of propagation. The study of spin precession in neutrinos requires a careful treatment of the neutrino's magnetic properties and their interactions with magnetic fields.

● Gravitational Fields and General Relativity: Neutrinos, like all particles, are influenced by gravitational fields according to the principles of general relativity. In the presence of a gravitational field, the curvature of spacetime affects the propagation of neutrinos. This can lead to modifications in their oscillation behavior and introduce additional complexities.

The gravitational interaction can cause the trajectory of neutrinos to deviate and can induce effects such as gravitational redshift and time dilation. These gravitational effects can impact the neutrino oscillation probabilities and potentially generate spin oscillations as well. The study of neutrino oscillations in the context of general relativity requires a combination of quantum field theory, general relativity, and the development of suitable theoretical frameworks.

● Experimental Probes and Future Directions: Experimental efforts play a crucial role in studying the spin oscillations of massive Dirac neutrinos in various physical environments. Neutrino oscillation experiments, conducted at particle accelerators, underground laboratories, or using astrophysical neutrino sources, provide valuable data for testing theoretical predictions and exploring the properties of neutrinos.

Future experiments, such as the Deep Underground Neutrino Experiment (DUNE) and the Jiangmen Underground Neutrino Observatory (JUNO), aim to study neutrino oscillations with higher precision and investigate matter effects, electromagnetic field interactions, and the impact of gravitational fields on neutrino behavior.

The authors of the publication in [37] have investigated the neutrino spin oscillations within external fields in curved spacetime. Their contributions have been highly valuable to the field. The researchers conducted a study on the evolution of neutrino spin in the presence of background matter and an external electromagnetic field within a curved spacetime. The primary motivation behind this study was to provide evidence supporting the validity of the quasiclassical equation governing neutrino spin evolution. They successfully derived a covariant equation for this purpose, starting from the Dirac equation that describes the interaction between a massive neutrino and external fields in a curved spacetime as

$$\left[i\hbar\gamma^\mu\,\mathcal{D}_\mu - \frac{\mu}{2}F_{\mu\nu}\sigma^{\mu\nu} - \frac{V^\mu}{2}\gamma_\mu\left(1-\gamma^5\right)\right]\psi(x) = mc\,\psi(x)\,, \quad (22)$$

where $F_{\mu\nu}$ is the Faradey tensor defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad (23)$$

and, A_μ is the 4-vector potential of the electromagnetic field. In the Eq.(15), $\mathcal{D}_\mu = \left(\partial_\mu + \Gamma_\mu\right)$ represents the covariant derivative, where Γ_μ denotes the spin connection. The symbols $\gamma^\mu = \gamma^\mu(x)$, $\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right)$, and $\gamma^5 = -\frac{i}{4!}E^{\mu\nu\alpha\beta}\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta$ refer to the coordinate-dependent Dirac matrices. Here, $E^{\mu\nu\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta}/\sqrt{-g}$ represents the covariant antisymmetric tensor in the context of curved spacetime, where $g = det(g_{\mu\nu})$ and $g_{\mu\nu}$ denotes the metric tensor. To conclude, the symbol μ represents the magnetic moment of a neutrino. The expression for $V^\mu = (V^0, \mathbf{V})$, which serves as the effective potential governing the interaction of the neutrino with arbitrarily polarized and moving matter, can be found in the reference [38].

The relationship between Eq. (15) and Eq. (1) becomes apparent when we observe that neglecting the impact of background matter and electromagnetic fields results in considering only gravitational effects. Consequently, Eq. (15) can be expressed equivalently to Eq. (1) as

$$\left[i\hbar\gamma^\mu\,\left(\partial_\mu + \Gamma_\mu\right) - mc\right]\psi(x) = 0\,. \quad (24)$$

The expression for the Dirac equation in a locally Minkowskian frame has been derived by the authors in the publication cited as [38]. Recognizing their work in this area, we believe it is unnecessary to duplicate their efforts. Instead, we can focus on obtaining the Dirac equation that specifically characterizes the interaction between a massive neutrino and external fields within a curved spacetime, tailored to our unique case.

$$\Big[i\hbar\gamma^\mu\,\partial_\mu + \frac{\hbar}{2}\gamma^\mu\gamma_5\sqrt{-g}\,\mathcal{A}_\mu - \frac{\mu}{2}F_{\mu\nu}\sigma^{\mu\nu} - \frac{V^\mu}{2}\gamma_\mu\left(1-\gamma^5\right) - mc\Big]\psi(x) = 0\,, \quad (25)$$

In Reference [39], the covariant equation governing the quasiclassical evolution of the neutrino spin, denoted as S^μ in the presence of general external fields, is derived. This derivation relies on the Heisenberg equation applied to the corresponding spin operator, taking into account the influence of the external fields. Subsequently, the equation is subjected to an averaging process over the neutrino wave packet. By employing Equation (25), we obtain the Lorentz invariant expression for the evolution equation of the neutrino spin S^μ, which accounts for the general interactions with external fields as

$$\frac{dS^\mu}{d\tau} = 2\mu\left(F^{\mu\nu}S_\nu - u^\mu F^{\lambda\rho}u_\lambda S_\rho\right) + \sqrt{2}G_F\mathcal{K}^{\mu\nu}S_\nu + \mathcal{G}^{\mu\nu}S_\nu \quad (26)$$

where $G_F = 1.17\cdot 10^{-5}GeV^{-2}$ is the Fermi constant. Additionally, the expressions for the tensors $\mathcal{K}^{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ are provided as

$$\mathcal{K}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}V_\alpha u_\beta\,,$$
$$\mathcal{G}^{\mu\nu} = \left(\gamma^{\mu\nu\lambda} + \gamma^{\lambda\mu\nu} + \gamma^{\nu\lambda\mu}\right)u_\lambda\,. \quad (27)$$

