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Fermion Fields in Soliton Background of Six-dimensional Pure Kaluza-Klein Theories*

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Dirac wave functions in the six-dimensional Abelian monopole geometry exhibit a singularity around $r = 0$, which is in contrast to the five-dimensional theory. Field equations for fermions in soliton background of a six-dimensional pure Kaluza-Klein theory are derived and illustrated by the spherically symmetric soliton case. Neutral fermion equations are decoupled into ordinary differential equations which agree with those obtained by the nulltetrad formalism in the Schwarzschild black-hole case.

The five-dimensional soliton solution of the Kaluza-Klein theories was first discovered by Gross-Perry [1] and Sorkin [2]. Recently, we found the soliton solution of $(4 + k)$-dimensional Abelian theories (it is also called pure Kaluza-Klein theory) [3]. Moreover, we obtained the most general spherically symmetric monopole of the six-dimensional Kaluza-Klein theory. We also found that there are substantial differences between the dynamical property of a charged fermion in the six-dimensional Abelian monopole geometry and that in the four- or five-dimensional theories [4]. Dirac wave functions in the six-dimensional theory exhibit a singularity around $r = 0$, but there is no phenomenon analogous to the Rubakov-Callan effect. On the other hand, the interaction of fermions with gravitational fields has attracted much attention. Brill

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and Wheeler [5] separated the Dirac equation in a spherically symmetric background space-time. So did Chandrasekhar [6] in Kerr geometry, Page and Güven [7] in Kerr-Newmann geometry. Soh and Pac [8] investigated fermion fields in soliton background of the five-dimensional pure Kaluza-Klein theories. Combining the two research aspects mentioned above, we shall present a general method to derive the fermion fields in soliton background of a six-dimensional pure Kaluza-Klein theory. This is the generalization of our discussions in Ref. [4]. Using Clifford algebra and spinor representation of $SO(5,1)$, we construct the spin-connection one-form which is necessary for the field equation in six-dimensional curved space-time and illustrated by the spherically symmetry soliton case. Neutral fermion equations are decoupled into ordinary differential equations which agree with those obtained by the nulltetrad formalism in the Schwarzschild black-hole case.

In this paper, the notation we used are the same as those in Ref. [4]. The signatary of $g_{\hat{\mu}\hat{\nu}}$ is $-+++++$, $x^{\hat{\mu}}=(x^{\mu}, y^{i})$, $\mu=0,1,2,3$; $i=1,2$.

The actions are:

$$
\begin{aligned}
S &= S_G + S_F,\\
S_G &= -\frac{1}{16\pi G_6}\int d^6x\sqrt{-g_6}\; R_6 \\
S_F &= i\int d^6x\sqrt{-g_6}\;\bar{\psi}\gamma^{\hat{c}}e_{\hat{c}}^{\hat{\mu}}(\partial_{\hat{\mu}}-\omega_{\hat{\mu}})\psi,
\end{aligned}
\tag{1}
$$

where $\gamma^{\hat{c}}$ is the six-dimensional Dirac matrix in flat space-time. $\omega_{\hat{\mu}}$ the spin-connection one-form, $e_{\hat{c}}^{\hat{\mu}}$ the sechs-bein. It is assumed that the vacuum geometry is $M^4\otimes S^1_{(1)}\otimes S^1_{(2)}$, where M^4 denotes the four-dimensional Minkowski space and $S^1_{(i)}$ is a circle of radius R_i. The metric $g_{\hat{\mu}\hat{\nu}}$ the fermion fields ψ become periodic functions of y_i with periodicity of $2\pi R_i$.

$$
\begin{aligned}
g_{\hat{\mu}\hat{\nu}}(x, y) &= \sum_{n_1,n_2} g_{\hat{\mu}\hat{\nu}}^{(n_1,n_2)}(x)\exp[i(n_1y^1/R_1+n_2y^2/R_2)],\\
\psi(x, y) &= \sum_{n_1,n_2} \psi^{(n_1,n_2)}(x)\exp[i(n_1y^1/R_1+n_2y^2/R_2)],
\end{aligned}
\tag{2}
$$

Where $n_1, n_2 = 1,2,3,\ldots$. Now consider the metric of a six-dimensional Kaluza-Klein theory

$$
ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} + V(dy^1 + A_{\mu}dx^{\mu})^2 + (dy^2)^2. \tag{3}
$$

