Rephasing invariant formulae for CP phases in general parameterizations of flavor mixing matrix and exact sum rules with unitarity triangles*

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Abstract: In this letter, we present rephasing invariant formulae $\delta^{(\alpha i)} = \arg[V_{\alpha 1} V_{\alpha 2} V_{\alpha 3} V_{1i} V_{2i} V_{3i} / V_{\alpha i}^3 \det V]$ for CP phases $\delta^{(\alpha i)}$ associated with nine Euler-angle-like parameterizations of a flavor mixing matrix. Here, α and i denote the row and column carrying the trivial phases in a given parameterization. Furthermore, we show that the phases $\delta^{(\alpha i)}$ and the nine angles $\Phi_{\alpha i}$ of unitarity triangles satisfy compact sum rules $\delta^{(\alpha, i+2)} - \delta^{(\alpha, i+1)} = \Phi_{\alpha+1, i} - \Phi_{\alpha+2, i}$ and $\delta^{(\alpha+1, i)} - \delta^{(\alpha+2, i)} = \Phi_{\alpha, i+2} - \Phi_{\alpha, i+1}$ where all indices are taken cyclically modulo three. These relations are natural generalizations of the previous result $\delta_{\rm PDG} + \delta_{\rm KM} = \pi - \alpha + \gamma$.

Keywords: CP violation, Unitarity triangles, CKM matrix

DOI: **CSTR:**

CPC Accepted

I. INTRODUCTION

The flavor mixing of quarks and leptons has been one of the essential components for the development of particle physics, and it is also an inherently intriguing subject for mathematical exploration. Among various representations of the mixing matrix [1−9], Euler-angle-like parameterizations using three rotation matrices and a distinct CP phase allow for nine possible forms [10]. However, at first glance, these nine phases appear to take chaotic values, showing no apparent pattern.

On the other hand, the unitarity conditions of the mixing matrix give rise to six unitarity triangles [11−17], and numerous papers have also investigated the connections between the CP phases and the triangles [18−30]. These nine angles also seem to have independent values.

More recently, rephasing invariant formulae for the CP phases [31−33] produce a simple sum rule relating the CP phases and angles as $\delta_{\rm PDG} + \delta_{\rm KM} = \pi - \alpha + \gamma$. This led to the discovery of an underlying order between the otherwise random CP phases and angles.

In this letter, to organize relations connecting the nine CP phases and the nine angles of unitarity triangles, we first derive rephasing invariant formulae for each CP phase in terms of the elements of the mixing matrix V and $\det V$. Next, we present exact sum rules between the phases and angles, demonstrating that the previous sum rule is a special case of these relations.

II. REPHASING INVARIANT FORMULAE FOR CP PHASES IN GENERAL PARAMETERIZATIONS OF FLAVOR MIXING MATRIX

In this section, following our previous works [31−33], we derive rephasing invariant formulae for CP phases in Euler-angle-like parameterizations of the flavor mixing matrix V. For this purpose, we adopt the nine parameterizations proposed by Fritzsch and Xing [34].

We begin by defining the 2×2 rotation matrix R_{ij} as follows:

$$R_{12}(\theta) = \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$R_{23}(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\sigma & s_\sigma \\ 0 & -s_\sigma & c_\sigma \end{pmatrix},$$
$$R_{31}(\tau) = \begin{pmatrix} c_\tau & 0 & s_\tau \\ 0 & 1 & 0 \\ -s_\tau & 0 & c_\tau \end{pmatrix}, \quad (1)$$

Where $s_\theta = \sin\theta, c_\theta = \cos\theta$. Furthermore, complex rotation matrices $R_{12}(\theta,\delta)$, $R_{23}(\sigma,\delta)$, and $R_{31}(\tau,\delta)$ are defined by replacing $1 \to e^{-i\delta}$ in Eq. (1). With these matrices, the