IV. NEUTRINOS

The production and detection of neutrinos occur in various flavor eigenstates, which are represented by $|\nu_\alpha\rangle$. These flavor eigenstates are combinations of mass eigenstates, represented by $|\nu_i\rangle$. Therefore, a flavor eigenstate can be expressed in terms of mass eigenstates, as discussed in references [16, 40]

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|\nu_i\rangle\,, \quad (28)$$

for a set of three neutrino flavors $\alpha = \{e, \mu, \tau\}$ and a set of three generations $i = \{1, 2, 3\}$, the Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix U, also known as the neutrino flavor mixing unitary matrix, plays a role similar to the Cabibbo-Kobayashi-Maskawa matrix in governing quark mixing. In the case of three generations of neutrinos, the MNSP matrix is characterized by three mixing angles θ_i, a phase δ that describes CP-violation, and two additional phases α_1 and α_2, which can only be non-zero if neutrinos are Majorana particles. If neutrinos are Dirac particles, then $\alpha_1 = \alpha_2 = 0$.

In the process of propagation, the neutrino is moving from source S to detector D, positioned at x_S and x_D correspondingly. The amplitude associated with the event of detecting a neutrino of flavor α at position x_S and observing it as a neutrino of flavor β at position x_D is described by

$$\mathcal{A}_{\beta\alpha} = |\langle\nu_\beta(x_D)|\nu_\alpha(x_S)\rangle| = \sum_i U^*_{\alpha i} U_{\beta i} |\langle\nu_i(x_D)|\nu_i(x_S)\rangle| \quad (29)$$

To estimate the spinor ν_i, the WKB approximation in Equation (11) incorporates both the action $\mathcal{S}_i(x)$ for the i-th mass eigenstate and the spin connection Γ_μ. Previous studies [41, 42, 43] have explored three distinct cases that give rise to neutrino oscillation: (*a*) occurring in a flat spacetime, (*b*) taking place in a curved spacetime within a non-rotating frame, and (*c*) happening in a curved spacetime within a rotating frame. In the cases (*a*) and (*b*), the phase difference of the neutrinos relies solely on $\mathcal{S}(x)$ and does not involve the spin connection, resulting in the following phase difference [44]

$$\mathcal{S}(m_i, x_D - x_S) \approx \mathcal{S}_i(x_D) - \mathcal{S}_i(x_S) = \int_{x_S}^{x_D} p_\mu dx^\mu \quad (30)$$

The presence of the parity-violating matrix γ_5 in the representation of the spin connection [14] in Eq. (13) reveals that in the case (*c*), there is an additional contribution to the phase shift when there are differences in spin orientation between the two mixing eigenstates ν_i. Moving forward, we will exclusively examine the massive neutrinos with identical spin orientations, disregarding any contributions stemming from different spin orientations. This decision is made as our research is primarily centered around neutrino flavor oscillations that occur under the influence of the action $\mathcal{S}(x)$. Consequently, we will solely consider the same spin orientation for the massive neutrinos, without taking into account any interaction between the neutrino spin and the metric that arises from Γ_μ.

The expression for the amplitude of the neutrino flavor transition can be formulated as follows:

$$\mathcal{A}_{\beta\alpha} = \sum_i U^*_{\alpha i} U_{\beta i} e^{-i\mathcal{S}(m_i, x_D - x_S)} \quad (31)$$

The difference in phase between two mass eigenstates can be written in the following form [35]

$$\Phi_{ij} = \mathcal{S}(m_i, x_D - x_S) - \mathcal{S}\left(m_j, x_D - x_S\right) \quad (32)$$

The probability of a transition in neutrino flavor, from the initially produced α flavor to the detection points, can be calculated as [37]

$$\mathcal{P}_{\beta\alpha} = \left|\mathcal{A}_{\beta\alpha}\right|^2 = \sum_{i,j} U_{\beta i} U^*_{\beta j} U_{\alpha j} U^*_{\alpha i} e^{-i\Phi_{ij}} . \quad (33)$$

It is important to note that the action assumes the form when m is small

$$\mathcal{S}(m_i, x_D - x_S) = \sum_{n=0}^{\infty} \frac{(m_i^2)^n}{n!} \mathcal{S}^{(n)}(x_D - x_S) \quad (34)$$

where

$$\mathcal{S}^{(n)}(x_D - x_S) = \frac{\partial^{(n)} \mathcal{S}(m_i, x_D - x_S)}{\partial^{(n)} m_i^2} \quad (35)$$

As a result, using the Taylor series one can express the phase difference as

$$\Phi_{ij} = \Delta m_{ij}^2 \mathcal{S}^{(1)}(x_D - x_S) + \frac{\Delta m_{ij}^4}{2} \mathcal{S}^{(2)}(x_D - x_S) + ... \quad (36)$$

where

$$\Delta m_{ij}^2 = m_i^2 - m_j^2 , \qquad \Delta m_{ij}^4 = \left(m_i^2 + m_j^2\right) \Delta m_{ij}^2 \quad (37)$$

