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Quantization of Pure Gauge Fields on Coset Space of Abelian Chiral Group

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Due to the change of the path-integral measure for fermion fields under chiral transformation, an extra term (chiral anomalous term) is added to the action of the generating functional for the coset pure gauge field theory of Abelian chiral group. The path-integral quantization of this theory is realized and the BRS invariance of the generating functional is restored with the aid of a chiral anomalous term. Ward identities, consistent with the classical PCAC equations, are also deduced by using the functional derivative technique.

1. INTRODUCTION

In Ref. [1] the quantization of the coset pure gauge fields with general Lie group G was studied. When G is a chiral group. However, owing to the change of the path integral measure for the fermion fields under chiral transformation [2], the usual BRS invariance of the generating functional will be broken. Therefore, for this special case, we must go a step further to study the quantization problem on coset pure gauge fields.

Consider the simplest chiral group $G = U(1)\times U(1)_5$ and take $H = U(1)$ as its subgroup. A Lagrangian which is locally gauge invariant under the subgroup H is

$$\mathscr{L}^{(A)} = -\frac{1}{4} F_{\mu\nu}^2 - \bar{\psi}\gamma_\mu(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi, \tag{1.1}$$

where A_μ is the vector gauge field in the subgroup H, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We introduce

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coset pure gauge field $\phi_0(x) \in G/H$ which can be parametrized as $\phi_0(x) = e^{i\theta(x)\gamma_5}$ and make transformation

$$\psi'(x) = e^{i\theta(x)\gamma_5}\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{i\theta(x)\gamma_5}, \tag{1.2}$$

Under Eq.(1.2), $\mathscr{L}^{(A)}$ is transformed into

$$\mathscr{L}^{(A)} = -\frac{1}{4}F_{\mu\nu}^2 - \bar{\psi}'\gamma_\mu(\partial_\mu - ieA_\mu - i\gamma_5\partial_\mu\theta)\psi' - m\bar{\psi}'e^{-2i\theta\gamma_5}\psi'. \tag{1.3}$$

For the sake of convenience, we omit the prime in Eq.(1.3) and rewrite it as

$$\mathscr{L}^{(B)} = -\frac{1}{4}F_{\mu\nu}^2 - \bar{\psi}\gamma_\mu(\partial_\mu - ieA_\mu - i\gamma_5\partial_\mu\theta)\psi - m\bar{\psi}e^{-2i\theta\gamma_5}\psi. \tag{1.4}$$

It can be easily proved that $\mathscr{L}^{(B)}$ is locally invariant under the chiral group G. Clearly, if $\theta(x) = 0$, $\mathscr{L}^{(B)}$ will go back to $\mathscr{L}^{(A)}$ and the difference between $\mathscr{L}^{(A)}$ and $\mathscr{L}^{(B)}$ is just a gauge transformation. So we call the gauge $\theta = 0$ a renormalizable gauge [3].

The generating functional corresponding to $\mathscr{L}^{(A)}$ can be written briefly as (the gauge fixing term etc. are omitted temporarily)

$$Z^{(A)} = \int [d\psi d\bar{\psi} dA_\mu] e^{i\int d^4x \mathscr{L}^{(A)}} \tag{1.5}$$

Obviously, both $\mathscr{L}^{(A)}$ and $Z^{(A)}$ are all locally invariant in the subgroup H. In the first sight, according to the conventional way, it seems that the generating functional corresponding to $\mathscr{L}^{(B)}$ can also be written as

$$Z^{(B)\prime} = \int [d\psi d\bar{\psi} dA_\mu d\theta] e^{i\int d^4x \mathscr{L}^{(B)}}, \tag{1.6}$$

where $[d\theta]$ is the invariant integral measure of the coset pure gauge field [1]. However, although $\mathscr{L}^{(B)}$ is locally invariant in the group G, $Z^{(B)\prime}$ is not a generating function with the corresponding BRS invariance, because the path-integral measure of Eq.(1.6) is not invariant under a chiral transformation. Thus, it is difficult to realize the path-integral quantization from the generating functional $Z^{(B)\prime}$ by virtue of the ordinary Faddeev-Popov technique.

