Note on single-trace EYM amplitudes with MHV configuration

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Abstract: In the maximally-helicity-violating (MHV) configuration, tree-level single-trace Einstein-Yang-Mills (EYM) amplitude with one and two gravitons have been shown to satisfy a formula where each graviton splits into a pair of collinear gluons. In this paper, we extend this formula to more general cases. We provide a general formula which expresses tree-level single-trace MHV amplitudes in terms of pure gluon amplitudes, where each graviton turns into a pair of collinear gluons.

Keywords: Scattering Amplitudes, MHV amplitudes, collinear limit

DOI: **CSTR:**

I. INTRODUCTION

In four dimensional spacetime, tree-level single-trace maximally-helicity-violating (MHV) amplitudes of Einstein-Yang-Mills (EYM) theory have been shown to satisfy the Selivanov-Bern-De Freitas-Wong [1−3] (SBDW) formula, which expresses the amplitude via a generating function. On another hand, Cachazo-He-Yuan (CHY) [4−6] formula gives a general approach to EYM amplitudes, which is independent of the dimension of spacetime and the helicity configuration. In four dimensions, the CHY formula has been shown to provide a *spanning forest formula* (first proposed in gravity, along the line of [7], [8] and [9]) for the single-trace MHV amplitude [10], which was further proven to be equivalent with the SBDW formula [10] and was generalized to double-trace MHV amplitudes [11] via the recursion expansion formula [12−16].

From another perspective, as pointed out in earlier literatures [17−21], each graviton in an EYM amplitude could be considered as a pair of collinear gluons which carry the same momentum and the same helicity. Particularly, inspired by the SBDW formula, [18] pointed out that the single-trace MHV amplitude with one and two gravitons can be explicitly expressed in terms of the MHV amplitudes where each graviton splits into a pair of collinear gluons [18]. This explicit formula of the single-trace MHV amplitudes was not extended into cases with an arbitrary number of gravitons yet. In this note, we take a small step forward in this direction: *we provide a general formula for single-trace MHV amplitudes where each graviton splits into a pair of collinear gluons.* When the number of gravitons is one or two, this formula turns back into the known results [18]. We hope this approach may provide a new insight for the study of helicity amplitudes in EYM.

The structure of this note is arranged as follows. In section 2, a helpful review of spinor-helicity formalism and the SBDW formula is presented. We study the amplitude with three gravitons in section 3 and sketch the general proof in section 4. Further discussions and conclusions are presented in section 5.

II. BACKGROUNDS

In this section, we provide a brief review of the spinor-helicity formalism in four dimensions [22], as well as the SBDW [1−3] formula and the spanning forest formula [10] for single-trace EYM amplitudes.

A. Spinor-helicity formalism in four dimensions

The momentum k_i^μ of each on-shell massless particle i is expressed by two copies of Weyl spinors $\lambda_i^a\widetilde{\lambda}_i^{\dot{a}}$. We define the spinor products as

$$\langle i,j\rangle \equiv \epsilon_{ab}\lambda_i^a\lambda_j^b, \qquad [i,j] \equiv \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_i^{\dot{a}}\widetilde{\lambda}_j^{\dot{b}},$$

where ϵ_{ab} and $\epsilon_{\dot{a}\dot{b}}$ are totally antisymmetric tensors. Apparently, the spinor products are antisymmetric objects under the exchanging of the two spinors. With this expression, the Lorentz contraction of two momenta k_a^μ and

Received 12 February 2025; Accepted 18 March 2025

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k_b^μ reads:

$$k_a \cdot k_b = \frac{1}{2}\langle a,b\rangle [b,a].$$

More helpful properties in spinor-helicity formalsim are displayed as follows.

- Momentum conservation for an n-point amplitude:

$$\sum_{\substack{i\neq j,k\\ i=1}}^{n} [j,i]\langle i,k\rangle = 0.$$

- Schouten identity:

$$\langle a,b\rangle\langle c,d\rangle = \langle a,c\rangle\langle b,d\rangle + \langle b,c\rangle\langle d,a\rangle,$$
$$[a,b][c,d] = [a,c][b,d] + [b,c][d,a].$$

- The eikonal identity resulted by Schouten identity

$$\sum_{i=j}^{k-1} \frac{\langle i,i+1\rangle}{\langle i,q\rangle\langle q,i+1\rangle} = \frac{\langle j,k\rangle}{\langle j,q\rangle\langle q,k\rangle}. \quad (1)$$

Finally, the n-gluon MHV amplitude $A(1,...,n)$ at tree level satisfies the famous Parke-Taylor formula [23][1)]:

$$A(1,...,n) \sim \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle},$$

where i, j denote the two negative-helicity gluons and other gluons are supposed to be positive-helicity ones.

