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Analytical Study of $U(1)$-Higgs System On a Lattice*

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The Higgs-U (1) gauge system ($Q = 1$) with radial degrees of freedom in $d = 4$ is studied by using variational-cumulant expansion method. Equations for determining the phase transitions of the model are given. Our result is qualitatively in agreement with Monte Carlo result in the case of $d = 4$. The result of $d = 5$ is also given.

1. INTRODUCTION

The importance of the Higgs mechanism has aroused our great interest in the research of gauge systems coupled with Higgs fields. A deeper understanding of such systems will be helpful to the study of cosmological problems and may bring some insight into the methodology of the study of gauge systems coupled with fermions.

The early analytical studies of coupled $U(1)$-Higgs systems are devoted to the case with Higgs fields of fixed length. Fradkin and Shenker [1] proved that when the Higgs fields transform like the fundamental representation of the gauge group, the confinement and the Higgs phases are continuously connected. This was confirmed later by the Monte Carlo simulations. However, when the length of the Higgs fields are allowed to fluctuate, the situation is changed. Monte Carlo studies by Munehisa [2,3] and Koutsoumbas [4] showed that the confinement and the Higgs phases are no longer continuously connected when the self-coupling constant λ of Higgs fields is small enough. In the weak coupling region the Higgs and the Coulomb phases are separated by a phase transition line. But there is some divergency in terms of the order of the phase transition. [2] ([4]) stated that the transition is of the first (second) order. Later Munehisa [3] pointed out that the disagreement is superficial, since the two studies used different parameters in their

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models. Toussaint and Sugar [5] demonstrated a first order phase transition by Monte Carlo simulations. Since few analytical studies have been made of Higgs fields with a radial degrees of freedom, and in order to have a more clear physical understanding of the Monte Carlo results, it is necessary to have an analytical study, even though the results may be still rough.

Traditionally in order to get a phase diagram by analytical method one chooses and calculates some order parameter. From the discontinuities of the order parameter one can determine the phase transition points, thus getting the phase transition lines and the phase diagram. Based on the variational-cumulant expansion method [6], we propose in this paper a method of deriving equations of phase transition line direct without detailed calculations of order parameters.

2. METHOD

We used the following action for the $U(1)$-Higgs system[2].

$$S = -\frac{\beta}{2}\sum_p (\mathrm{tr}U_p + \mathrm{tr}U_p^+) + \sum_i [\lambda(\phi_i\phi_i^+)^2 + m^2\phi_i\phi_i^+]$$
$$+ \sum_{i,\mu} [2\phi_i\phi_i^+ - (\phi_i U_i \phi_{i+\mu}^+ + \mathrm{h.c.})], \tag{1}$$

where ϕ_i is the Higgs field defined on the site i on the lattice. U_p is the product of $U_l \in U(1)$ along the four links of a plaquette, ϕ_i and U_l are in the fundamental representation. $\sum_p$, $\sum_i$, $\sum_{i,\mu}$ are sums over all plaquettes, sites and links respectively. Let $U_{i,\mu} \equiv U_l = e^{i\theta_l}$, then $U_p = e^{i\theta_p}$, $\theta_p = \theta_i + \theta_j - \theta_k - \theta_l$, where i,j,k,l denote four links of a plaquette. Let $\phi_i = \rho_i V_i$ with $V_i = e^{i\varphi_i}$ and ρ_i is real, $0 \le \rho_i \le \infty$, the corresponding integral measure $[d\phi^+ d\phi]$ becomes $[\rho d\rho dV]$. After a gauge transformation $U_{i,\mu} \to V_i U_{i,\mu} V_{i+\mu}$, we have

$$S = -\beta\sum_p \cos\theta_p + \sum_i (\lambda\rho_i^4 + m^2\rho_i^2) + \sum_{i,\mu} [2\rho_i^2 - 2\rho_i\rho_{i+\mu}\cos\theta_l], \tag{2}$$

For simplicity we adopt an approximation $\rho_i = \rho_{i+\mu} = \rho$, then

$$S = -\beta\sum_p \cos\theta_p + M(\lambda\rho^4 + m^2\rho^2) + \sum_l (2\rho^2 - 2\rho^2\cos\theta_l), \tag{3}$$

where M is the total number of sites. $\sum_l$ runs over all links. The partition function of the system can be written as

$$Z \equiv e^{-W_1 M} = \int [\rho d\rho]\, e^{-M(\lambda\rho^4 + m^2\rho^2) - 2Md\rho} \cdot \int \left[\frac{d\theta}{2\pi}\right] e^{\beta\sum_p \cos\theta_p} \cdot e^{2\rho^2 \sum_l \cos\theta_l}, \tag{4}$$

where d is the dimensionality of the system. The $U(1)$ group integral in Eq.(4) may be treated by using the variational-cumulant expansion technique. Let $J = 2\rho^2$, then $2\rho^2 \sum_l \cos\theta_l = J\sum_l \cos\theta_l$.