In this paper, a generating functional $Z^{(B)}$ which is locally invariant under the group G is derived strictly from the generating functional $Z^{(A)}$. Comparing this $Z^{(B)}$ with Eq. (1.6), we find that there is an additional term in the action of $Z^{(B)}$. Such an additional term may be called chiral anomalous term, for it relates closely to the Adler anomaly in axial-vector Ward identity. By virtue of the new generating functional $Z^{(B)}$, we realize the quantization of the coset pure gauge field under Abelian chiral group G, and prove that BRS invariance of $Z^{(B)}$ under G will be restored again. Furthermore, according to the familiar functional derivative technique, we can obtain the Ward identities in the non-perturbation approach. The Ward identities exactly coincide with the classical PCAC equation which is directly derived from the action of $Z^{(B)}$. Thus we have a divergence equation of the axial-vector current which has a consistent form both in classical and in quantum version. In this sense, we arrive at an anomaly-free theory.

This paper is arranged as follows: in Sec. 2 the expression of the generating functional $Z^{(B)}$ is derived; in Sec. 3 the path-integral quantization of this theory is realized; in Sec. 4 the BRS invariance of the generating functional $Z^{(B)}$ under the group G is proved, and in the last Sec., the consistent Ward identities are obtained both in classical and in quantum (non-perturbative) approach, and the main results in this paper are briefly discussed.

2. DERIVATION OF GENERATING FUNCTIONAL

The path-integral quantization of a gauge theory which is described by the Lagrangian (1.1) in subgroup $H = U(1)$ can be realized with the aid of the Faddeev-Popov technique [4]:

$$Z^{(A)} = \int [d\psi d\bar{\psi} dA_\mu]\delta(F_A)\Delta_A e^{i\int d^4x \mathscr{L}^{(A)}}, \tag{2.1}$$

where $F_A[A_\mu] = 0$ is the gauge condition, for example, $F_A = \partial_\mu A_\mu = 0$, and Δ_A is the Faddeev-Popov determinant, which satisfies the following relations:

$$\Delta_A \int [d\mu(h)]\delta(F_A^h) = 1, \quad F_A^h = F_A[h(\partial_\mu - ieA_\mu)h^{-1}], \tag{2.2}$$

where $h = e^{i\beta}$ is an element of the $U(1)$ group, and $[d\mu(h)]$ is the invariant integral measure of the $U(1)$ group,

$$[d\mu(h)] = [d\beta]. \tag{2.3}$$

In order to investigate the behaviour of the generating functional (2.1) under transformation (1.2), we introduce the invariant integral measure of coset space G/H

$$[d\mu(\phi_0)] = [d\theta], \tag{2.4}$$

so that

$$\int [d\theta]\delta(\theta) = 1. \tag{2.5}$$

Inserting Eq.(2.5) into the right hand side of Eq.(2.1), we find

$$Z^{(A)} = \int [d\psi d\bar{\psi} dA_\mu d\theta]\delta(F_A)\delta(\theta)\Delta_A e^{i\int d^4x \mathscr{L}^{(A)}}. \tag{2.6}$$

Fujikawa has pointed out that the path-integral measure of fermion $[d\psi d\bar{\psi}]$ is not invariant under a chiral transformation [2]. But his discussion was limited to the Euclidean space as well as to the infinitesimal transformation. Now, we can discuss this problem in the Minkowskian space and for any finite transformation. To this end we rewrite Eq.(1.2) as

$$\psi'_\sigma(x) = \int d^4x' (e^{i\theta(x)\gamma_5})_{\sigma\rho}\delta(x - x')\psi_\rho(x'), \tag{2.7}$$

and a similar expression for $\bar{\psi}'_\sigma(x)$. In terms of the discrete indices x and x' in coordinate space, we have

$$\psi'_\sigma(x) = \sum_{x'} A(x, x')_{\sigma\rho}\psi_\rho(x'), \tag{2.8}$$

where

$$A(x, x')_{\sigma\rho} = (e^{i\theta(x)\gamma_5})_{\sigma\rho}\delta_{x,x'}. \tag{2.9}$$