B. SBDW formula and the spanning forest formula

In EYM, there are two possible situations of the tree-level single-trace MHV amplitudes: the (g^-,g^-) and (h^-,g^-) configurations, which correspond to amplitudes with two negative-helicity gluons, and one negative-helicity gluon plus one negative-helicity graviton. In the following, we focus on the (g^-,g^-) configuration. The (h^-,g^-) can be studied similarly.

The SBDW [1–3] formula expresses the single trace (g^-g^-)-MHV amplitude $A(1,...,i,...,j,...,N|\mathrm{H})$ in EYM as:

$$A(1,...,i,...,j,...,N|\mathrm{H}) \sim \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle N1\rangle} S(i,j,\mathrm{H},\{1,...,N\}) \quad (2)$$

where $1,...,N$ are gluons arranged in a fixed ordering, $\mathrm{H}=\{n_1,...,n_M\}$ are gravitons which are independent of color orderings. The negative-helicity gluons are supposed to be i and j. The $S(i,j,\mathrm{H},\{1,...,N\})$ factor is generated by an exponential generating function, particularly

$$S(\mathrm{H};\{1,...,N\}) = \left(\prod_{m\in \mathrm{H}} \frac{d}{da_m}\right) \exp\Bigg[\sum_{n_1\in \mathrm{H}} a_{n_1} \sum_{l\in \mathrm{G}} \psi_{ln_1} \times \exp\Bigg[\sum_{n_2\in \mathrm{H}, n_2\neq n_1} a_{n_2}\psi_{n_1n_2}\exp(...)\Bigg]\Bigg]\Bigg|_{a_m=0}, \quad (3)$$

in which

$$\psi_{ab} \equiv \frac{[ab]\langle a\xi\rangle\langle a\eta\rangle}{\langle ab\rangle\langle b\xi\rangle\langle b\eta\rangle} \quad (4)$$

where ξ, η are arbitrarily chosen reference spinors and G is the gluon set. *In this note, we set $\xi=1$ and $\eta=N$ when studying the (g^-,g^-) configuration.*

It was shown in [10] that $S(\mathrm{H};\mathrm{G})$ could be expanded by spanning forest form. Particularly:

$$S(\mathrm{H};\mathrm{G}) = \sum_{F\in\mathcal{F}_{\mathrm{G}}(\mathrm{G}\cup\mathrm{H})} \Big(\prod_{ab\in E(F)} \psi_{ab}\Big), \quad (5)$$

where we have summed over all possible forests F, where gluons and gravitons are considered as vertices, and the gluons are considered as the root set. Each edge ab is dressed by ψ_{ab}, and multiply all such edges in a given forest F together.

In the case of (h^-,g^-), the formulas (2) (3) and (5) are slightly changed [3, 10, 11] via (i). replacing i,j in (2) by the negative helicity graviton and the negative-helicity gluon, (ii). replacing the gravitons set H in (3) and (5) by the positive-helicity graviton set H^+, while the root set is still the gluon set. (ii). introducing an extra minus (-1).

III. AMPLITUDES WITH THREE GRAVITONS

In this section, we extend the study of one and two graviton single-trace MHV amplitudes [18], where each gravition is presented as a pair of collinear gluons, to the cases with an arbitrary number of gravitons. We demonstrate this by the example with three gravitons in the current section, and then provide a general formula in the next section.