It corresponds just to the trial action S_0 in the variational treatment of the pure $U(1)$ gauge model [6], then

$$Z_0 = \int_{-\pi}^{\pi}\left[\frac{d\theta}{2\pi}\right] e^{J\sum_l \cos\theta_l} = [I_0(J)]^{Md}, \tag{5}$$

$$\int_{-\pi}^{\pi}\left[\frac{d\theta}{2\pi}\right] e^{\beta\sum_p \cos\theta_p}\cdot e^{J\sum_l \cos\theta_l} \equiv Z_0\left\langle e^{\beta\sum_p \cos\theta_p}\right\rangle_0 \tag{6}$$

$$= Z_0 e^{\sum_{n=1}^{\infty}\frac{1}{n!}K_n(\beta,J)}, \tag{7}$$

Here Eq.(7) is just the cumulant expansion. Calculation of $K_n(\beta,J)$ is presented in the Appendix. Now we have

$$Z \equiv e^{-WM} = \int [dJ]\, e^{-2M\ln 2 - M(\frac{1}{4}\lambda J^2+\frac{1}{2}m^2 J) - MJd + M\cdot F(\beta,J)}, \tag{8}$$

where

$$M F(\beta,J) = M d \ln I_0(J) + \sum_{n=1}^{\infty}\frac{1}{n!}K_n(\beta,J), \tag{9}$$

The integral in Eq.(8) may be taken by using the saddle point method. Then the free energy per site is

$$W = 2\ln 2 + \frac{1}{4}\lambda J^2 + \frac{1}{2}m^2 J + Jd - F(\beta,J), \tag{10}$$

where J is determined by the saddle point equation

$$\frac{\lambda}{2}J + \frac{1}{2}m^2 + d - \frac{\partial}{\partial J}F(\beta,J) = 0 \tag{11}$$

Usually to determine the phase diagram one directly calculates the order parameter

$$\langle|\phi|^2\rangle = -\frac{\partial W}{\partial m^2}, \tag{12}$$

and from its discontinuity to find the phase transition point. Now, instead of calculating $\langle|\phi|^2\rangle$, we try to find directly where its singularities will take place. Substituting Eq.(10) into Eq.(12), we have

$$\langle|\phi|^2\rangle = -\frac{1}{2}J - \left[\frac{\lambda}{2}J + \frac{1}{2}m^2 + d - \frac{\partial F(\beta,J)}{\partial J}\right]\frac{\partial J}{\partial m^2}, \tag{13}$$

Obviously its singularity can happen only when $\frac{\partial J}{\partial m^2}$ becomes infinite. The dependence of J on m^2 can be obtained from Eq.(11) by taking the partial derivative with respect to m^2

$$\frac{1}{2} + \left[\frac{1}{2}\lambda - \frac{\partial^2}{\partial J^2}F(\beta,J)\right]\frac{\partial J}{\partial m^2} = 0, \tag{14}$$

Then $\frac{\partial J}{\partial m^2}$ becomes infinite only when

$$\frac{1}{2}\lambda - \frac{\partial^2}{\partial J^2}F(\beta, J) = 0, \tag{15}$$

thus the equation of phase transition line is determined from the system of Equations (15) and (11). After we have got $F(\beta,J)$ (see the appendix), calculation of $\frac{\partial J}{\partial m^2}F(\beta,J)$ is straightforward. In this paper $F(\beta,J)$ is calculated to the 4-th order of cumulant expansion.

3. RESULTS AND DISCUSSION

In the four dimension the phase structure is presented in Fig. 1 for $\lambda = 0.8$. The dashed line is the Monte Carlo result [2]. When β varies from 0 to 0.55, only one solution is found and presented by a solid line. When $\beta > 0.55$ three branches of solutions appear. One of them with the lowest free energy is presented by the solid line in the $\beta > 0.55$ region. It shows the existence of the first order phase transition in the weak coupling region. The second branch of solutions is unphysical, since it has higher free energy and m^2 quickly approaches infinity with the increasing β. The third branch of solutions with higher energy is presented in Fig. 1 by dot-dashed line in the $\beta > 0.55$ region. It separates the strong coupling region and the weak coupling region. Physically, this branch with a higher free energy corresponds to an unstable solution of the system. This unstability suggests the possibility of the existence of a phase transition of higher orders, which is in agreement with the Monte Carlo estimation that there is a phase transition of 2-nd order between the confinement phase and Coulomb phase. The Higgs and Coulomb phases are connected in the vicinity of $\beta \sim 0.5$ (Fig. 1). We think it is due to the approximation we have made, and we expect that including higher order cumulant expansions will provide a better result. Certainly the error also may be caused by the approximation $\rho_x = \rho_{x+\mu} = \rho$. Our calculation shows when $0 \leq \lambda \leq 3.985$ the Higgs and the confinement phases are separated by a phase transition line of the first order while when $\lambda > 3.985$ the two phases