Because the fermion field quantities are elements of the Grassmann algebra in the path-integral theory, the integral measure for ψ is transformed into

$$[d\psi'] = \prod_x d\psi'(x) = (\mathrm{Det}A(x, x'))^{-1} \prod_x d\psi(x) = (\mathrm{Det}A)^{-1}[d\psi]. \tag{2.10}$$

under the chiral transformation (2.8). From Eq.(2.9) it can be easily seen that A is a unitary and diagonalizable matrix, and its determinant has the form

$$\det A = \mathrm{Det}[(e^{i\theta(x)\gamma_5})_{\sigma\rho}\delta_{x,x'}] = e^{i\mathrm{Tr}\theta\gamma_5}, \tag{2.11}$$

where we take a formal sign

$$\begin{aligned}\mathrm{Tr}\theta\gamma_5 &= \mathrm{tr}\gamma_5 \sum_x \theta(x) = \mathrm{tr}\gamma_5 \sum_x \theta(x)\delta_{x,x'}\big|_{x'=x} \\ &= \int d^4x\,\mathrm{tr}\theta(x)\gamma_5\delta(x - x')\big|_{x'=x}.\end{aligned} \tag{2.12}$$

It is clear that $\mathrm{Tr}\theta\gamma_5$ is an ambiguous quantity in the form $0 \times \infty$ mathematically. In order to give the calculating result a definite physical meaning, we must take a correct limiting procedure to calculate this ambiguous quantity, in other words, we must assign a regularization scheme. Therefore, we introduce the following regularization factor

$$e^{i\frac{\not{D}^2(x)}{M^2}}\big|_{M\to\infty} = 1 \tag{2.13}$$

to calculate $\mathrm{Tr}\theta\gamma_5$ where $\not{D}(x) = \gamma_\mu(\partial_\mu - ieA_\mu(x))$ is a Dirac operator determined by Lagrangian (1.1). Thus,

$$\mathrm{Tr}\theta\gamma_5 = \int d^4x\,\mathrm{tr}\theta(x)\gamma_5 e^{i\frac{\not{D}^2(x)}{M^2}}\delta(x - x')\big|_{\substack{x'=x,\\ M\to\infty}} \tag{2.14}$$

Inserting the plane wave expansion for the δ-function in Eq. (2.14) and making further calculation, we have

$$\mathrm{Tr}\theta\gamma_5 = -\frac{ie^2}{16\pi^2}\int d^4x\theta(x)F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x), \quad (2.15)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$. In terms of Eq.(2.15), Eq.(2.10) can be written as

$$[d\psi'] = [d\psi]e^{i\int d^4x \frac{ie^2}{16\pi^2}\theta F_{\mu\nu}\tilde{F}_{\mu\nu}}. \quad (2.16)$$

Similarly, under the transformation (1.2), the measure $[d\bar{\psi}]$ transforms into

$$[d\bar{\psi}'] = [d\bar{\psi}]e^{i\int d^4x \frac{ie^2}{16\pi^2}\theta F_{\mu\nu}\tilde{F}_{\mu\nu}}. \quad (2.17)$$

and then $[d\psi d\bar{\psi}]$ can be written as

$$[d\psi d\bar{\psi}] = [d\psi' d\bar{\psi}']\, e^{i\int d^4x \frac{-ie^2}{8\pi^2}\theta F_{\mu\nu}\tilde{F}_{\mu\nu}}. \quad (2.18)$$

Thus, under Eq.(1.2), the generating functional (2.6) becomes

$$Z^{(A)} = \int [d\psi' d\bar{\psi}' dA_\mu d\theta]\delta(F_A)\delta(\theta)\Delta_A \exp\Big\{i\int d^4x\Big(-\frac{1}{4}F^2_{\mu\nu} - \bar{\psi}'\gamma_\mu(\partial_\mu - ieA_\mu - i\gamma_5\partial_\mu\theta)\psi' - m\bar{\psi}'e^{-2i\theta\gamma_5}\psi' - \frac{ie^2}{8\pi^2}\theta F_{\mu\nu}\tilde{F}_{\mu\nu}\Big)\Big\}. \quad (2.19)$$

Omitting the prime in Eq.(2.19) and denoting it by $Z^{(B)}$, we have

$$Z^{(B)} = \int [d\psi d\bar{\psi} dA_\mu d\theta]\delta(F_A)\delta(\theta)\Delta_A e^{i\int d^4x(\mathscr{L}^{(B)}+\theta G)}, \quad (2.20)$$

where $G = \frac{-ie^2}{8\pi^2}F_{\mu\nu}\tilde{F}_{\mu\nu}$. Comparing Eq.(1.6) with Eq.(2.20) it can be seen there is an additional term θG in Eq.(2.20).