We introduce the sechs-bein field $e_{\hat{c}}^{\hat{\mu}}$ by

$$
\eta_{\hat{a}\hat{b}}e^{\hat{a}}_{\hat{\mu}}e^{\hat{b}}_{\hat{\nu}} = g_{\hat{\mu}\hat{\nu}}, \tag{4}
$$

so that

$$
e_{\hat{c}}^{\hat{\mu}} = \begin{bmatrix} e_{\mu}{}^{c} & 0 & 0 \\ -A_{\mu}e_{\mu}{}^{c} & \frac{1}{\sqrt{V}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}
$$

Using, a conformal transformation $g_{\mu\nu} \to V^{-\frac{1}{3}} g_{\mu\nu}$, $V \to V^{-\frac{1}{3}} V$ and $\psi \to V^{\frac{1}{12}} \psi$, so that action (1) becomes

$$S = \frac{-1}{16\pi G_4} \int d^4x \sqrt{-g_4} \left(R_4 + \frac{1}{6} g^{\mu\nu} \cdot \frac{\partial_\mu V \partial_\nu V}{V} + \frac{1}{4} V^{\frac{1}{3}} F_{\mu\nu} F^{\mu\nu} \right)$$
$$+ i \int d^4x \sqrt{-g_4} \left[\bar{\psi} \gamma^c e_{\mu}{}^{c} \left(\partial_\mu - \omega'_\mu - A_\mu \frac{in_1}{R_1} + A_\mu \omega'_5 \right) \right] \psi \qquad (6)$$
$$+ \bar{\psi} \gamma^5 \left(\frac{in_1}{R_1} - \omega'_5 \right) \frac{1}{\sqrt{V}} \psi + \bar{\psi} \gamma^6 \left(\frac{in_2}{R_2} - \omega'_6 \right) \psi \Big],$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength and ω' is the new ω after this rescaling transformation. In order to derive the spin-connection one-form ω(and ω') in the orthonormal frame, we use the relation: $\theta^{\hat{a}} = e^{\hat{a}}_{\hat{\mu}} dx^{\hat{\mu}}, d\theta^{\hat{a}} = -\Gamma^{\hat{a}}_{\hat{b}} \Lambda \theta^{\hat{b}}$ $\Gamma_{\hat{a}\hat{b}} + \Gamma_{\hat{b}\hat{a}} = 0$.

The connection one-form Γ can be calculated accordingly. Under the scale transformation, Γ changes to Γ',

$$\Gamma'^{\hat{a}}_{\hat{b}} = \Gamma^{\hat{a}}_{\hat{b}} - (f'^{\hat{a}} \theta_{\hat{b}} - f'_{\hat{b}} \theta^{\hat{a}}), \qquad (7)$$

where

$$f'_{\hat{a}} = e^{\hat{\mu}}_{\hat{a}} \partial_{\hat{\mu}} (\ln V^{-\frac{1}{6}}).$$

If we consider connection Γ' as a SO(5,1) Lie algebra valued one form, we can directly construct ω'.

As an application of the general formalism discussed above, we consider a six-dimensional spherically symmetric metric[1,3].

$$g_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} = - \left[\frac{1 - \frac{m}{r}}{1 + \frac{m}{r}} \right]^{\frac{2}{\alpha}} (dt)^2 + \left[1 + \frac{m}{r} \right]^4 \left[\frac{1 - \frac{m}{r}}{1 + \frac{m}{r}} \right]^{2(\alpha - \beta - 1)/\alpha} (dr^2 + r^2 d\Omega^2)$$
$$+ \left[\frac{1 - \frac{m}{r}}{1 + \frac{m}{r}} \right]^{\frac{2\beta}{\alpha}} (dy^1)^2 + (dy^2)^2, \qquad (8)$$

where

$$\alpha = \left(\beta^2 + \beta + \frac{1}{2} \right)^{\frac{1}{2}}, \quad d\Omega^2 = d\theta^2 + \mathrm{Sin}^{-2} \theta \, d\phi^2.$$