According to (2) and (5), the MHV amplitude with gluons $1,...,N$ and three gravitons n_1, n_2, n_3 is presented by

$$A(1,...,N|n_1,n_2,n_3) \sim \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle N1\rangle} S_3,$$

1) In this work, we use $\sim$ to neglect the coupling constant and an overall normalization factor.

where S_3 is the abbreviation of the factor (5) with three gravitons. Specifically, S_3 is expressed as

$$\begin{aligned} S_3 = &\psi_1\psi_2\psi_3+\psi_1\psi_2(\psi_{13}+\psi_{23})+\psi_1\psi_3(\psi_{12}+\psi_{32}) \\ &+\psi_2\psi_3(\psi_{21}+\psi_{31})+\psi_1(\psi_{12}\psi_{23}+\psi_{13}\psi_{32}+\psi_{12}\psi_{13}) \\ &+\psi_2(\psi_{21}\psi_{13}+\psi_{23}\psi_{31}+\psi_{21}\psi_{23}) \\ &+\psi_3(\psi_{31}\psi_{12}+\psi_{32}\psi_{21}+\psi_{31}\psi_{32}), \end{aligned} \tag{6}$$

which are characterized by all possible spanning forests with structures Fig. 1. Each ψ_{ab} $(a\neq b, a,b=1,2,3)$ in the above expression is defined by (4) and is associating to an edge in the graphs Fig. 1, while the ψ_i $(i=1,2,3)$, associating to the graviton n_i, is defined by

$$\psi_i \equiv \sum_{l\in \mathrm{G}} \psi_{ln_i}.$$

In the following, we analyze the contribution of each term in eq. (6).

First, let us deal with the term $\psi_1\psi_{12}\psi_{23}$, which is characterized by Fig. 1 (a) (with $a=1$, $b=2$, $c=3$), on the right hand side of eq. (6). Noting that

$$\begin{aligned} \psi_1 &= \sum_{l\in\mathrm{G}} \frac{[ln_1]\langle l1\rangle\langle lN\rangle}{\langle ln_1\rangle\langle n_1 1\rangle\langle n_1 N\rangle} \\ &= \sum_{l\in\mathrm{G}}[ln_1]\langle ln_1\rangle \frac{-\langle 1l\rangle}{\langle 1n_1\rangle\langle n_1 l\rangle}\frac{\langle lN\rangle}{\langle ln_1\rangle\langle n_1 N\rangle} \\ &= \sum_{l\in\mathrm{G}} s_{ln_1}\times\sum_{r_1=1}^{l-1}\frac{\langle r_1,r_1+1\rangle}{\langle r_1,n_1\rangle\langle n_1,r_1+1\rangle}\sum_{t_1=l}^{N-1}\frac{\langle t_1,t_1+1\rangle}{\langle t_1,n_1\rangle\langle n_1,t_1+1\rangle}, \end{aligned} \tag{7}$$

where the eikonal identity (1) and the fact that $s_{ln_1}=[ln_1]\langle n_1 l\rangle$ are applied, we write the Parke-Taylor factor accompanied by $\psi_1\psi_{12}\psi_{23}$ as

$$\begin{aligned} &\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle N1\rangle}\psi_1\psi_{12}\psi_{23} = \psi_{12}\psi_{23} \\ &\Bigg[\sum_{l\in\mathrm{G}} s_{ln_1}\sum_{r_1=1}^{l-1}\sum_{t_1=l}^{N-1}\langle ij\rangle^4\frac{1}{\langle 12\rangle\cdots\langle r_1,n_1\rangle\langle n_1,r_1+1\rangle\cdots\langle l-1,l\rangle} \\ &\times\frac{1}{\langle l,l+1\rangle\cdots\langle t_1,\widetilde{n}_1\rangle\langle\widetilde{n}_1,t_1+1\rangle\cdots\langle N1\rangle}\Bigg], \end{aligned} \tag{8}$$

where the factors $\langle r_1,r_1+1\rangle$ and $\langle t_1,t_1+1\rangle$ in the denominator of the Parke-Taylor factor have been replaced by $\langle r_1,n_1\rangle\langle n_1,r_1+1\rangle$ and $\langle t_1,n_1\rangle\langle n_1,t_1+1\rangle$, respectively. The n_1 in the second Parke-Taylor factor is further denoted by $\widetilde{n}_1$. Hence, the graviton n_1 splits into two gluons n_1 and $\widetilde{n}_1$ with the same momentum and helicity, which are respectively inserted between 1, l and l, N. Now we further express ψ_{12} and ψ_{23} by