Received 2 September 2025; Accepted 15 September 2025
* The study is partly supported by the MEXT Leading Initiative for Excellent Young Researchers Grant Number JP2023L0013
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flavor mixing matrix V admits nine different parameterizations [34].

$$
\begin{aligned}
P1:\ V^{(33)} &= R_{12}(\theta)R_{23}(\sigma,\delta^{(33)})R_{12}^{-1}(\theta'),\\
P2:\ V^{(11)} &= R_{23}(\sigma)R_{12}(\theta,\delta^{(11)})R_{23}^{-1}(\sigma'),\\
P3:\ V^{(13)} &= R_{23}(\sigma)R_{31}(\tau,\delta^{(13)})R_{12}(\theta),\\
P4:\ V^{(31)} &= R_{12}(\theta)R_{31}(\tau,\delta^{(31)})R_{23}^{-1}(\sigma),\\
P5:\ V^{(22)} &= R_{31}(\tau)R_{12}(\theta,\delta^{(22)})R_{31}^{-1}(\tau'),\\
P6:\ V^{(32)} &= R_{12}(\theta)R_{23}(\sigma,\delta^{(32)})R_{31}(\tau),\\
P7:\ V^{(12)} &= R_{23}(\sigma)R_{12}(\theta,\delta^{(12)})R_{31}^{-1}(\tau),\\
P8:\ V^{(21)} &= R_{31}(\tau)R_{12}(\theta,\delta^{(21)})R_{23}(\sigma),\\
P9:\ V^{(23)} &= R_{31}(\tau)R_{23}(\sigma,\delta^{(23)})R_{12}^{-1}(\theta). \qquad (2)
\end{aligned}
$$

Here, in each of these parameterizations, a row α and a column i have trivial phases; thus, $V_{\alpha j}, V_{\beta i} \in \mathbb{R}$. The notation $V^{(\alpha i)}$ and $\delta^{(\alpha i)}$ distinguishes the corresponding matrix and CP phase. More explicitly, the conditions are

$$
\begin{aligned}
\arg V_{\alpha 1}^{(\alpha i)} &= \arg V_{\alpha 2}^{(\alpha i)} = \arg V_{\alpha 3}^{(\alpha i)}\\
&= \arg V_{1i}^{(\alpha i)} = \arg V_{2i}^{(\alpha i)} = \arg V_{3i}^{(\alpha i)} = 0 \text{ or } \pi. \qquad (3)
\end{aligned}
$$

Since there is a duplication of the element $V_{\alpha i}^{(\alpha i)}$, there are effectively five independent conditions, and the phase π appears either zero or two times. As the final condition, the argument of the determinant is

$$
\arg\det V^{(\alpha i)} = -\delta^{(\alpha i)}. \qquad (4)
$$

In other words, the phase structures of these parameterizations, $V^{(\alpha i)}$, are specified by these six conditions.

Let us consider transforming the mixing matrix V in an arbitrary basis into one of these parameterizations, $V^{(\alpha i)}$. Suppose that general rephasing transformations remove unphysical phases as

$$
V^{(\alpha i)} = \Psi_L^\dagger V \Psi_R, \qquad (5)
$$

where $(\Psi_L)_{\alpha\beta} = e^{i\gamma_{L\alpha}}\delta_{\alpha\beta}$ and $(\Psi_R)_{ij} = e^{i\gamma_{Ri}}\delta_{ij}$ are diagonal phase matrices. The inverse transformation from $V^{(\alpha i)}$ to the original matrix V is given by

$$
V = \Psi_L V^{(\alpha i)} \Psi_R^\dagger, \quad V_{\beta j} = e^{i\gamma_{L\beta}} V_{\beta j}^{(\alpha i)} e^{-i\gamma_{Rj}}. \qquad (6)
$$