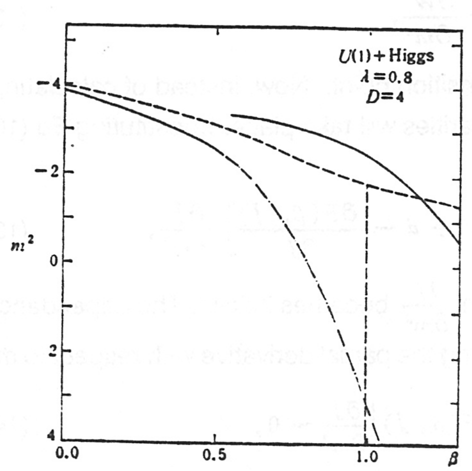


FIG. 1

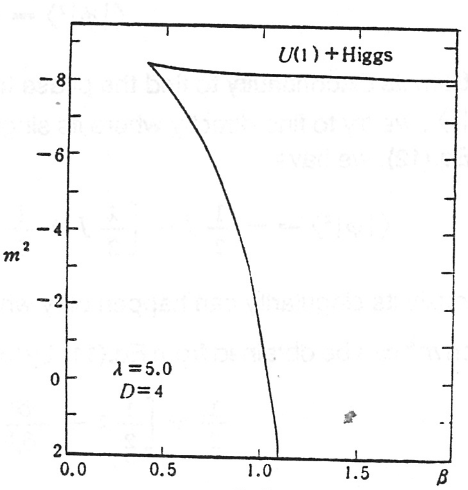


FIG. 2

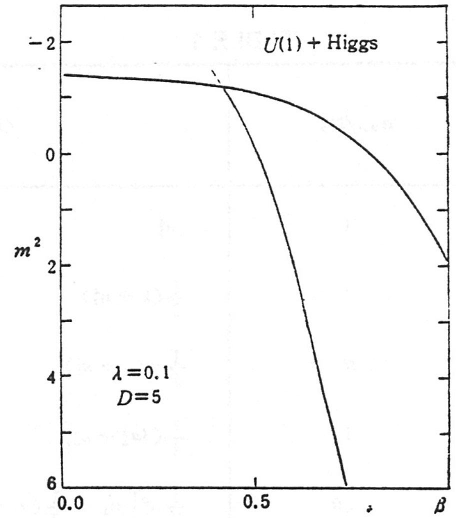


FIG. 3

are analytically connected.

In Fig. 2 the result for $\lambda = 5.0$ is presented. It approximates to the case of fixed length ($\lambda \to \infty$) and is in agreement with the theoretical expectation by Fradkin and Shenker [1].

In the five dimension when $0 \leq \lambda \leq 4.981$ the Higgs phase is separated from the confinement phase by a first order phase transition line, while $\lambda > 4.981$ the two phases are analytically connected. The phase diagram for $\lambda = 0.1$ is plotted in Fig. 3. No corresponding Monte Carlo results are available.

Appendix. The calculation of $K_n(\beta, J)$

It is convenient to use the diagrammatic notation in the calculation of $K_n(\beta,J)$, for example

$$K_1(\beta, J) = \beta\left\langle\sum_p \cos\theta_p\right\rangle_0 = \beta\left\langle\sum_p \Box\right\rangle_0,$$

$$K_2(\beta,J) = \beta^2\left\langle\left(\sum_p \cos\theta_p\right)^2\right\rangle_0 - \beta^2\left\langle\sum_p \cos\theta_p\right\rangle_0^2$$

$$= \beta^2\left\langle\left(\sum_p \Box\right)^2\right\rangle_0 - \beta^2\left\langle\sum_p \Box\right\rangle_0^2 = \beta^2\left\langle\left(\sum_p \Box\right)^2\right\rangle_c,$$

In $K_n(\beta,J)$ only connected diagrams contribute. Thus in the calculation of higher order K_n, one should draw all possible connected diagrams and count the number of equivalent diagrams as in the case of $SU(2)$ [6], then calculate the statistical average of each equivalent diagram with respect to S_0. For $U(1)$ gauge group the calculation of these statistical averages is much easier than the one for the $SU(2)$ case. For example