V. TWO-FLAVOUR NEUTRINO OSCILLATIONS

The finding in Eq. (33) is applicable to all numbers of neutrino generations and neutrino energies. However, the likelihood of converting to a specific neutrino flavor is diminished in certain scenarios, including solar neutrino mixing. In these situations, the MNSP matrix [46, 47] simplifies to a member of the SO(2) group and can be represented by a single mixing angle, denoted as Θ

$$U = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \quad (38)$$

In the case of two flavors of neutrino oscillations, there is a sing\-le difference in mass, denoted as $\Delta m_{12}^2 = m_1^2 - m_2^2$, and another difference $\Delta m_{12}^4 = \left(m_1^2 + m_2^2\right) \Delta m_{12}^2$. By simplifying equation (33), the final form of the probability of neutrino oscillation can be expressed as [48, 49]

$$\mathcal{P}_{\beta\alpha} = \begin{cases} \sin^2\Theta \sin^2\Phi_{12} , & \beta \neq \alpha \\ 1 - \sin^2\Theta \sin^2\Phi_{12} , & \beta = \alpha \end{cases} \quad (39)$$

where the phase shift Φ_{12} up to second order can be determined using the equation (36) as

$$\Phi_{12} = \Delta m_{12}^2\, \mathcal{S}^{(1)}(r_D - r_S) + \frac{\Delta m_{12}^4}{2}\mathcal{S}^{(2)}(r_D - r_S) \tag{40}$$

As a consequence, the probability expression takes the form as

$$\mathcal{P}_{\beta\alpha} = \begin{cases} \sin^2\Theta \sin^2\left[\mathcal{S}^{(1)}(r_D - r_S)\Delta m_{12}^2 \right. \\ \left. \quad + \frac{1}{2}\mathcal{S}^{(2)}(r_D - r_S)\Delta m_{12}^4\right], \quad \beta \neq \alpha \\ 1 - \sin^2\Theta \sin^2\left[\mathcal{S}^{(1)}(r_D - r_S)\Delta m_{12}^2 \right. \\ \left. \quad + \frac{1}{2}\mathcal{S}^{(2)}(r_D - r_S)\Delta m_{12}^4\right], \quad \beta = \alpha \end{cases} \tag{41}$$

and here, we can take the expressions for $\mathcal{S}^{(1)}(r_D - r_S)$ and $\mathcal{S}^{(2)}(r_D - r_S)$, which are represented by (53) and (54), correspondingly and the given expression (41) is a general formula used to calculate the probability of a neutrino undergoing flavor transition as it propagates from its source to a receiver.

VI. JOHANNSEN SPACETIME

We are currently concentrating on evaluating the Johannsen spacetime, which is a more general version of the Kerr spacetime and can be defined using the following metric [50]

$$\begin{aligned} ds^2 = &-\frac{\tilde{\Sigma}\left(\Delta - a^2 A_2^2(r)\sin^2\theta\right)}{B^2}dt^2 + \frac{\tilde{\Sigma}}{A_5(r)\Delta}dr^2 + \frac{\tilde{\Sigma}}{A_6(\theta)}d\theta^2 - \\ &-\frac{2a\left[\left(r^2+a^2\right)A_1(r)A_2(r) - \Delta\right]\tilde{\Sigma}\sin^2\theta}{B^2}dt d\phi + \\ &+\frac{\left[\left(r^2+a^2\right)^2 A_1^2(r) - a^2\Delta\sin^2\theta\right]\tilde{\Sigma}\sin^2\theta}{B^2}d\phi^2 , \end{aligned} \tag{42}$$

where

$$\begin{aligned} &B = A_1(r)A_3(\theta)(r^2+a^2) - A_2(r)A_4(\theta)a^2\sin^2\theta \\ &\tilde{\Sigma} = \Sigma + f(r) + g(\theta), \quad \Delta = r^2 - 2Mr + a^2, \\ &\Sigma = r^2 + a^2\cos^2\theta \end{aligned} \tag{43}$$

To be clear, generally speaking, it is not possible to locate the stationary points of the function's effective potential within the given background spacetime (42). Nevertheless, if we make a specific selection of the profile functions, it may become feasible

$$\begin{aligned} &A_1(r) = 1 + \sum_{k=3}^{\infty}\alpha_{1k}\left(\frac{M}{r}\right)^k, \qquad A_2(r) = 1 + \sum_{k=2}^{\infty}\alpha_{3k}\left(\frac{M}{r}\right)^k, \\ &A_5(r) = 1 + \sum_{k=2}^{\infty}\alpha_{5k}\left(\frac{M}{r}\right)^k, \qquad f(r) = r^2\sum_{k=3}^{\infty}\epsilon_k\left(\frac{M}{r}\right)^k, \\ &A_3(\theta) = A_4(\theta) = A_6(\theta) = 1, \qquad g(\theta) = 0, \end{aligned} \tag{44}$$

The equatorial plane at $\theta_0 = \pi/2$ is where the stationary points of the function effective potential can be found. It is crucial to highlight that the Johannsen spacetime is distinguished by a set of parameters, namely α_{1k}, α_{3k}, α_{5k}, and ϵ_k, which vary based on the mass and spin of the black hole. It is important to emphasize that the key features of this parametrization are: (a) the metric retains its smoothness at all points both inside and outside the event horizon, and (b) it has been convincingly demonstrated that it can accurately reproduce certain black hole solutions in alternative theories of gravity by appropriately choosing the deformation parameters [51]. In addition, the spacetime is derived by enforcing the separability of the Hamilton-Jacobi equations, despite lacking a theoretical basis for doing so. It is worth noting that alternative theories of gravity exist where non-Kerr black hole solutions do not meet this condition. However, maintaining separability may aid in certain calculations. This spacetime model finds application in conducting phenomenological calculations in the field of black hole astrophysics [52, 53].