By considering the arguments of the determinants, a relation $\arg\det V = -\delta^{(\alpha i)} + \sum_{\beta,j}(\gamma_{L\beta} - \gamma_{Rj})$ is obtained, and the CP phase $\delta^{(\alpha i)}$ is determined to be

$$
\begin{aligned}
\delta^{(\alpha i)} &= (\gamma_{L1} + \gamma_{L2} + \gamma_{L3} - \gamma_{R1} - \gamma_{R2} - \gamma_{R3}) - \arg\det V\\
&= \arg[V_{\alpha 1}V_{\alpha 2}V_{\alpha 3}V_{1i}V_{2i}V_{3i}/V_{\alpha i}^3] - \arg\det V\\
&= \arg\left[\frac{V_{\alpha 1}V_{\alpha 2}V_{\alpha 3}V_{1i}V_{2i}V_{3i}}{V_{\alpha i}^3 \det V}\right]. \qquad (7)
\end{aligned}
$$

In practice, since $V_{\alpha i}$ appears twice in the numerator, cancellations occur between the numerator and the denominator as

$$
\begin{aligned}
\delta^{(\alpha i)} &= \arg\left[\frac{V_{\alpha 1}V_{\alpha 2}V_{\alpha 3}V_{1i}V_{2i}V_{3i}}{V_{\alpha i}^3 \det V}\right]\\
&= \arg\left[\frac{(\prod_{j\neq i} V_{\alpha j})(\prod_{\beta\neq\alpha} V_{\beta i})}{V_{\alpha i}\det V}\right]. \qquad (8)
\end{aligned}
$$

Since six phases are required to eliminate the unphysical phases $\gamma_{L\alpha}$ and γ_{Ri}, any other combinations have nontrivial phases of $V_{\beta j}^{(\alpha i)}$ and are not explicitly solved for $\delta^{(\alpha i)}$. This derivation remains valid even in the presence of Majorana phases for leptons because the resulting CP phases are manifestly rephasing invariant.

Explicitly, the formulae are expressed as

$$
\begin{aligned}
\delta^{(11)} &= \arg\left[\frac{V_{12}V_{13}V_{21}V_{31}}{V_{11}\det V}\right], \quad \delta^{(12)} = \arg\left[\frac{V_{11}V_{13}V_{22}V_{32}}{V_{12}\det V}\right],\\
\delta^{(13)} &= \arg\left[\frac{V_{11}V_{12}V_{23}V_{33}}{V_{13}\det V}\right], \quad \delta^{(21)} = \arg\left[\frac{V_{22}V_{23}V_{11}V_{31}}{V_{21}\det V}\right],\\
\delta^{(22)} &= \arg\left[\frac{V_{21}V_{23}V_{12}V_{32}}{V_{22}\det V}\right], \quad \delta^{(23)} = \arg\left[\frac{V_{21}V_{22}V_{13}V_{33}}{V_{23}\det V}\right],\\
\delta^{(31)} &= \arg\left[\frac{V_{32}V_{33}V_{11}V_{21}}{V_{31}\det V}\right], \quad \delta^{(32)} = \arg\left[\frac{V_{31}V_{33}V_{12}V_{22}}{V_{32}\det V}\right],\\
\delta^{(33)} &= \arg\left[\frac{V_{31}V_{32}V_{13}V_{23}}{V_{33}\det V}\right]. \qquad (9)
\end{aligned}
$$

They provide a necessary and sufficient set of "irreducible fifth-order" invariants, which cannot be decomposed into second- and third-order invariants, as only nine such invariants are constructed using $\det V$.