$$\langle \boxed{4} \rangle_0 = \langle \cos^4\theta_p \rangle_0$$

TABLE 1

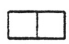
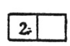
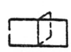
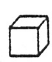
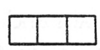
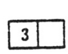
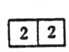
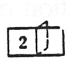
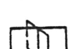
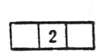
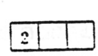
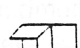
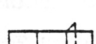
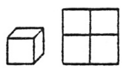
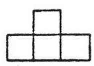
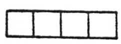
n	i	$D_{n,i}$	$\alpha_{n,i}/N_p$	$\langle D_{n,i}\rangle_0$
1	1		1	u_1^4
2	1		1	$\frac{1}{2}(1+u_2^4)$
	2		R	$\frac{1}{2}u_1^6(1+u_2)$
3	1		1	$\frac{1}{4}(3u_1^4+u_3^4)$
	2		$3R$	$\frac{1}{2}u_1^8\left[u_1+\frac{1}{2}(u_1+u_3)u_2^3\right]$
	3		RR_1	$\frac{1}{4}u_1^9(3u_1+u_3)$
	4		$8R_1$	$\frac{1}{4}u_1^6(1+3u_2^2)$
	5		$3(3RR_0-8R_1)$	$\frac{1}{4}u_1^8(1+u_2)^2$
4	1		1	$\frac{1}{8}(3+4u_2^4+u_4^4)$
	2		$4R$	$\frac{1}{8}u_1^8[3u_1^3(1+u_2)+u_3^3(u_2+u_4)]$
	3		$3R$	$\frac{1}{4}\left(1+2u_2^4+\frac{1}{2}u_2^6+\frac{1}{2}u_2^6u_4\right)$
	4		$6RR_1$	$\frac{1}{4}u_1^4\left(1+u_2+\frac{1}{2}u_2^7+u_2^4+\frac{1}{2}u_2^3u_4\right)$
	5		RR_1R_2	$\frac{1}{8}u_1^{12}(3+4u_2+u_4)$
	6		$48R_1$	$\frac{1}{4}u_1^4\left[(1+u_2)u_1^2+\frac{1}{2}u_2^2(u_1^2+u_3^2)+u_1u_2^3u_3\right]$
	7		$6(3RR_0-8R_1)$	$\frac{1}{4}u_1^6\left[2u_1^2+\frac{1}{2}u_2^2(u_1+u_3)^2\right]$
	8		$12(3RR_0-8R_1)$	$\frac{1}{4}u_1^5(1+u_2)\left[u_1+\frac{1}{2}(u_1+u_3)u_2^3\right]$
	9		$96R_1^2$	$\frac{1}{4}u_1^9(u_1+u_2u_3+u_1u_2+u_1u_2^2)$
	10		$48R_1(R_0^2+4R_1+2R_2^2)$	$\frac{1}{8}u_1^{11}(3u_1+u_3+u_2u_3+3u_1u_2)$

TABLE 1
Continued

n	i	$D_{n,i}$	$\alpha_{n,i}/N_p$	$\langle D_{n,i}\rangle_0$
	11		$48R_1$	$\frac{1}{8}u_1^6(1+2u_2^2+4u_2^3+u_2^4)$
	12		$192R_1^2$,	$\frac{1}{8}u_1^8(1+u_2+3u_2^2+3u_2^3)$
	13		$12(2R_0+R_1+2R_1R_2)$	$\frac{1}{8}u_1^8(1+6u_2^2+u_2^4)$
	14		$96(R_0^3-2R_1^2-R_1)$	$\frac{1}{8}u_1^{10}(1+3u_2+3u_2^2+u_2^3)$
	15		$48(9R_0^3-14R_0^2+19R_0-7)$	$\frac{1}{8}u_1^{10}(1+3u_2+3u_2^2+u_2^3)$

$$= \frac{1}{z_0}\int_{-\pi}^{\pi}\left[\frac{d\theta}{2\pi}\right]\cos^4(\theta_1+\theta_2-\theta_3-\theta_4)e^{J\sum_l \cos\theta_l}$$

$$= \frac{1}{z_0}\int_{-\pi}^{\pi}\left[\frac{d\theta}{2\pi}\right]\left\{\frac{1}{8}\cos 4\theta_p+\frac{1}{2}\cos 2\theta_p+\frac{3}{8}\right\}e^{J\sum_l \cos\theta_l}$$

$$= \frac{1}{8}\langle 3+4\cos 2\theta_p+\cos 4\theta_p\rangle_0$$

$$= \frac{1}{8}(3+4u_2^4+u_4^4),$$

where $u_n = \frac{I_n(J)}{I_0(J)}$. Let

$$K_n(\beta,J) = \left\langle\left(\sum_p \square\right)^n\right\rangle_c = N_p\beta^n\sum_i \alpha_{n,i}\langle D_{n,i}\rangle_c,$$

$\alpha_{n,i}$, $D_{n,i}$ are listed to $n=4$ in Table 1, where the following notation has been used: $R = 4R_0$, $R_0 = 2d-3$, $R_1 = 2d-4$, $R_2 = 2d-5$, $N_p = Md(d-1)/2$.

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