The equation for massive neutrinos, known as the Hamilton-Jacobi equation, can be expressed as [54]

$$-2\frac{\partial S}{d\tau} = g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} \tag{45}$$

Using the metric geometry (42), Hamilton-Jacobi equation (45) take the form as

$$\begin{aligned} -2\frac{\partial S}{d\tau} = g^{tt}\left(\frac{\partial S_t}{\partial t}\right)^2 + 2g^{t\phi}\frac{\partial S_t}{\partial t}\frac{\partial S_\phi}{\partial \phi} + g^{\phi\phi}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 + \\ + g^{rr}\left(\frac{\partial S_r}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 \end{aligned} \tag{46}$$

The given form of the Hamilton-Jacobi function is as follows [50]

$$\mathcal{S} = \frac{1}{2}m^2\tau - Et + L\phi + \mathcal{S}_r + \mathcal{S}_\theta \tag{47}$$

After performing certain algebraic calculations, the HJ equation can be separated in the form of

$$\frac{1}{\Delta}\left[(r^2+a^2)A_1(r)E-aA_2(r)L\right]^2 -m^2\left[r^2+f(r)\right]-A_5(r)\Delta\left(\frac{\partial S_r}{\partial r}\right)^2 =\left(\frac{\partial S_\theta}{\partial\theta}\right)^2+\left(\frac{A_3(\theta)L}{\sin\theta}-aA_4(\theta)E\sin\theta\right)^2+m^2a^2\cos^2\theta\,, \tag{48}$$

The same constant K which is well-known as the Carter constant can be assigned to both sides of the equation

$$\left(\frac{\partial S_\theta}{\partial\theta}\right)^2=K-\left(\frac{A_3(\theta)L}{\sin\theta}-aA_4(\theta)E\sin\theta\right)^2-m^2a^2\cos^2\theta\,,$$
$$\left(\frac{\partial S_r}{\partial r}\right)^2=\frac{\left[(r^2+a^2)A_1(r)E-aA_2(r)L\right]^2-\Delta\left[K-m^2\left(r^2+f(r)\right)\right]}{A_5(r)\Delta^2} \tag{49}$$

It is possible to achieve motion on a plane with a fixed angle $\theta=\theta_0$ by selecting the indicated angle

$$K=\left(\frac{A_3(\theta_0)L}{\sin\theta_0}-aA_4(\theta_0)E\sin\theta_0\right)^2+m^2a^2\cos^2\theta_0 \tag{50}$$

If that's the case, the action is interpreted as

$$\mathcal{S}=-Et+L\phi+\int\frac{\sqrt{R(r)}}{\Delta}dr \tag{51}$$

where

$$R(r)=\frac{\left[(r^2+a^2)A_1(r)E-aA_2(r)L\right]^2-\Delta\left[K-m^2\left(r^2+f(r)\right)\right]}{A_5(r)} \tag{52}$$

As mentioned in Equation (34), if we consider the action up to the second order in m^2 and expand it from r_S (source distance) to r_D (detector distance) as

$$\mathcal{S}^{(1)}(r_D-r_S)=\int_{r_S}^{r_D}\frac{\left(r^2+f(r)\right)\sqrt{A_5(r)}}{2\sqrt{((r^2+a^2)A_1(r)E-aA_2(r)L)^2-\Delta K}}dr \tag{53}$$

and

$$\mathcal{S}^{(2)}(r_D-r_S)=\int_{r_S}^{r_D}\frac{\left(r^2+f(r)\right)^2A_5^{3/2}(r)}{4\left[((r^2+a^2)A_1(r)E-aA_2(r)L)^2-\Delta K\right]^{3/2}}dr \tag{54}$$

The Johannsen spacetime reduces to the Kerr spacetime when parameterizations in Eq. (44) are not taken into account. The expression for $\mathcal{S}^{(1)}$ becomes the same as the one obtained by the authors of [36], where the first-order result for neutrino oscillations in a Kerr metric is derived. Now, in the upcoming section, we will explore the process of determining the probability for the two-flavor case of neutrino oscillations.

A. Radial propagation

The reason why purely radial motion is not possible in the Kerr spacetime is due to the influence of the black hole's rotation, which introduces a phenomenon known as frame-dragging. Frame-dragging is a consequence of the spacetime curvature caused by the rotating black hole. In general relativity, the presence of mass or energy curves the surrounding spacetime, affecting the motion of objects within it. In the case of a rotating black hole described by the Kerr spacetime, the rotation creates a twisting or dragging effect on the nearby spacetime. When a test particle moves radially, aiming directly toward or away from the black hole, it is still subject to the curvature of spacetime caused by the rotating black hole. This curvature, combined with the rotation, causes the test particle to experience an additional angular momentum component. As a result, the particle's trajectory deviates from a purely radial path, leading to a combination of radial and angular motion. Think of it as if the rotating black hole "drags" the nearby spacetime around with it, causing objects to be influenced by the rotation even if they are initially moving radially. This effect prevents purely radial motion in the Kerr spacetime and results in a spiraling or helical trajectory for the test particle. Therefore, in the Kerr spacetime, a test particle cannot move along purely radial trajectories. Due to the curvature of spacetime caused by the rotating black hole, the particle will experience a combination of radial and angular motion, even if it initially moves radially.