The solutions are singular at $r = m$ except the Schwarzschild case ($\alpha = 1, \beta = 0$) and the soliton solution ($\alpha = \beta = \infty$). The one-forms are simply:

$$\theta'^{\hat{t}} = g_{tt}^{\frac{1}{2}} g_{55}^{-\frac{1}{6}} dt, \quad \theta'^{\hat{r}} = g_{rr}^{\frac{1}{2}} g_{55}^{-\frac{1}{6}} dr, \quad \theta'^{\hat{\theta}} = g_{rr}^{\frac{1}{2}} g_{55}^{-\frac{1}{6}} r d\theta, \qquad (9)$$
$$\theta'^{\hat{\phi}} = g_{rr}^{\frac{1}{2}} g_{55}^{-\frac{1}{6}} r \sin\theta d\phi, \quad \theta'^{\hat{5}} = g_{55}^{\frac{2}{3}} dy^1, \quad \theta'^{\hat{6}} = g_{55}^{-\frac{1}{3}} dy^2.$$

The connection one-form Γ' can be directly calculated,

$$\Gamma' = a_0\theta^{\hat t}L^{\hat t\hat r} + a_1\theta^{\hat\theta}L^{\hat r\hat\theta} + a_1\theta^{\hat\phi}L^{\hat r\hat\phi} + a_2\theta^{\hat\phi}L^{\hat\theta\hat\phi} + a_3\theta^{\hat 5}L^{\hat 5\hat 5} + a_4\theta^{\hat 6}L^{\hat 6\hat 6}, \tag{10}$$

where $L^{\hat c\hat b}$ is matrix generator of SO(5,1), and

$$\begin{aligned} a_0 &= (g_{tt}g_{rr})^{-\frac{1}{2}}(dg_{tt}^{\frac{1}{2}}/dr) - \frac{1}{3}(g_{rr}g_{55})^{-\frac{1}{2}}(dg_{55}^{\frac{1}{2}}/dr), \\ a_1 &= r^{-2}g_{rr}^{-\frac{1}{2}} + r^{-1}g_{rr}^{-1}(dg_{rr}^{\frac{1}{2}}/dr) - \frac{1}{3}(g_{rr}g_{55})^{-\frac{1}{2}}(dg_{55}^{\frac{1}{2}}/dr), \\ a_2 &= (rg_{rr}^{\frac{1}{2}})^{-1}\cot\theta,\quad a_3 = \frac{2}{3}(g_{rr}g_{55})^{-\frac{1}{2}}(dg_{55}^{\frac{1}{2}}/dr), \\ a_4 &= \frac{1}{3}(g_{rr}g_{55})^{-\frac{1}{2}}(dg_{55}^{\frac{1}{2}}/dr). \end{aligned} \tag{11}$$

We may define aggregates of a M^6 multi-vector [4,11] to build up the Clifford algebra C_6. A M^6 spinor is an eight-component vector of S_8. In particular, corresponding to the orthonormal vector basis, $e_0, e_1, \ldots, e_6$ of M^6 we have $\gamma^0, \gamma^1, \ldots \gamma^6$, where $\gamma^{\hat c}$ are 8 x 8 matrices satisfying $\{\gamma^{\hat c}, \gamma^{\hat b}\} = 2\eta^{\hat c\hat b}$. Let us choose the Dirac-type basis, then

$$\gamma^t = \begin{bmatrix} 0 & -I & & \\ I & 0 & & \\ & & 0 & -I \\ & & I & 0 \end{bmatrix}, \quad \gamma^r = \begin{bmatrix} -\sigma_3 & 0 & & \\ 0 & \sigma_3 & & \\ & & -\sigma_3 & 0 \\ & & 0 & \sigma_3 \end{bmatrix},$$

$$\gamma^\theta = \begin{bmatrix} -\sigma_1 & 0 & & \\ 0 & \sigma_1 & & \\ & & -\sigma_1 & 0 \\ & & 0 & \sigma_1 \end{bmatrix}, \quad \gamma^\phi = \begin{bmatrix} -\sigma_2 & 0 & & \\ 0 & \sigma_2 & & \\ & & -\sigma_2 & 0 \\ & & 0 & \sigma_1 \end{bmatrix}, \tag{12}$$

$$\gamma^5 = \begin{bmatrix} & & 0 & -I \\ & & -I & 0 \\ 0 & -I & & \\ -I & 0 & & \end{bmatrix}, \quad \gamma^6 = \begin{bmatrix} & & 0 & iI \\ & & iI & 0 \\ 0 & -iI & & \\ -iI & 0 & & \end{bmatrix},$$