When $\theta = 0$, i.e. in the renormalizable gauge, this term is vanishing automatically. Besides, integrating over θ in Eq.(2.20) gives the generating functional $Z^{(A)}$. Therefore, we call Eq.(2.20) the generating functional in the renormalizable gauge, and also call θG the chiral anomalous term because of its relation with Adler anomaly.

It must be pointed out that the use of Eq.(2.14) to calculate $\mathrm{Tr}\theta\gamma_5$, where θ may be any finite parameter, is good only for Abelian group such as $U(1)$. If the group G is a non-Abelian chiral group $SU(N) \times SU(N)$, Eq.(2.14) must be generalized [5] to

$$\mathrm{Tr}\alpha\gamma_5 = \int_0^1 dt\int d^4x e^{t\alpha\cdot\gamma}\mathrm{tr}\alpha(x)\gamma_5\frac{1}{2}\Big(e^{i\frac{\not{D}_L(x)\not{D}_R(x)}{M^2}} + e^{i\frac{\not{D}_R(x)\not{D}_L(x)}{M^2}}\Big)\delta(x-x')\Big|_{\substack{x'=x\\ M\to\infty}} \quad (2.21)$$

Where $\alpha(x) = \frac{\alpha^i(x)\lambda_i}{2}$ are the chiral transformation parameters and $\not{D}_L(x)$ and $\not{D}_R(x)$ are the left- and right-handed Dirac operators

$$\not{D}_L(x) = \gamma_\mu(\partial_\mu + V_\mu(x) + A_\mu(x)),\quad \not{D}_R(x) = \gamma_\mu(\partial_\mu + V_\mu(x) - A_\mu(x)), \quad (2.22)$$

respectively, $V_\mu(x)$ and $A_\mu(x)$ are the vector and axial-vector gauge fields respectively. In addition, $\alpha \cdot Y = \int d^4y\alpha^i(y)Y_i(y)$, where $Y_i(y)$ is a pseudoscalar gauge operator. Here appears a 5-dimensional integral, so we call this regularization scheme a 5-dimensional regularization scheme.

3. PATH-INTEGRAL QUANTIZATION

In the previous section, by virtue of introducing the coset pure gauge field $\theta(x)$, we derive the generating functional $Z^{(B)}$ in the renormalizable gauge from the generating functional $Z^{(A)}$. Now, let us turn to arbitrary gauge conditions

$$F_B^i[\psi, A_\mu, \theta] = 0, \quad (i = 1, 2) \tag{3.1}$$

from the renormalizable gauge $F_A[A_\mu] = 0, \theta = 0$. Introduce the Faddeev-Popov determinant $\Delta_B(\psi, A_\mu, \theta)$ which satisfies

$$\Delta_B \int [d\mu(g)]\delta(F_B^g) = 1, \tag{3.2}$$

where $[d\mu(g)]$ is the invariant measure of group $G = U(1) \times U(1)_5$ and $g = e^{i\alpha\gamma_5}e^{i\beta}$ is an element of G,

$$[d\mu(g)] = [d\beta d\alpha], \quad F_B^g = F_B\left[e^{i\alpha\gamma_5}e^{i\beta}\psi, A_\mu + \frac{1}{e}\partial_\mu\beta, \theta + \alpha\right], \tag{3.3}$$

Moreover, Δ_B is a gauge invariant quantity, i.e., $\Delta_B^g = \Delta_B$. We can write Δ_B as

$$\Delta_B = \mathrm{Det}M, \quad M = (M_{ij}(x, y)) = \left(\frac{\delta F_B^{ig}(x)}{\delta u_j(y)}\right)_{g=1}$$

$$= \begin{pmatrix} \dfrac{\delta F_B^{1g}(x)}{\delta\beta(y)} & \dfrac{\delta F_B^{1g}(x)}{\delta\alpha(y)} \\ \dfrac{\delta F_B^{2g}(x)}{\delta\beta(y)} & \dfrac{\delta F_B^{2g}(x)}{\delta\alpha(y)} \end{pmatrix}_{g=1}. \tag{3.4}$$

where $\delta u_1(x) = \delta\beta(x)$, $\delta u_2(x) = \delta\alpha(x)$.