The Johannsen spacetime, on the other hand, is a modification of the Kerr spacetime that introduces additional parameters to account for potential deviations from the standard Kerr geometry. The specific effects of these deviations on the motion of test particles would depend on the particular form of the Johannsen metric and the values of the parameters. Therefore, it is possible that in certain cases or parameter regimes, the Johannsen spacetime may allow for different types of motion, including potentially radial motion. The specific effects of these deviations on the motion of test particles would depend on the particular form of the Johannsen metric and the values of the parameters. Without knowing the specific form of the Johannsen metric and the parameter values, it is challenging to provide a definitive answer regarding the existence or nature of purely radial motion in the Johannsen spacetime. The additional parameters introduced in the Johannsen metric could potentially alter the gravitational field in such a way that purely radial motion becomes possible. However, it is also possible that the additional forces or curvature effects introduced by the deviations from the Kerr geometry would still prevent purely radial motion.

The specific form of the Johannsen metric, which incorporates these additional parameters, determines the exact nature of the deviations from the Kerr spacetime and how they affect the motion of test particles. Depending on the values of these parameters, it is conceivable that the additional forces or curvature effects introduced in the Johannsen spacetime could allow for purely radial motion in certain cases. For example, if the additional parameters in the Johannsen metric introduce modifications to the gravitational field that counteract or weaken the frame-dragging effect caused by the black hole's rotation, it could potentially allow for purely radial motion. This could occur if the additional parameters change the geometry of spacetime in such a way that the angular momentum component induced by the rotation becomes negligible or is canceled out. The characterization of Φ_{ij}^m will be influenced by the geodesic parameters, including energy and angular momentum. Next, we consider the scenario of radial propagation with $L=K=0$, wherein we find

$$\mathcal{S}=-Et+\int\frac{\sqrt{R_*(r)}}{\Delta}dr \tag{55}$$

where

$$R_*(r)=\frac{(r^2+a^2)^2A_1^2(r)E^2+m^2\Delta\left(r^2+f(r)\right)}{A_5(r)} \tag{56}$$

By expanding the action to its second order in m^2, we obtain

$$\mathcal{S}_*^{(1)}(r_D-r_S)=\frac{1}{2E}\int_{r_S}^{r_D}\frac{\left(r^2+f(r)\right)\sqrt{A_5(r)}}{A_1(r)(r^2+a^2)}dr \tag{57}$$

and

$$\mathcal{S}_*^{(2)}(r_D-r_S)=\frac{1}{4E^3}\int_{r_S}^{r_D}\frac{\Delta\left(r^2+f(r)\right)^2A_5^{3/2}(r)}{A_1^3(r)(r^2+a^2)^3}dr \tag{58}$$

As a consequence of Equation (41), the probability expression for radial propagation assumes the following form

$$\mathcal{P}_{\beta\alpha}=\begin{cases}\sin^2\Theta\sin^2\left[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2\right.\\ \quad\left.+\frac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\right], & \beta\neq\alpha\\ 1-\sin^2\Theta\sin^2\left[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2\right.\\ \quad\left.+\frac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\right], & \beta=\alpha\end{cases} \tag{59}$$

Consequently, from these equations, it becomes evident that in the case of radial propagation with $L=K=0$, the contribution of ϕ to the phase of oscillations is not discernible. This implies that the phase of oscillations remains independent of ϕ. However, it's important to note that the specific implications of the Johannsen metric on test particle dynamics and the conditions under which purely radial motion might be allowed would depend on the precise form of the metric and the values of the additional parameters. The exploration of these effects and conditions would require detailed analysis and numerical simulations specific to the Johannsen metric [52]. Numerical simulations are used to study the behavior of the Johannsen metric and its effect on various physical phenomena. By simulating the motion of test particles, electromagnetic fields, or other relevant quantities in the Johannsen spacetime, they can compare the results with observations or theoretical predictions [53]. These simulations can help constrain the values of the additional parameters that best match the observed data or desired physical behavior.

Subsequently, conducting a comparative analysis of the outcomes obtained from the Schwarzschild, Kerr, and Johannsen spacetimes would be of great interest. The Johannsen metric is determined by the mass (M) and spin (a) of the black hole, along with four independent functions that account for potential deviations from the Kerr solution. When $\alpha_{1k}=\alpha_{3k}=\alpha_{5k}=\epsilon_k=0$, the metric simplifies to the Kerr solution [50]. For the sake of simplicity in this study, we will specifically examine two scenarios: one where only α_{52} is non-zero, and another where only ϵ_3 is set to zero. Consequently, we can derive the expressions for $\mathcal{S}_*^{(1)}(r_D-r_S)$ and $\mathcal{S}_*^{(2)}(r_D-r_S)$ as follows:

$$\begin{aligned}\mathcal{S}_*^{(1)}(r_D-r_S)=&\frac{1}{2E}\cdot\frac{1}{a^3\cdot M^2\cdot r_D^2}\cdot\left[a\cdot M^2\cdot\sqrt{\alpha_{52}\cdot M^2+r_D^2}\cdot\left(a^2\cdot r_D-\epsilon_3\cdot M^3\right)+\epsilon_3\cdot M^5\cdot r_D\cdot\sqrt{\alpha_{52}M^2-a^2}\cdot\cot^{-1}\right.\\&\left.\left(\frac{a\cdot\sqrt{\alpha_{52}\cdot M^2-a^2}}{a^2-r_D\cdot\sqrt{\alpha_{52}\cdot M^2+r_D^2}+r_D^2}\right)\right]-\frac{1}{2E}\cdot\frac{1}{a^3\cdot M^2\cdot r_S^2}\cdot\left[a\cdot M^2\cdot\sqrt{\alpha_{52}\cdot M^2+r_S^2}\cdot\left(a^2\cdot r_S-\epsilon_3\cdot M^3\right)\right.\\&\left.+\epsilon_3\cdot M^5\cdot r_S\cdot\sqrt{\alpha_{52}M^2-a^2}\cdot\cot^{-1}\left(\frac{a\cdot\sqrt{\alpha_{52}\cdot M^2-a^2}}{a^2-r_S\cdot\sqrt{\alpha_{52}\cdot M^2+r_S^2}+r_S^2}\right)\right].\end{aligned} \tag{60}$$

and

$$
\begin{aligned}
\mathcal{S}_*^{(2)}(r_D-r_S) &= \frac{1}{4E^3}\cdot\frac{r_D^2\sqrt{\left(\alpha_{52}\cdot M^2+r_D^2\right)}}{8\left(\epsilon_3\cdot M^3+r_D^3\right)^2}\cdot\left(\frac{\epsilon_3\cdot M^3}{r_D}+r_D^2\right)^2\cdot\left[8+\frac{12\left(a^2+3Mr_D-\epsilon_3 M^2\right))}{3\cdot\left(a^2+r_D^2\right)\cdot}\right.\\
&-\frac{M^3\cdot(\alpha_{52}\cdot r_D+2\epsilon_3\cdot(M-2r_D))}{3\cdot a^2\cdot\left(a^2+r_D^2\right)}-\frac{2\cdot\epsilon_3\cdot M^5\cdot(2\alpha_{52}\cdot(2M+r_D)+\epsilon_3\cdot M)}{3\cdot a^4\cdot\left(a^2+r_D^2\right)}\\
&+\frac{2a^2\epsilon_3^2M^7(\alpha_{52}M+3r_D)-11\alpha_{52}\epsilon_3^2M^9r_D}{3a^8\left(a^2+r_D^2\right)}-\frac{16M\log\left(\sqrt{\alpha_{52}M^2+r_D^2}+r_D\right)}{\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}+\frac{1}{a^9\cdot\sqrt{a^2-\alpha_{52}\cdot M^2}\cdot\left(\alpha_{52}\cdot M^2+r_D^2\right)^{1/2}}\\
&+\frac{\epsilon_3\cdot M^5\cdot\log(r_D)\cdot\left(3\epsilon_3a^4-8\alpha_{52}a^2M^2(4\alpha_{52}+3\epsilon_3)+24\alpha_{52}^2\epsilon_3M^4\right)}{a^8\cdot\sqrt{\alpha_{52}}\cdot\left(a_{52}\alpha_{52}M^2+r_D^2\right)^{1/2}}\\
&+\frac{\epsilon_3\cdot M^5\cdot\left(8a^2\alpha_{52}M^2(4\alpha_{52}+3\epsilon_3)-3\epsilon_3a^4-24\alpha_{52}^2\epsilon_3M^4\right)\cdot\log\left(\sqrt{\alpha_{52}}\cdot\sqrt{\alpha_{52}M^2+r_D^2}+a_{52}\cdot M\right)}{a^8\sqrt{\alpha_{52}}\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}\\
&-\frac{12M\left(a^2-\alpha_{52}M^2\right)\left(a^6r_D-2a^4\epsilon_3M^3-\epsilon_3^2M^6r_D\right)}{\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}-\frac{16\epsilon_3M^5\left(3a^4\alpha_{52}-4a^2\epsilon_3M^2+9\alpha_{52}\epsilon_3M^4\right)}{a^8r_D\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}\\
&\left.+\frac{3\epsilon_3^2M^6\left(8\alpha_{52}M^2-5a^2\right)}{a^6r_D^2\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}-\frac{6\alpha_{52}\epsilon_3^2M^8}{a^4r_D^4\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}+\frac{16\alpha_{52}\epsilon_3^2M^9}{a^6r_D^3\left(\alpha_{52}M^2+r_D^2\right)^{1/2}}\right]\\
&-\frac{1}{4E^3}\cdot\frac{r_S^2\sqrt{\left(\alpha_{52}\cdot M^2+r_S^2\right)}}{8\left(\epsilon_3\cdot M^3+r_S^3\right)^2}\cdot\cdot\left(\frac{\epsilon_3\cdot M^3}{r_S}+r_S^2\right)^2\cdot\left[8+\frac{12\left(a^2+3Mr_S-\epsilon_3M^2\right))}{3\cdot\left(a^2+r_S^2\right)\cdot}\right.\\
&+\frac{2a^2\epsilon_3^2M^7(\alpha_{52}M+3r_S)-11\alpha_{52}\epsilon_3^2M^9r_S}{3a^8\left(a^2+r_S^2\right)}-\frac{M^3\cdot(\alpha_{52}\cdot r_S+2\epsilon_3\cdot(M-2r_S))}{3\cdot a^2\cdot\left(a^2+r_S^2\right)}\\
&-\frac{2\cdot\epsilon_3\cdot M^5\cdot(2\alpha_{52}\cdot(2M+r_S)+\epsilon_3\cdot M)}{3\cdot a^4\cdot\left(a^2+r_S^2\right)}-\frac{16M\log\left(\sqrt{\alpha_{52}M^2+r_S^2}+r_S\right)}{\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}+\frac{1}{a^9\cdot\sqrt{a^2-\alpha_{52}\cdot M^2}\cdot\left(\alpha_{52}\cdot M^2+r_S^2\right)^{1/2}}\\
&+\frac{\epsilon_3\cdot M^5\cdot\log(r_S)\cdot\left(3\epsilon_3a^4-8\alpha_{52}a^2M^2(4\alpha_{52}+3\epsilon_3)+24\alpha_{52}^2\epsilon_3M^4\right)}{a^8\cdot\sqrt{\alpha_{52}}\cdot\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}\\
&+\frac{\epsilon_3\cdot M^5\cdot\left(8a^2\alpha_{52}M^2(4\alpha_{52}+3\epsilon_3)-3\epsilon_3a^4-24\alpha_{52}^2\epsilon_3M^4\right)\cdot\log\left(\sqrt{\alpha_{52}}\cdot\sqrt{\alpha_{52}M^2+r_S^2}+\alpha_{52}\cdot M\right)}{a^8\sqrt{\alpha_{52}}\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}\\
&-\frac{12M\left(a^2-\alpha_{52}M^2\right)\left(a^6r_S-2a^4\epsilon_3M^3-\epsilon_3^2M^6r_S\right)}{\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}-\frac{16\epsilon_3M^5\left(3a^4a_{52}-4a^2\epsilon_3M^2+9\alpha_{52}\epsilon_3M^4\right)}{a^8r_S\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}\\
&\left.+\frac{3\epsilon_3^2M^6\left(8\alpha_{52}M^2-5a^2\right)}{a^6r^2\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}-\frac{6\alpha_{52}\epsilon_3^2M^8}{a^4r_S^4\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}+\frac{16\alpha_{52}\epsilon_3^2M^9}{a^6r_S^3\left(\alpha_{52}M^2+r_S^2\right)^{1/2}}\right].
\end{aligned}
\tag{61}
$$