We set $\tilde\Psi = (g,f,h,j)$. Using these $\gamma^{\hat c}$ matrices, we construct the generators of SO(5,1) Lie algebra, $\Sigma^{\hat c\hat b} = \frac{1}{2i}[\gamma^{\hat c}, \gamma^{\hat b}]$, which give the spinor representation. The spin-connection one-form ω' is then

$$\begin{aligned} \omega' = \frac{i}{2}\,(a_0\hat\theta^t\Sigma^{\hat t\hat r} &+ a_1\theta^{\hat\theta}\Sigma^{\hat r\hat\phi} + a_1\theta^{\hat\phi}\Sigma^{\hat r\hat\phi} + a_2\theta^{\hat\phi}\Sigma^{\hat\theta\hat\phi} \\ &+ a_3\theta^{\hat 5}\Sigma^{\hat 5} + a_4\theta^{\hat 6}\Sigma^{\hat r\hat 6}). \end{aligned} \tag{13}$$

and the field equations of ψ are

$$\begin{aligned} -g_{tt}^{-\frac{1}{2}}\partial_t g - uf - vj &= 0, \\ g_{tt}^{-\frac{1}{2}}\partial_t f + uf - vh &= 0, \\ -g_{tt}^{-\frac{1}{2}}\partial_t j - uh - vg &= 0, \\ g_{tt}^{-\frac{1}{2}}\partial_t h + u_j - vf &= 0, \end{aligned} \tag{14}$$

where $u = g_{rr}^{-\frac{1}{2}}\sigma_2\partial_r + r^{-1}g_{rr}^{-\frac{1}{2}}\sigma_1\partial_\theta + (r\sin\theta)^{-1}g_{rr}^{-1}\sigma_3\partial_\phi + \Big[\frac{1}{4}a_0 + \frac{1}{2}a_1 + \frac{1}{4}a_3 + \frac{1}{12}$ $g_{55}^{-\frac{1}{2}} \times (dg_{55}^{\frac{1}{2}}/dr)\Big]\sigma_3 + \frac{1}{2}a_1\sigma_1$, $v = g_{55}^{-\frac{1}{2}}\frac{in_1}{R_1} - \frac{n_2}{R_2}$.

when $n_1 = n_2 = 0$, which corresponds to the case of neutral massless fermions, Eqs.(14) are reduced to $-g_{tt}^{-\frac{1}{2}}\partial_t g - uf = 0$ and $g_{tt}^{-\frac{1}{2}}\partial_t f - ug = 0$, and $-g_{tt}^{-\frac{1}{2}}\partial_t j - uh = 0$, $g_{tt}^{-\frac{1}{2}}\partial_t h - uj = 0$. The differences among these equations are only in the signs of the operator $g_{tt}^{-\frac{1}{2}}\partial_t$. We should note that the equations can be decoupled. For example, we can immediately obtain the decoupled equation for g_1 which is one of the two components of g:

$$\Big\{\Big(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + \frac{1}{2}\operatorname{ctg}\theta\Big)\Big(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi + \frac{1}{2}\operatorname{ctg}\theta\Big) - (rg_{rr}^{\frac{1}{2}}g_{tt}^{-\frac{1}{2}}\partial_t - r\partial_r - s)(rg_{rr}^{\frac{1}{2}}g_{tt}^{-\frac{1}{2}} + r\partial_r + s)\Big\}g_1 = 0, \tag{15}$$

The equation for the other component g_2 is similarly obtained by changing the order of differential operators in Eq.(15), where

$$s = 1 + \frac{r}{2}g_{tt}^{-\frac{1}{2}}\frac{dg_{tt}^{\frac{1}{2}}}{dt} + rg_{rr}^{-\frac{1}{2}}\frac{dg_{rr}^{\frac{1}{2}}}{dr}. \tag{A}$$

Eq.(15) is consistent with that obtained by the nulltetrad formalism in the Schwarzschild black-hole case. It is also a simple generalization of the Teukolsky and Boses neutrino equations [9,10].

An important application of Eq.(7) is given in Ref.[4], namely, the Rubakov-Callan effect of six-dimensional Abelian monopole obtained through a direct calculation.

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