From the latest UTfit results of the CKM matrix parameters [35],

$$
\begin{aligned}
\sin\theta_{12} &= 0.22519 \pm 0.00083,\\
\sin\theta_{23} &= 0.04200 \pm 0.00047,\\
\sin\theta_{13} &= 0.003714 \pm 0.000092,\\
\delta &= 1.137 \pm 0.022 = 65.15^\circ \pm 1.3^\circ, \qquad (10)
\end{aligned}
$$

The numerical evaluation at the best-fit values yields

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$$\begin{pmatrix} \delta^{(11)} & \delta^{(12)} & \delta^{(13)} \\ \delta^{(21)} & \delta^{(22)} & \delta^{(23)} \\ \delta^{(31)} & \delta^{(32)} & \delta^{(33)} \end{pmatrix} = \begin{pmatrix} 92.44^\circ & 115.91^\circ & 65.15^\circ \\ 157.51^\circ & 1.089^\circ & 114.89^\circ \\ 23.54^\circ & 156.49^\circ & 93.45^\circ \end{pmatrix}. \tag{11}$$

In particular, the standard PDG phase $\delta_{\rm PDG}$ and the original Kobayashi–Maskawa phase $\delta_{\rm KM}$ are given by

$$\delta_{\rm PDG} = \delta^{(13)}, \quad \delta_{\rm KM} = \pi - \delta^{(11)}. \tag{12}$$

Interestingly, the sum over any row or column of the phases $\delta^{(ij)}$ yields the same value of a ninth-order invariant.

$$\sum_i \delta^{(ij)} = \sum_j \delta^{(ij)} = \arg\left[\frac{\prod_{\alpha,i} V_{\alpha i}}{\det V^3}\right] = 273.49^\circ. \tag{13}$$

For a cross-check, we numerically compute $\cos\delta^{(ij)}$.

$$\begin{pmatrix} \cos\delta^{(11)} & \cos\delta^{(12)} & \cos\delta^{(13)} \\ \cos\delta^{(21)} & \cos\delta^{(22)} & \cos\delta^{(23)} \\ \cos\delta^{(31)} & \cos\delta^{(32)} & \cos\delta^{(33)} \end{pmatrix} = \begin{pmatrix} -0.0425 & -0.437 & 0.420 \\ -0.924 & 0.9998 & -0.421 \\ 0.917 & -0.917 & -0.0603 \end{pmatrix}. \tag{14}$$

These values agree with those obtained from the moduli of the matrix elements.

$$\begin{aligned}
\cos\delta^{(11)} &= \frac{(1-|V_{11}|^2)^2|V_{22}|^2 - |V_{11}V_{12}V_{21}|^2 - |V_{13}V_{31}|^2}{2|V_{11}V_{12}V_{13}V_{21}V_{31}|} = -0.0425, \\
\cos\delta^{(12)} &= \frac{(1-|V_{12}|^2)^2|V_{23}|^2 - |V_{12}V_{13}V_{22}|^2 - |V_{11}V_{32}|^2}{2|V_{11}V_{12}V_{13}V_{22}V_{32}|} = -0.437, \\
\cos\delta^{(13)} &= -\frac{(1-|V_{13}|^2)^2|V_{22}|^2 - |V_{12}V_{23}V_{13}|^2 - |V_{11}V_{33}|^2}{2|V_{11}V_{12}V_{13}V_{23}V_{33}|} = 0.420, \\
\cos\delta^{(21)} &= -\frac{(1-|V_{21}|^2)^2|V_{12}|^2 - |V_{11}V_{21}V_{22}|^2 - |V_{23}V_{31}|^2}{2|V_{21}V_{22}V_{23}V_{11}V_{31}|} = -0.924, \\
\cos\delta^{(22)} &= \frac{(1-|V_{22}|^2)^2|V_{33}|^2 - |V_{22}V_{23}V_{32}|^2 - |V_{12}V_{21}|^2}{2|V_{21}V_{22}V_{23}V_{12}V_{32}|} = 0.9998, \\
\cos\delta^{(23)} &= -\frac{(1-|V_{23}|^2)^2|V_{32}|^2 - |V_{22}V_{23}V_{33}|^2 - |V_{13}V_{21}|^2}{2|V_{21}V_{22}V_{23}V_{13}V_{33}|} = -0.421, \\
\cos\delta^{(31)} &= -\frac{(1-|V_{31}|^2)^2|V_{22}|^2 - |V_{21}V_{32}V_{31}|^2 - |V_{11}V_{33}|^2}{2|V_{11}V_{21}V_{31}V_{32}V_{33}|} = 0.917, \\
\cos\delta^{(32)} &= -\frac{(1-|V_{32}|^2)^2|V_{23}|^2 - |V_{22}V_{32}V_{33}|^2 - |V_{12}V_{31}|^2}{2|V_{31}V_{32}V_{33}V_{12}V_{22}|} = -0.917, \\
\cos\delta^{(33)} &= \frac{(1-|V_{33}|^2)^2|V_{11}|^2 - |V_{13}V_{31}V_{33}|^2 - |V_{23}V_{32}|^2}{2|V_{13}V_{31}V_{33}V_{23}V_{32}|} = -0.0603.
\end{aligned} \tag{15}$$