Inserting the left-hand side of Eq.(3.2) into Eq.(2.20), the generating functional becomes

$$Z^{(B)} = \int [d\psi d\bar{\psi} dA_\mu d\theta]\delta(F_A)\delta(\theta)\Delta_A e^{i\int d^4x(\mathscr{L}^{(B)}+\theta G)} \int [d\mu(g)]\delta(F_B^g)\Delta_B$$

$$= \int [d\mu(g)d\psi d\bar{\psi} dA_\mu d\theta]\delta(F_A)\delta(\theta)\delta(F_B^g)\Delta_A\Delta_B e^{i\int d^4x(\mathscr{L}^{(B)}+\theta G)}. \tag{3.5}$$

It can be easily seen that only the factor $\delta(F_B^g)$ includes the group element g in the integrand of Eq.(3.5). To eliminate the group element g, we perform a transformation in the integration variables:

$$\psi' = e^{i\alpha\gamma_5}e^{i\beta}\psi,\ \bar{\psi}' = \bar{\psi}e^{-i\beta}e^{i\alpha\gamma_5},\ A_\mu' = A_\mu + \frac{1}{e}\partial_\mu\beta,\ \theta' = \theta + \alpha. \tag{3.6}$$

In this transformation, Δ_A and Δ_B are invariant, and

$$F_B^g = F[\psi',\ A_\mu',\ \theta'] \equiv F_B',$$

$$\delta(F_A) = \delta(F_A[A_\mu]) = \delta\left(F_A\left[A_\mu' - \frac{1}{e}\,\partial_\mu\beta\right]\right),\ \delta(\theta) = \delta(\theta' - \alpha)$$

$$\mathscr{L}^{(B)} = -\frac{1}{4}\,F_{\mu\nu}'^2 - \bar{\psi}'\gamma_\mu(\partial_\mu - ieA_\mu' - i\gamma_5\partial_\mu\theta')\psi' - m\bar{\psi}'e^{-2i\theta'\gamma_5}\psi' = \mathscr{L}^{(B)\prime},$$

$$\theta G = -\frac{ie^2}{8\pi^2}\,(\theta' - \alpha)F_{\mu\nu}'\tilde{F}_{\mu\nu}' \equiv \theta' G' + \frac{ie^2}{8\pi^2}\,\alpha F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{3.7}$$

Besides, it can be easily seen that $[dA_\mu d\theta] = [dA_\mu' d\theta']$. We have also

$$[d\psi d\bar{\psi}] = [d\psi' d\bar{\psi}']e^{2i\mathrm{Tr}\alpha\gamma_5}. \tag{3.8}$$

In evaluating $\mathrm{Tr}\alpha\gamma_5$, considering that the effective Lagrangian now is $\mathscr{L}^{(B)} + \theta G$, we must use the following limiting procedure to replace Eq.(2.14) [5]:

$$2\,\mathrm{Tr}\alpha\gamma_5 = \int d^4x\mathrm{tr}\alpha(x)\gamma_5(e^{i\frac{\not{D}_L(x)\not{D}_R(x)}{M^2}} + e^{i\frac{\not{D}_R(x)\not{D}_L(x)}{M^2}})\delta(x - x')_{\substack{x'=x\\ M\to\infty}}, \tag{3.9}$$

where $\not{D}_L(x) = \gamma_\mu(\partial_\mu - ieA_\mu(x) - i\partial_\mu\theta(x))$ and $\not{D}_R(x) = \gamma_\mu(\partial_\mu - ieA_\mu(x) + i\partial_\mu\theta(x))$ are the left- and right-handed Dirac operators respectively, which are determined by Lagrangian $\mathscr{L}^{(B)}$. Then we make further calculation of the right-hand side of (3.9) and have

$$2\mathrm{Tr}\alpha\gamma_5 = -\frac{ie^2}{8\pi^2}\int d^4x\alpha F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{3.10}$$