$$\psi_{12} = s_{n_1n_2}\sum_{r_2=1}^{n_1-1}\frac{\langle r_2,r_2+1\rangle}{\langle r_2,n_2\rangle\langle n_2,r_2+1\rangle}\sum_{t_2=\widetilde{n}_1}^{N-1}\frac{\langle t_2,t_2+1\rangle}{\langle t_2,n_2\rangle\langle n_2,t_2+1\rangle}, \tag{9}$$

$$\psi_{23} = s_{n_2n_3}\sum_{r_3=1}^{n_2-1}\frac{\langle r_3,r_3+1\rangle}{\langle r_3,n_3\rangle\langle n_3,r_3+1\rangle}\sum_{t_3=\widetilde{n}_2}^{N-1}\frac{\langle t_3,t_3+1\rangle}{\langle t_3,n_3\rangle\langle n_3,t_3+1\rangle}, \tag{10}$$

respectively. When (9) is substituted into (8), we find that the graviton n_2 splits into two gluons n_2 and $\widetilde{n}_2$, which are respectively inserted to the left side of n_1 and to the right side of $\widetilde{n}_1$. Similarly, (10) finally inserts two gluons n_3 and $\widetilde{n}_3$ corresponding to the graviton n_3 to the left side of n_2 and the right side of $\widetilde{n}_2$. The term $\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle N1\rangle}\psi_1\psi_{12}\psi_{23}$ is then written as

$$\sum_{l\in\mathrm{G}} s_{n_1l}s_{n_2n_1}s_{n_3n_2}\sum_{\boldsymbol{\rho}^{(l)}}\mathrm{PT}\left(1,\boldsymbol{\rho}^{(l)},N\right),$$

in which, we introduced $\mathrm{PT}(a_1,...,a_m)$ to denote the PT factor $\frac{\langle ij\rangle^4}{\langle a_1a_2\rangle\langle a_2a_3\rangle\cdots\langle a_ma_1\rangle}$ for short. Permutations $\boldsymbol{\rho}^{(l)}$ for a given $l\in\mathrm{G}$ are given by

$$\boldsymbol{\rho}^{(l)}\in\Big\{\{2,...,l-1\}\sqcup\!\sqcup\{n_3,n_2,n_1\},l,\{l+1,...,N-1\}\sqcup\!\sqcup\{\widetilde{n}_1,\widetilde{n}_2,\widetilde{n}_3\}\Big\},$$

where the $A\sqcup\!\sqcup B$ for two ordered sets A, B denotes all possible permutations by merging A and B together so that the relative ordering of elements in each of A and B is preserved. The above permutations can be characterized by the graph Fig. 2 (a).

Second, we investigate the term with $\psi_1\psi_{12}\psi_{13}$, which is associated to the graph Fig. 1 (b) (with $a=1$, $b=2$, $c=3$). When the factor ψ_1 and ψ_{12} are expressed by (7) and (9), and ψ_{13} is expressed as follows

$$\psi_{13} = s_{n_1n_3}\sum_{r_3=1}^{n_1-1}\frac{\langle r_3,r_3+1\rangle}{\langle r_3,n_3\rangle\langle n_3,r_3+1\rangle}\sum_{t_3=\widetilde{n}_1}^{N-1}\frac{\langle t_3,t_3+1\rangle}{\langle t_3,n_3\rangle\langle n_3,t_3+1\rangle},$$

we just split the gravitons n_1, n_2 and n_3 into three pairs of gluons $\{n_1,\widetilde{n}_1\}$, $\{n_2,\widetilde{n}_2\}$ and $\{n_3,\widetilde{n}_3\}$, respectively. The two gluons n_1, $\widetilde{n}_1$ coming from the graviton n_1 are inserted to the left and the right sides of l, while the n_2 and $\widetilde{n}_2$ (and also n_3 and $\widetilde{n}_3$) are further inserted to the left of n_1 and the right of $\widetilde{n}_1$. Thus this term turns into

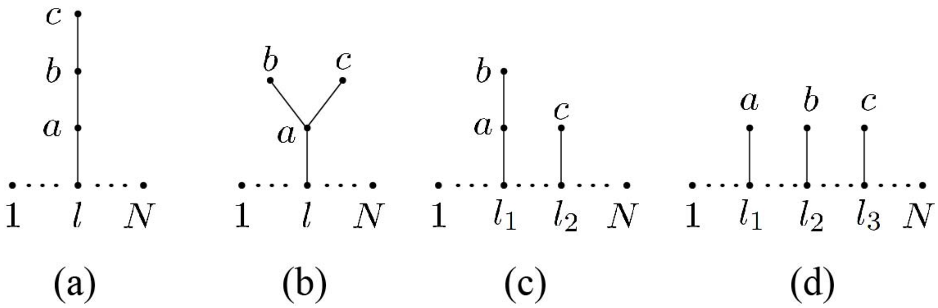

Fig. 1. All possible topologies of spanning forests for the three-graviton example. The a, b and c refer to different gravitons.