Since $\delta^{(\alpha i)}$ also carries the sign of $\sin\delta^{(\alpha i)}$, it contains more information than an evaluation based solely on the absolute values of the matrix elements.

III. EXACT SUM RULES BETWEEN CP PHASES IN GENERAL PARAMETERIZATIONS AND UNITARITY TRIANGLES

It is noteworthy to reconsider the relations between the results and the angles of unitarity triangles. The three angles of the unitarity triangle are rephasing invariants.

$$\begin{aligned}
\alpha &= \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right] = 92.40^\circ, \\
\beta &= \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right] = 22.49^\circ, \\
\gamma &= \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right] = 65.11^\circ.
\end{aligned} \tag{16}$$

A general phase matrix Φ of these angles is defined in [13]. In particular, the elements in the second column of Φ correspond to α, β, and γ, respectively.

$$\Phi \equiv \begin{pmatrix} \arg(-\Pi_{ud}^*) & \arg(-\Pi_{us}^*) & \arg(-\Pi_{ub}^*) \\ \arg(-\Pi_{cd}^*) & \arg(-\Pi_{cs}^*) & \arg(-\Pi_{cb}^*) \\ \arg(-\Pi_{td}^*) & \arg(-\Pi_{ts}^*) & \arg(-\Pi_{tb}^*) \end{pmatrix} = \begin{pmatrix} 1.054^\circ & 22.49^\circ & 156.46^\circ \\ 64.09^\circ & 92.40^\circ & 23.50^\circ \\ 114.85^\circ & 65.11^\circ & 0.0370^\circ \end{pmatrix}. \tag{17}$$

Here, $\Pi_{\alpha i} \equiv V_{\beta j}V_{\beta k}^* V_{\gamma k}V_{\gamma j}^*$, and the indices are defined cyclically. For better readability, all flavor indices are rewritten using numbers as follows:

$$\Pi = \begin{pmatrix} V_{33}V_{32}^*V_{22}V_{23}^* & V_{31}V_{33}^*V_{23}V_{21}^* & V_{32}V_{31}^*V_{21}V_{22}^* \\ V_{13}V_{12}^*V_{32}V_{33}^* & V_{11}V_{13}^*V_{33}V_{31}^* & V_{12}V_{11}^*V_{31}V_{32}^* \\ V_{23}V_{22}^*V_{12}V_{13}^* & V_{21}V_{23}^*V_{13}V_{11}^* & V_{22}V_{21}^*V_{11}V_{12}^* \end{pmatrix}. \tag{18}$$

The sum over each row and column of the matrix Φ equals 180°.

$$\sum_i \Phi_{ij} = \sum_j \Phi_{ij} = \pi. \tag{19}$$

Since solving general sum rules is a challenging task, we employ a transfer matrix approach. We define a phase matrix Δ and a transfer matrix T,