Therefore, under transformation (3.6), the generating functional (3.5) can be written as

$$Z^{(B)} = \int[d\mu(g)d\psi' d\bar{\psi}' dA_\mu' d\theta']\delta\left(F_A\left[A_\mu' - \frac{1}{e}\,\partial_\mu\beta\right]\right)$$
$$\times\,\delta(\theta' - \alpha)\delta(F_B')\Delta_A\Delta_B e^{i\int d^4x(\mathscr{L}^{(B)\prime} + \theta' G')}. \tag{3.11}$$

Omitting all the primes in the above expression, we have

$$Z^{(B)} = \int[d\psi d\bar{\psi} dA_\mu d\theta]\delta(F_B)\Delta_B e^{i\int d^4x(\mathscr{L}^{(B)} + \theta G)}$$
$$\cdot\int[d\mu(g)]\delta\left(F_A\left[A_\mu - \frac{1}{e}\,\partial_\mu\beta\right]\right)\delta(\theta - \alpha)\Delta_A. \tag{3.12}$$

Noticing $[d\mu(g)] = [d\beta d\alpha]$, we obtain from (2.2) and (2.5)

$$\int[d\beta d\alpha]\Delta_A\delta\left(F_A\left[A_\mu-\frac{1}{e}\,\partial_\mu\beta\right]\right)\delta(\theta-\alpha)=1. \tag{3.13}$$

Thus, we arrive at

$$Z^{(B)}=\int[d\psi d\bar{\psi}dA_\mu d\theta]\delta(F_B)\Delta_B e^{i\int d^4x(\mathscr{L}^{(B)}+\theta G)}. \tag{3.14}$$

This is the path-integral quantization of the theory in an arbitrary gauge.

4. BRS INVARIANCE

By using the Faddeev-Popov ghost fields $\xi^i(x)$ and $\eta^i(x)$, $(i=1,2)$, we can rewrite (3.14) as

$$Z^{(B)}=\int[d\psi d\bar{\psi}dA_\mu d\theta d\xi^i d\eta^i]\exp i\left\{\int d^4x\left[\mathscr{L}^{(B)}+\theta G-\frac{1}{2\alpha}\,(F_B^i)^2\right]\right.$$
$$\left.+\int d^4xd^4y\xi^i(x)M_{ij}(x,\ y)\eta^j(y)\right\}. \tag{4.1}$$

where $M_{ij}(x,y)$ is given by (3.4). For the sake of simplicity of writing, we denote

$$[d\mu]=[d\psi d\bar{\psi}dA_\mu d\theta d\xi^i d\eta^i],$$
$$S_{\rm eff}=\int d^4x\left[\mathscr{L}^{(B)}+\theta G-\frac{1}{2\alpha}(F_B^i)^2\right]+\int d^4xd^4y\xi^i(x)M_{ij}(x,\ y)\eta^j(y). \tag{4.2}$$

So Eq.(4.1) becomes

$$Z^{(B)}=\int[d\mu]e^{iS_{\rm eff}}. \tag{4.3}$$

It can be easily proved that the generating functional (4.3) is BRS invariant. To do this, consider the following BRS transformation

$$\delta\psi(x)=i(\eta_1(x)+\gamma_5\eta_2(x))\delta\lambda\psi(x),$$
$$\delta\bar{\psi}(x)=i\bar{\psi}(x)(-\eta_1(x)+\gamma_5\eta_2(x))\delta\lambda,$$
$$\delta A_\mu(x)=\frac{1}{e}\,\partial_\mu\eta_1(x)\delta\lambda,\ \delta\theta(x)=\eta_2(x)\delta\lambda,$$
$$\delta\xi^i(x)=-\frac{1}{\alpha}F_B^i(x)\delta\lambda,\ \delta\eta^i(x)=0. \tag{4.4}$$