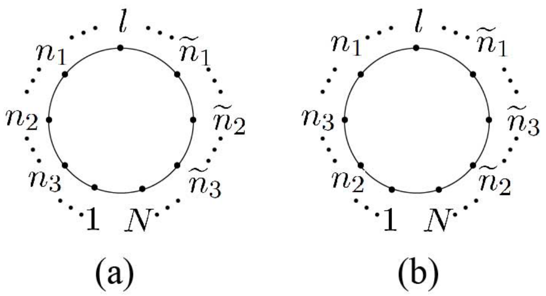

Fig. 2. (a). Permutations with the relative orderings $1,...,n_3,...,n_2,...,n_1,...,l,...,\widetilde{n}_1,...,\widetilde{n}_2,...,\widetilde{n}_3,...,N$. (b). Permutations with the relative orderings $1,...,n_2,...,n_3,...,n_1,...,l,...,\widetilde{n}_1,...,\widetilde{n}_3,...,\widetilde{n}_2,...,N$.

$$\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle N1\rangle}\psi_1\psi_{12}\psi_{13} = \sum_{l\in \mathrm{G}} s_{n_1 l}s_{n_2 n_1}s_{n_3 n_1}\sum_{\boldsymbol{\rho}^{(l)}}\mathrm{PT}\left(1,\boldsymbol{\rho}^{(l)},N\right),$$

where $\boldsymbol{\rho}^{(l)}$ for a given l is now given by

$$\boldsymbol{\rho}^{(l)} \in \Big\{\{2,...,l-1\}\shuffle\{\{n_3\}\shuffle\{n_2\},n_1\},l, \{l+1,...,N-1\}\shuffle\{\widetilde{n}_1,\{\widetilde{n}_2\}\shuffle\{\widetilde{n}_3\}\}\Big\}, \tag{11}$$

which are characterized by Fig. 2 (a) and (b).

Third, we calculate the term with $\psi_1\psi_3\psi_{12}$ (see Fig. 1 (c) with $a=1$, $b=2$, $c=3$). When the same trick with the previous examples is applied, ψ_1 and ψ_{12} are expressed by (7) and (9), while ψ_3 is obtained via replacing n_1 in (7) by n_3. Again, these factors are used to insert gluon pairs into the Parke-Taylor factor. The result is

$$\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle N1\rangle}\psi_1\psi_3\psi_{12} = \sum_{l_1,l_2\in \mathrm{G}} s_{n_1 l_1}s_{n_2 n_1}s_{n_3 l_2}\sum_{\boldsymbol{\rho}^{(l_1,l_2)}}\mathrm{PT}\left(1,\boldsymbol{\rho}^{(l_1,l_2)},N\right),$$

in which, $\boldsymbol{\rho}^{(l_1,l_2)}$ for given (l_1,l_2) satisfies

$$\boldsymbol{\rho}^{(l_1,l_2)} \in \Big\{\boldsymbol{\rho}_L^{(l_1)}\shuffle\{n_3\},l_2,\boldsymbol{\rho}_R^{(l_1)}\shuffle\{\widetilde{n}_3\}\Big\},$$
$$\text{where}\quad \boldsymbol{\rho}^{(l_1)} \in \Big\{\{2,...,l_1-1\}\shuffle\{n_2,n_1\},l_1, \{l_1+1,...,N-1\}\shuffle\{\widetilde{n}_1,\widetilde{n}_2\}\Big\}. \tag{12}$$