As a result, the probability expression for radial propagation in Johannsen spacetime takes on the following form

$$
\mathcal{P}_{\beta\alpha}^{Johannsen}=\begin{cases}\sin^2\Theta\sin^2\left[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2\right.\\ \quad\left.+\dfrac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\right], & \beta\neq\alpha\\ 1-\sin^2\Theta\sin^2\left[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2\right.\\ \quad\left.+\dfrac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\right], & \beta=\alpha\end{cases}
\tag{62}
$$

When $a_{52}=\epsilon_3=0$, the metric (42) is simplified and reduces to the Kerr solution. Consequently, the expressions for $\mathcal{S}*^{(1)}(r_D-r_S)$ and $\mathcal{S}*^{(2)}(r_D-r_S)$ also reduce in a manner similar to that observed in Kerr spacetime [36, 18, 29]

$$
\mathcal{S}_*^{(1)}(r_D-r_S)=\frac{1}{2E}\cdot\left[r_D-r_S-a\left(\arctan\frac{r_D}{a}-\arctan\frac{r_S}{a}\right)\right].
\tag{63}
$$

and

$$\mathcal{S}_*^{(2)}(r_D - r_S) = \frac{1}{8E^3}\cdot\Bigg[2(r_D - r_S) - 3a\left(\arctan\frac{r_D}{a} - \arctan\frac{r_S}{a}\right) + \frac{a^2(r_D-4M)}{a^2+r_D^2} - \frac{a^2(r_S-4M)}{a^2+r_S^2} - 2M\ln\left(\frac{a^2+r_D^2}{a^2+r_S^2}\right) + \frac{a^4M}{(a^2+r_D^2)^2} - \frac{a^4M}{(a^2+r_S^2)^2}\Bigg] \tag{64}$$

As a result, the probability expression for radial propagation in Kerr spacetime takes on the following form

$$\mathcal{P}_{\beta\alpha}^{Kerr} = \begin{cases} \sin^2\Theta\sin^2\Big[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2 + \frac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\Big], & \beta\neq\alpha \\ 1-\sin^2\Theta\sin^2\Big[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2 + \frac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\Big], & \beta=\alpha \end{cases} \tag{65}$$

If we set a_{52}, ϵ_3, and a to zero, the metric given by equation (42) is simplified and we obtain the Schwarzschild solution. Consequently, the expressions for $\mathcal{S}_*^{(1)}(r_D-r_S)$ and $\mathcal{S}_*^{(2)}(r_D-r_S)$ also simplify, resulting in the recovery of the same result as in Schwarzschild spacetime [16, 11]

$$\mathcal{S}_*^{(1)}(r_D-r_S) = \frac{1}{2E}\cdot(r_D-r_S)\,. \tag{66}$$

and

$$\mathcal{S}_*^{(2)}(r_D-r_S) = \frac{1}{4E^3}\cdot\left[r_D-r_S-2M\cdot\ln\frac{r_D}{r_S}\right]. \tag{67}$$