where $\delta\lambda$ is a Grassmann number. Eq.(4.4) indicates that the transformations of $A_\mu(x)$, $\theta(x)$ and $\xi^i(x)$ are all translations and $\eta^i(x)$ is invariant, so the path-integral measure corresponding to these fields is invariant. In addition, the transformations of $\psi(x)$ and $\bar{\psi}(x)$ can also be written as

$$\psi'(x)=e^{i(\eta_1(x)+\gamma_5\eta_2(x))\delta\lambda}\psi(x),\quad \bar{\psi}'(x)=\bar{\psi}(x)e^{i(-\eta_1(x)+\gamma_5\eta_2(x))\delta\lambda}, \tag{4.5}$$

and the path-integral measure of the fermion fields is transformed into

$$[d\psi' d\bar{\psi}'] = [d\psi d\bar{\psi}] e^{-2i\mathrm{Tr}\eta_2\delta\lambda\gamma_5}. \tag{4.6}$$

By means of Eq.(3.9) we can calculate $\mathrm{Tr}\eta_2\delta\lambda\gamma_5$ and obtain

$$2\,\mathrm{Tr}\eta_2\delta\lambda\gamma_5 = \frac{-ie^2}{8\pi^2}\int d^4x\eta_2(x)F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x)\delta\lambda. \tag{4.7}$$

Moreover, under the transformation (4.4), $\mathscr{L}^{(B)}$ is invariant and the variation of the chiral anomalous term is

$$\delta\int d^4x\theta(x)G(x) = \frac{-ie^2}{8\pi^2}\int d^4x\eta_2(x)F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x)\delta\lambda. \tag{4.8}$$

This indicates that the variations of the chiral anomalous term and the fermion integral measure are exactly cancelled one another. Therefore, in order to demonstrate that the generating functional (4.3) is BRS invariant, we only need to prove that is invariant under transformation (4.4). This proof is just the same as that in ordinary gauge theories. Thus, we complete the proof of BRS invariance of the generating functional (4.3).

5. WARD IDENTITIES

From the effective Lagrangian of the generating functional (3.14)

$$\mathscr{L}_{\mathrm{eff}} = \mathscr{L}^{(B)} + \theta G = -\frac{1}{4}F^2_{\mu\nu} - \bar{\psi}\gamma_\mu(\partial_\mu - ieA_\mu - i\gamma_5\partial_\mu\theta)\psi - m\bar{\psi}e^{-2i\theta\gamma_5}\psi - \frac{ie^2}{8\pi^2}\theta F_{\mu\nu}\tilde{F}_{\mu\nu} \tag{5.1}$$

we can directly derive the classical equation of motion for the coset pure gauge field θ

$$\frac{\partial\mathscr{L}_{\mathrm{eff}}}{\partial\theta} = \partial_\mu\frac{\partial\mathscr{L}_{\mathrm{eff}}}{\partial\partial_\mu\theta}, \tag{5.2}$$

or

$$\partial_\mu(i\bar{\psi}\gamma_\mu\gamma_5\psi) = 2im\bar{\psi}\gamma_5e^{-2i\theta\gamma_5}\psi - \frac{ie^2}{8\pi^2}F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{5.3}$$

This is just the axial-vector Ward identity in coordinate representation, i.e. the PCAC equation, where $\frac{-ie^2}{8\pi^2}F_{\mu\nu}\tilde{F}_{\mu\nu}$ is the Adler anomalous term. When $\theta = 0$, (5.3) becomes

$$\partial_\mu(i\bar{\psi}\gamma_\mu\gamma_5\psi) = 2\,im\bar{\psi}\gamma_5\psi - \frac{ie^2}{8\pi^2}F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{5.4}$$

It can be seen that Eq.(5.4) is exactly the anomalous Ward identity derived from the Lagrangian $\mathscr{L}^{(A)}$ in the perturbation approach.