On the second line, the $\boldsymbol{\rho}^{(l_1)}$ denotes the permutations established by inserting the collinear gluons corresponding to n_1 and n_2 into the original gluon set, while $\boldsymbol{\rho}_L^{(l_1)}$ and $\boldsymbol{\rho}_R^{(l_1)}$ are the sectors separated by the gluon l_2 in the permutation $\boldsymbol{\rho}^{(l_1)}$. Possible relative positions of l_2 in $\boldsymbol{\rho}^{(l_1)}$ are displayed by Fig. 3 (a)-(g). Since the choices of l_1 and l_2 are independent of each other and we finally summed over all possible choices of l_1 and l_2, one can exchange the roles of l_1, l_2 in (12) as follows

$$\boldsymbol{\rho}^{(l_1,l_2)} \in \Big\{\boldsymbol{\rho}_L^{(l_2)}\shuffle\{n_2,n_1\},l_1,\boldsymbol{\rho}_R^{(l_2)}\shuffle\{\widetilde{n}_1,\widetilde{n}_2\}\Big\},$$
$$\text{where}\quad \boldsymbol{\rho}^{(l_2)} \in \Big\{\{2,...,l_2-1\}\shuffle\{n_3\},l_2, \{l_2+1,...,N-1\}\shuffle\{\widetilde{n}_3\}\Big\}. \tag{13}$$

When all possible spanning forests for amplitude with three gravitons are considered, the full MHV amplitude with three gravitons is finally expressed by the following formula:

$$A(1,...,N|n_1,n_2,n_3) \sim \sum_{\substack{\text{Spanning Forests}\\ \{\mathcal{T}_1,...,\mathcal{T}_i\}}}\ \sum_{l_1,...,l_i\in \mathrm{G}} K(\mathcal{T}_1)\ldots K(\mathcal{T}_i)\,\mathrm{PT}\left(1,\boldsymbol{\rho}^{(l_1,...,l_i)},N\right).\quad (i\leq 3) \tag{14}$$

In the above expression, we have summed over all possible spanning forests where the original gluon set G plays as the root set. For a given spanning forest with i ($i\leq 3$) trees $\mathcal{T}_1,...,\mathcal{T}_i$ planted at gluons $l_1,...,l_i\in$ G (l_j and l_k with distinct labels may be identical), each $K(\mathcal{T}_j)$ ($j=1,...,i$) is given by

$$K(\mathcal{T}_j) = \prod_{ab\in E(\mathcal{T}_j)} s_{ab},$$

where $ab\in E(\mathcal{T}_j)$ is an edge of the tree $\mathcal{T}_j$ with vertices a

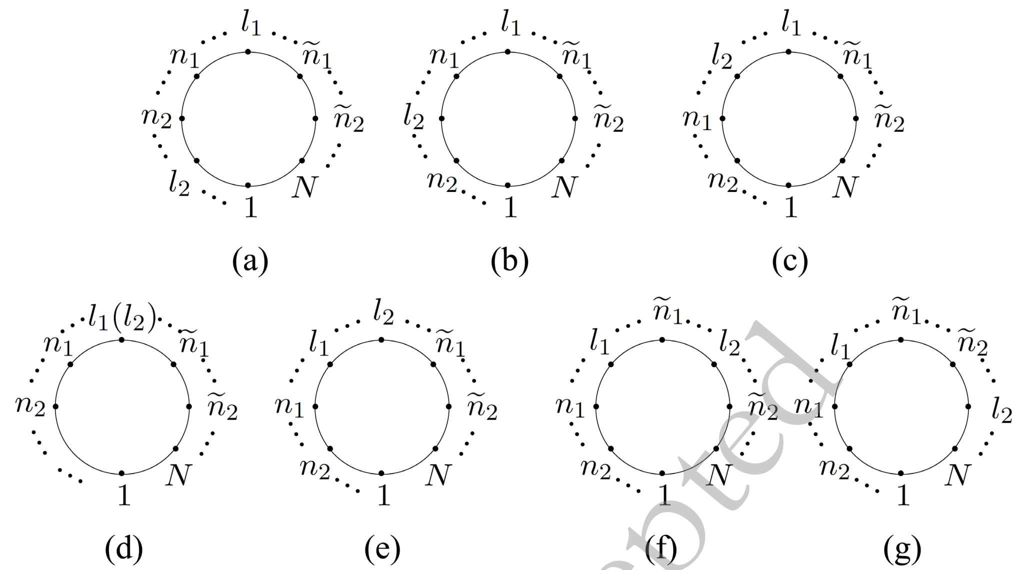

Fig. 3. Possible relative positions of l_2 in the permutations $\boldsymbol{\rho}^{(l_1)}$ in (12)