As a result, the probability expression for radial propagation in Schwarzschild spacetime takes on the following form

$$\mathcal{P}_{\beta\alpha}^{Schwrzd} = \begin{cases} \sin^2\Theta\sin^2\Big[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2 + \frac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\Big], & \beta\neq\alpha \\ 1-\sin^2\Theta\sin^2\Big[\mathcal{S}_*^{(1)}(r_D-r_S)\Delta m_{12}^2 + \frac{1}{2}\mathcal{S}_*^{(2)}(r_D-r_S)\Delta m_{12}^4\Big], & \beta=\alpha \end{cases} \tag{68}$$

VII. PROPER DICTANCE

The neutrino propagates across its proper distance, while dr is simply a coordinate. The proper distance can be expressed as [14]

$$dL_p = \sqrt{\left(\frac{g_{0\nu}g_{0\mu}}{g_{00}} - g_{\nu\mu}\right)dx^\nu dx^\mu} \tag{69}$$

In the context of Johannsen spacetime, there exists

$$dL_p = \sqrt{-g_{rr}dr^2 + \left(\frac{g_{t\phi}^2}{g_{tt}} - g_{\phi\phi}\right)d\phi^2} \tag{70}$$

The following expression is obtained by multiplying dr^2

$$L_p = \int_{r_S}^{r_D}\sqrt{-g_{rr} + \left(\frac{g_{t\phi}^2}{g_{tt}} - g_{\phi\phi}\right)\frac{\dot\phi^2}{\dot r^2}}\,dr \tag{71}$$

With the use of four-velocity normalization, $g_{\mu\nu}\dot x^\mu\dot x^\nu = -1$, and considering the aforementioned expressions, $\dot r$ and $\dot\phi$ can be expressed as

$$\dot r = \frac{A_5(r)}{m\tilde\Sigma}\sqrt{R(r)} = \frac{\sqrt{A_5(r)}}{m\tilde\Sigma}\sqrt{\left[(r^2+a^2)A_1(r)E - aA_2(r)L\right]^2 - \Delta\left[K - m^2(r^2+f(r))\right]}\,,$$
$$\dot\phi = \frac{L}{\tilde\Sigma\sin^2\theta}\frac{\left[A_1(r)(r^2+a^2) - A_2(r)a^2\sin^2\theta\right]^2}{\left[(r^2+a^2)^2A_1^2(r) - a^2\Delta\sin^2\theta\right]} \tag{72}$$

It is apparent that performing an analytical calculation of Eq. (71) is more complex.

VIII. FINDINGS AND PROSPECTS FOR THE FUTURE

In this work, we delved into the mathematical aspects of the Dirac equation within a curved space-time and investigated its application in analyzing neutrino oscillations. To achieve this objective we have used the WKB approximation. In particular, we devised a technique for determining the phase shift in flavor neutrino oscillations by employing a Taylor series expansion of the action, considering contributions up to fourth order in Δm^4. In section 4, we examined the intricate dynamics of transition probabilities within our framework, revealing their intricate nature despite fluctuations in mass representation. Furthermore, this method has been employed to evaluate the variation in the phase difference of neutrino mass eigenstates caused by the gravitational field explained by the Johannsen spacetime.

We know that the phenomenon of neutrino oscillation, which is a quantum phenomenon, takes place in both flat and curved spacetime metrics. It is anticipated that the presence of massive objects like stars and black holes, with their gravitational fields, can impact the way neutrinos propagate and alter their oscillation patterns. Research into neutrino oscillation in curved spacetime is currently being actively pursued, and its findings hold significant implications for the fields of astrophysics and cosmology. Furthermore, it is widely acknowledged that the application of gravitational lensing methods offers compelling proof of the presence of Dark Matter. According to the principles of general relativity, the trajectory of the light can be bent when encountering massive objects or gravitational fields. This deflection is closely linked to the mass of the object and can be likened to the focusing effect of a lens. In our upcoming research, we will investigate the gravitational lensing effect on neutrino oscillations to provide evidence of the existence of dark matter.

We would like to emphasize the significant progress made by the authors of the publication in [38] in investigating the intricacies of neutrino spin oscillations within a curved spacetime, particularly in the presence of background matter and an external electromagnetic field. The authors have successfully derived the Dirac equation that governs neutrino oscillations under these external field conditions. They have also conducted numerical solutions to explore the behavior of these oscillations. The utilization of numerical simulations and the analysis of experimental data are crucial in comprehending the behavior of neutrinos in realistic scenarios. Neutrino oscillation experiments, such as those conducted at particle accelerators or involving atmospheric and solar neutrinos, provide invaluable data for testing theoretical predictions and gaining insights into the spin oscillations of massive Dirac neutrinos in diverse physical environments. Their work would require a significant investment of time and extensive numerical calculations due to the complex nature of studying spin oscillations of massive Dirac neutrinos in the presence of background matter, electromagnetic fields, and gravitational fields. Looking ahead to our future papers, our objective is to explore and uncover the potential for spin precession in the presence of background matter and an external electromagnetic field within a curved spacetime. The derivation of the neutrino spin evolution equation provided in this context relies on the general spin evolution equation in the Heisenberg representation. Adopting this approach enables us to carefully analyze the contributions of various external fields mentioned earlier to the evolution of neutrino spin.

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