Similarly, from Eq.(5.1), we can derive the classical equation of motion for the subgroup gauge field $A\mu$

$$\partial_\nu F_{\mu\nu} + \frac{ie^2}{2\pi^2} \partial_\nu \tilde{F}_{\mu\nu} = ie\bar{\psi}\gamma_\mu\psi, \tag{5.5}$$

which leads to the CVC equation

$$\partial_\mu(i\bar{\psi}\gamma_\mu\psi) = 0. \tag{5.6}$$

Thus we can easily obtain the vector and axial-vector Ward identities in coordinate representation by virtue of the classical equations of motion of the field quantities. On the other hand, we can also derive these Ward identities from the BRS invariant generating functional (4.3) by means of the non-perturbative functional derivative technique. To this end, let us use the external sources $\bar{\chi}$、χ、J_μ and J corresponding to the field quantities ψ、$\bar{\psi}$、A_μ and θ respectively and write Eq.(4.3) as

$$Z[J] = \int [d\mu] e^{iS_{\rm eff}[J]}, \tag{5.7}$$

where

$$S_{\rm eff}[J] = S_{\rm eff} + \int d^4x(\bar{\psi}\chi + \bar{\chi}\psi + J_\mu A_\mu + J\theta). \tag{5.8}$$

By using the integral formula for the Grassmann variables

$$\int d\xi^i \xi^j = \delta_{ij}. \tag{5.9}$$

we have from (5.7)

$$\int [d\mu] \xi^i(x) e^{iS_{\rm eff}[J]} = 0. \tag{5.10}$$

Upon performing the BRS transformation and using standard functional derivative technique we can get the following vector and axial-vector Ward identities easily

$$\partial_\mu\langle T\psi(z) j_\mu(x)\bar{\psi}(z')\rangle_0 = \delta(x - z')\langle T\psi(z)\bar{\psi}(x)\rangle_0 - \delta(x - z)\langle T\psi(x)\bar{\psi}(z')\rangle_0, \tag{5.11}$$

$$\begin{aligned}\partial_\mu\langle T\psi(z) j_{5\mu}(x)\bar{\psi}(z')\rangle_0 = {} & 2\,im\langle T\psi(z)\bar{\psi}(x)\gamma_5 e^{-2i\theta(x)\gamma_5}\psi(x)\bar{\psi}(z')\rangle_0 \\ & - \frac{ie^2}{8\pi^2}\langle T\psi(z) F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x)\bar{\psi}(z')\rangle_0 \\ & - \delta(x - z')\langle T\psi(z)\bar{\psi}(x)\gamma_5\rangle_0 \\ & - \delta(x - z)\langle T\gamma_5\psi(x)\bar{\psi}(z')\rangle_0\end{aligned} \tag{5.12}$$

where $j_\mu = i\bar{\psi}\gamma_\mu\psi$, $j_{5\mu} = i\bar{\psi}\gamma_\mu\gamma_5\psi$. Comparing Eq.(5.11) with Eq.(5.6), and Eq.(5.12) with Eq.(5.3), it can be seen that they are consistent. Furthermore, when $\theta = 0$, Eq.(5.12) becomes

$$\begin{aligned}\partial_\mu\langle T\psi(z)j_{5\mu}(x)\bar{\psi}(z')\rangle_0 &= 2\,im\langle T\psi(z)\bar{\psi}(x)\gamma_5\psi(x)\bar{\psi}(z')\rangle_0 \\ &\quad -\frac{ie^2}{8\pi^2}\langle T\psi(z)F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x)\bar{\psi}(z')\rangle_0 \\ &\quad -\delta(x-z')\langle T\psi(z)\bar{\psi}(x)\gamma_5\rangle_0 \\ &\quad -\delta(x-z)\langle T\gamma_5\psi(x)\bar{\psi}(z')\rangle_0 \end{aligned} \tag{5.13}$$

and this equation coincides with Eq.(5.4) also. Thus, we obtain the consistent Ward identities by virtue of the classical and the quantum (non-perturbation) method. In this sense, we may say that Eq.(4.3) is the generating functional of an anomaly-free theory.

To sum up, considering the variation of the path-integral measure of the fermion fields under a chiral transformation, We have found new joint-invariant generating functional with a chiral anomalous term, which is BRS invariant and leads to consistent classical and quantum Ward identities. We can also deal with the quantization of the coset pure gauge field theory under a non-Abelian chiral group in a similar manner. This will be discussed in another paper.

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