and b. More explicity, there are four possible topologies for the three-graviton amplitude, as shown by Fig. 1 (a), (b), (c) and (d), which respectively provide factors

$$s_{cb}s_{ba}s_{al}, \quad s_{ba}s_{ca}s_{al}, \quad s_{ba}s_{al_1}s_{cl_2}, \quad s_{al_1}s_{bl_2}s_{cl_3},$$

while a, b, c represent distinct gravitons. Two graphs with exchanging the branches attached to a same vertex are considered as the same graph, e.g. Fig. 1 (b). The permutations associated to Fig. 1 (a) and (b) can be recursively defined by (11) and (12), via replacing the subscripts 1, 2 and 3 of gravitons in (12) by a, b and c, respectively. The permutations for Fig. 1 (c) satisfy

$$\begin{aligned}&\boldsymbol{\rho}^{(l_1,l_2)} \in \left\{\boldsymbol{\rho}^{(l_1)} \sqcup\!\sqcup \{n_c\}, l_2, \boldsymbol{\rho}^{(l_1)} \cup \{\widetilde{n}_c\}\right\},\\ &\text{where} \quad \boldsymbol{\rho}^{(l_1)} \in \Big\{\{2,\dots,l_1-1\} \cup \{n_b,n_a\}, l_1,\\ &\{l_1+1,\dots,N-1\} \sqcup\!\sqcup \{\widetilde{n}_a,\widetilde{n}_b\}\Big\}.\end{aligned}$$

Permutations accompanying to Fig. 1 (d) are presented by

$$\begin{aligned}&\boldsymbol{\rho}^{(l_1,l_2,l_3)} \in \left\{\boldsymbol{\rho}^{(l_1,l_2)} \sqcup\!\sqcup \{n_c\}, l_3, \boldsymbol{\rho}^{(l_1,l_2)} \sqcup\!\sqcup \{\widetilde{n}_c\}\right\},\\ &\text{where } \boldsymbol{\rho}^{(l_1,l_2)} \in \left\{\boldsymbol{\rho}_L^{(l_1)} \sqcup\!\sqcup \{n_b\}, l_2, \boldsymbol{\rho}_R^{(l_1)} \sqcup\!\sqcup \{\widetilde{n}_b\}\right\}\\ &\text{and } \boldsymbol{\rho}^{(l_1)} \in \left\{\{2,\dots,l_1-1\} \sqcup\!\sqcup \{n_a\}, l_1, \{l_1+1,\dots,N-1\} \sqcup\!\sqcup \{\widetilde{n}_a\}\right\}.\end{aligned}$$

Having displayed the example with three gravitons, we turn to the general formula in the next section.

IV. THE GENERAL FORMULA

Inspired by the example in the previous section, we propose the following general formula where gravitions split into pairs of collinear gluons

$$A(1,\dots,N|\mathrm{H}) \sim \sum_{l_1,\dots,l_i\in \mathrm{G}} \sum_{\substack{\text{Spanning Forests}\\ \{\mathcal{T}_1,\dots,\mathcal{T}_i\}}} K(\mathcal{T}_1)\dots K(\mathcal{T}_i)\,\mathrm{PT}\left(1,\boldsymbol{\rho}^{(l_1,\dots,l_i)},N\right). \tag{15}$$

Here we sum over all possible spanning forests in which trees are planted at gluons $l_1,\dots,l_i \in \mathrm{G}$. This summation is expressed by two summations:

- (i). summing over all possible choices of the roots $l_1,\dots,l_i$ $(i=1,\dots,M)$,

- (ii). for a given choice of roots $l_1,\dots,l_i$, summing over all possible configurations of forests, which consist of nontrivial trees $\mathcal{T}_1,\dots,\mathcal{T}_i$ planted at the gluons $l_1,\dots,l_i$.