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$$\Delta \equiv \begin{pmatrix} \delta^{(11)} & \delta^{(12)} & \delta^{(13)} \\ \delta^{(21)} & \delta^{(22)} & \delta^{(23)} \\ \delta^{(31)} & \delta^{(32)} & \delta^{(33)} \end{pmatrix}, \quad T \equiv \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{20}$$

one finds the following relations

$$\Delta T - \Delta = \begin{pmatrix} \arg \frac{V_{11}^2 V_{22} V_{32}}{V_{12}^2 V_{21} V_{31}} & \arg \frac{V_{12}^2 V_{23} V_{33}}{V_{13}^2 V_{22} V_{32}} & \arg \frac{V_{13}^2 V_{21} V_{31}}{V_{11}^2 V_{23} V_{33}} \\ \arg \frac{V_{12} V_{21}^2 V_{32}}{V_{11} V_{22}^2 V_{31}} & \arg \frac{V_{13} V_{22}^2 V_{33}}{V_{12} V_{23}^2 V_{32}} & \arg \frac{V_{11} V_{23}^2 V_{31}}{V_{13} V_{21}^2 V_{33}} \\ \arg \frac{V_{12} V_{22} V_{31}^2}{V_{11} V_{21} V_{32}^2} & \arg \frac{V_{13} V_{23} V_{32}^2}{V_{12} V_{22} V_{33}^2} & \arg \frac{V_{11} V_{21} V_{33}^2}{V_{13} V_{23} V_{31}^2} \end{pmatrix} = T^2 \Phi T^2 - T \Phi T^2 . \tag{21}$$

Alternatively, by multiplying T from the right and using $T^2 = T^T = T^{-1}$, we arrive at a more compact notation.

$$\Delta T^2 - \Delta T = T^2 \Phi - T \Phi . \tag{22}$$

This represents the transfer along columns of Δ and rows of Φ.

$$\delta^{(\alpha, i+2)} - \delta^{(\alpha, i+1)} = \Phi_{\alpha+1, i} - \Phi_{\alpha+2, i} . \tag{23}$$

Here, the indices are taken modulo three. Note that the transfer T^T from the left increases the indices of the rows. These relations are numerically verified as follows:

$$\begin{pmatrix} 65.15^\circ & 92.44^\circ & 115.91^\circ \\ 114.89^\circ & 157.51^\circ & 1.089^\circ \\ 93.45^\circ & 23.54^\circ & 156.49^\circ \end{pmatrix} - \begin{pmatrix} 115.91^\circ & 65.15^\circ & 92.44^\circ \\ 1.089^\circ & 114.89^\circ & 157.51^\circ \\ 156.49^\circ & 93.45^\circ & 23.54^\circ \end{pmatrix} = \begin{pmatrix} -50.76 & 27.29 & 23.47 \\ 113.80 & 42.62 & -156.42 \\ -63.04 & -69.91 & 132.95 \end{pmatrix}$$
$$= \begin{pmatrix} 64.09^\circ & 92.40^\circ & 23.50^\circ \\ 114.85^\circ & 65.11^\circ & 0.0370^\circ \\ 1.054^\circ & 22.49^\circ & 156.46^\circ \end{pmatrix} - \begin{pmatrix} 114.85^\circ & 65.11^\circ & 0.0370^\circ \\ 1.054^\circ & 22.49^\circ & 156.46^\circ \\ 64.09^\circ & 92.40^\circ & 23.50^\circ \end{pmatrix} . \tag{24}$$

Similarly, we can find another transfer as follows:

$$T\Delta - \Delta = \begin{pmatrix} \arg \frac{V_{11}^2 V_{32} V_{33}}{V_{12} V_{13} V_{31}^2} & \arg \frac{V_{12}^2 V_{31} V_{33}}{V_{11} V_{13} V_{32}^2} & \arg \frac{V_{13}^2 V_{31} V_{32}}{V_{11} V_{12} V_{33}^2} \\ \arg \frac{V_{12} V_{13} V_{21}^2}{V_{11}^2 V_{22} V_{23}} & \arg \frac{V_{11} V_{13} V_{22}^2}{V_{12}^2 V_{21} V_{23}} & \arg \frac{V_{11} V_{12} V_{23}^2}{V_{13}^2 V_{21} V_{22}} \\ \arg \frac{V_{22} V_{23} V_{31}^2}{V_{21}^2 V_{32} V_{33}} & \arg \frac{V_{21} V_{23} V_{32}^2}{V_{22}^2 V_{31} V_{33}} & \arg \frac{V_{21} V_{22} V_{33}^2}{V_{23}^2 V_{31} V_{32}} \end{pmatrix} = T^2 \Phi T^2 - T^2 \Phi T . \tag{25}$$

Alternatively, by multiplying T from the left, one obtains:

$$T^2 \Delta - T \Delta = \Phi T^2 - \Phi T , \tag{26}$$

and for matrix elements

$$\delta^{(\alpha+1, i)} - \delta^{(\alpha+2, i)} = \Phi_{\alpha, i+2} - \Phi_{\alpha, i+1} . \tag{27}$$

Iterating these two transfers, we can find the sum rules for all combinations between $\delta^{(\alpha i)}$ and $\delta^{(\beta j)}$.

Finally, we reproduce the results of previous work. Since it suffices to examine the relation between $\delta^{(13)}$ and $\delta^{(11)}$, the case of $\alpha = 1, i = 2$ in Eq. (23) becomes

$$\delta^{(11)} - \delta^{(13)} = \Phi_{22} - \Phi_{32} , \quad \pi - \delta_{\rm KM} - \delta_{\rm PDG} = \alpha - \gamma . \tag{28}$$

Thus, the relation $\delta_{\rm PDG} + \delta_{\rm KM} = \pi - \alpha + \gamma$ is confirmed. Furthermore, it has also been shown in previous work [32,33] that such a difference between phases and angles is expressed in terms of a third-order invariant [36].

$$\delta_{\rm PDG} - \gamma = \arg \left[- \frac{V_{us} V_{cd} V_{tb}}{\det V_{\rm CKM}} \right] = \pi - \delta_{\rm KM} - \alpha . \tag{29}$$

A comprehensive study of this kind of relation will be performed in the subsequent work. The results generalize the previous sum rule to the nine CP phases in the different parameterizations and angles of unitarity triangles.

IV. SUMMARY

In this letter, we present rephasing invariant formulae $\delta^{(\alpha i)} = \arg \left[V_{\alpha 1} V_{\alpha 2} V_{\alpha 3} V_{1i} V_{2i} V_{3i} / V_{\alpha i}^3 \det V \right]$ for CP phases $\delta^{(\alpha i)}$ associated with the nine parameterizations $V^{(\alpha i)}$ of a flavor mixing matrix. Here, α and i denote the row and column with trivial phases in $V^{(\alpha i)}$. In practice, since cancellations occur between the numerator and the denominator, these are a necessary and sufficient set of irreducible fifth-order invariants constructed using the determinant.

Furthermore, we find exact sum rules $\delta^{(\alpha, i+2)} - \delta^{(\alpha, i+1)} = \Phi_{\alpha+1, i} - \Phi_{\alpha+2, i}$ and $\delta^{(\alpha+1, i)} - \delta^{(\alpha+2, i)} = \Phi_{\alpha, i+2} - \Phi_{\alpha, i+1}$ between the nine phases $\delta^{(\alpha i)}$ and the nine angles $\Phi_{\alpha i}$ of the unitar-

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ity triangles. These relations revealed intrinsic underlying orders connecting the CP phases and angles, which seem to be random and independent. We show that a relation from the previous work $\delta_{\rm PDG}+\delta_{\rm KM}=\pi-\alpha+\gamma$ is immediately reproduced as a special case ($\alpha=1, i=2$) of the first sum rules. These results provide a general and concise framework for differences between angles and phases, and offer a valuable guide for future experimental tests of CP violation.

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