For a fixed forest, each tree $\mathcal{T}_k$ is associated with a factor $K(\mathcal{T}_k)$ where each edge between two vertices a, b is assigned by a factor s_{ab}. The permutations $\boldsymbol{\rho}^{(l_1,\dots,l_k)}$ in the PT factors can be defined recursively:

$$\boldsymbol{\rho}^{(l_1,\dots,l_k)} = \left\{\boldsymbol{\rho}_L^{(l_1,\dots,l_{k-1})} \sqcup\!\sqcup \boldsymbol{\sigma}^{\mathcal{T}_k}, l_k, \boldsymbol{\rho}_R^{(l_1,\dots,l_{k-1})} \sqcup\!\sqcup \left(\widetilde{\boldsymbol{\sigma}}^{\mathcal{T}_k}\right)^T\right\}. \quad (k\le i) \tag{16}$$

where $\boldsymbol{\rho}_L^{(l_1,\dots,l_{k-1})}$ and $\boldsymbol{\rho}_R^{(l_1,\dots,l_{k-1})}$ denote the two ordered sets which are separated by the gluon l_k in the permutation $\boldsymbol{\rho}^{(l_1,\dots,l_{k-1})}$. The $\boldsymbol{\sigma}^{\mathcal{T}_k}$ $(\widetilde{\boldsymbol{\sigma}}^{\mathcal{T}_k})$ stands for the permutations established by the tree graph $\mathcal{T}_k$ whose nodes are $\{n_i\}$ $(\{\widetilde{n}_i\})$, while $(\widetilde{\boldsymbol{\sigma}}^{\mathcal{T}_k})^T$ denotes the reverse of $\widetilde{\boldsymbol{\sigma}}^{\mathcal{T}_k}$.

Now we sketch the proof of the general formula (15):

• (i). *Step-1* Expand the MHV amplitude according to (2) and (5) in terms of spanning forests. Each forest F in general consists of i tree structures $\mathcal{T}_1,...,\ \mathcal{T}_i$ planted to gluons $l_1,...,l_i \in \mathrm{G}$.

• (ii). *Step-2* For a given forest $F=\{\mathcal{T}_1,...,\ \mathcal{T}_i\}$ and the tree $\mathcal{T}_1$, there are two types of edges (a). the edge between a graviton a and the root (a gluon $l_1 \in \mathrm{G}$), (b). The edge between two gravitons b and c. In the former case, the edge is associated with a factor ψ_a which is expressed according to (7), while an edge of the latter form is accompanied by a factor ψ_{bc}, which is further rewritten as (9). After this manipulation, the factor ψ_a splits the graviton n_a into collinear gluons n_a and $\widetilde{n}_a$ and then inserts them to the left and right of l_1, respectively. A factor ψ_{bc} splits the graviton n_c into collinear gluons n_c and $\widetilde{n}_c$ which are further inserted to the left of n_b and the right of $\widetilde{n}_b$ (n_b which is nearer to root than n_c has already been treated before). The factor assigned to each edge bc is s_{bc}, and the product of all these factors gives $K(\mathcal{T}_1)$. The permutations established by this step are given by

$$\rho^{(l_1)}=\left\{\{2,...,l_1-1\}\shuffle\sigma^{\mathcal{T}_1},l_1,\{l_1+1,...,N-1\}\shuffle\left(\widetilde{\sigma}^{\mathcal{T}_1}\right)^T\right\}.$$

• (iii). *Step-3* Insert the collinear gluons corresponding to the gravitons on trees $\mathcal{T}_2,...,\ \mathcal{T}_i$ in turn, by repeating step-2. We finally get the general formula (15) with permutations defined in (16).

V. CONCLUSION

In this note, we presented a formula (15) for single-trace EYM amplitudes in the MHV configuration (with two negative-helicity gluons). Each graviton in this formula splits into a pair of collinear gluons. Thus an N-gluon, M-graviton amplitude is finally expressed as a combination of $N+2M$ gluon amplitudes with M pairs of collinear gluons. When the adjustment pointed in section 2 is considered, the formula (15) is straightforwardly extended to the MHV amplitude with one negative-helicity gluon and one negative-helicity graviton via (i). replacing i, j in the numerator of the PT factor by the negative-helicity graviton and the negative-helicity gluon, (ii). using the positive-helicity graviton set instead of the full graviton set on the RHS of (15). (iii). dressing the expression by an extra sign (-1). It is worth extending the collinear expression in the current paper to the double-trace amplitudes and amplitudes with other helicity configurations in a future work.

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