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Alternative Regularization Scheme and Conformal Anomaly Strings

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An improved approach for regularizing ill-defined sum $\sum_{i=1}^{\infty} \varphi_i^+(x)\varphi_i(x)$ is presented, and the conformal anomaly in strings is discussed.

In the research of anomaly in quantum field theory, we need to handle the divergent infinite sum

$$A(x) = \sum_{i=1}^{\infty} \varphi_i^+(x)\varphi_i(x) \tag{1}$$

here $\{\varphi_i(x)\}$ is the eigenfunction of real Hermitian operator R^+R, satisfying

$$R^+ R\varphi_i(x) = \lambda_i^2\varphi_i(x), \quad \int d^D x\varphi_i^+(x)\varphi_j(x) = \delta_{ij}$$

and

$$\sum_{i=1}^{\infty} \varphi_i^+(x)\varphi_i(y) = \delta^D(x-y).$$

To control the divergence and obtain a meaningful result, regularization is needed. A widely used regularization method is to insert a regularization factor characterized by a parameter M^2 into the infinite sum. Thus, the infinite sum after regularization can be written into an expansion series of parameter M^2:

$$A_{reg}(x) = \sum_{i=1}^{\infty} \varphi_i^+(x)e^{-\lambda_i^2/M^2}\varphi_i(x) = a(x)M^2 + b(x) + O(1/M^2). \tag{2}$$

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M^2 plays a subtle role in this method: when M^2 in (2) goes to zero, the first term vanishes while the third term diverges; if M^2 is interpreted as a cut-off of a large momentum, the first term is finite but the third term has non-zero contribution. In fact, it is unnecessary to preset a parameter M^2. Our work aims at improving the regularization method. The calculation will be greatly simplified and the cut-off which controls the divergence will be more natural and clearer in our improved method. The direct regularization of Eq.(1) will be particularly suitable for the study of the conformal anomaly of strings, where the index theorem cannot be directly applied (The reason will be given later).

A relativistic string has the property of local Weyl scale invariance which results in null-trace of energy-momentum tensor of a classical string. In general, however, quantization destroys the Weyl scale invariance and causes conformal anomaly (also called trace anomaly), which leads inconsistency in the theory. Some authors have studied this problem. Polyakov [2] was the first to point out that the conformal anomaly would vanish if the string was in its critical dimension. Fujikawa [3] recalculated the critical dimension of bosonic string with the method of functional integration. Some people later applied his idea to many other string models. Alvarez [5] tried to understand the topological meaning of the conformal anomaly. He used class-index theorem and obtained the same result. The reason why he could use class-index theorem is unclear. We will come back to this point after presenting the regularization method. We can find that the conformal anomaly is a topological invariant, though the index theorem cannot be applied directly to calculate conformal anomaly of a string, because the latter is directly proportional to Eular character.

To regularize Eq.(1), we define

$$A_{reg}(x) = \oint_c \frac{dz}{2\pi i} \sum_{i=1}^{\infty} \varphi_i^+(x) \frac{1}{z + R^+R} \varphi_i(x) \tag{3}$$

Here the integration in the complex z-plane goes around contour c which contains the poles of the integrand. Notice the operating order: first summarizing $\sum_{i=1}^{\infty}$, then integrating $\oint_c dz$. If we reverse the order, we will go back to Eq.(1). After we define the sum, it is legal to transform the basis from $\{\varphi_i(x)\}$ to $\{e^{ik\cdot x}\}$. Under this transformation,

$$\sum_{i=1}^{\infty} \varphi_i^+(x) \frac{1}{z + R^+R} \varphi_i(x) = \mathrm{Tr} \frac{1}{z + R^+R} = \mathrm{tr} \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} e^{ik\cdot x} \frac{1}{z + R^+R} e^{-ik\cdot x}. \tag{4}$$

To be specific, we have already taken the dimension of space-time to be $D = 2$. It is not difficult to generalize the calculation to any $D = 2n$ (this is the case of strings). The tr on the right-hand side of Eq.(4) operates on both Dirac index and gauge index (if there is any). To commute $e^{-ik\cdot x}$ with $1/(z + R^+R)$, we assume that operator R contains only the first order of differential z_a. Hence, we have

$$R^+R = -k^2 - \hat{K} - \hat{Q}. \tag{5}$$

where operator $\hat{K} = k_a\hat{P}_a$, both $\hat{P}_a$ and $\hat{Q}$ are independent of k_a. There is a conformal factor $\rho^{-1}(x)$ on the right-hand side of Eq.(5) for string theory, but here it is omitted. Expanding $1/(z + R^+ R)$, we have

$$\frac{1}{z+R^{+}R}=\frac{1}{z-k^{2}}+\frac{1}{z-k^{2}}(\hat{R}+\hat{Q})\frac{1}{z-k^{2}}$$
$$+\frac{1}{z-k^{2}}(\hat{R}+\hat{Q})\frac{1}{z-k^{2}}(\hat{k}+\hat{Q})\frac{1}{z-k^{2}}+\cdots \tag{6}$$

Notice the formulas

$$\int_{0}^{\infty}dx\frac{x^{m_1}}{(z-x)^{m_2}}=\frac{(-)^{m_1+1}}{z^{m_2-m_1-1}}B(m_2-m_1-1,\ m_1+1)$$

and

$$\oint_{c}\frac{dz}{2\pi i}\frac{1}{zn}=\begin{cases}1 & n=1\\ 0 & n\neq 1\end{cases},$$

we can prove that only the first few terms contribute to Eq.(3), i. e.,

$$A_{reg}(x)=\oint_{c}\frac{dz}{2\pi i}\int_{0}^{\infty}\frac{dk^{2}}{4\pi}\left(\frac{1}{z-k^{2}}+\frac{1}{z-k^{2}}\hat{Q}\frac{1}{z-k^{2}}\right.$$
$$\left.+\frac{1}{z-k^{2}}\hat{R}\frac{1}{z-k^{2}}\hat{R}\frac{1}{z-k^{2}}\right). \tag{7}$$

In the above formula, integration $\int dk^{2}$ of the first term diverges logarithmically. This divergence can be simply controlled by a momentum cut-off μ^2:

$$\oint_{c}\frac{dz}{2\pi i}\int_{0}^{\infty}\frac{dk^{2}}{4\pi}\frac{1}{z-k^{2}}\rightarrow\oint_{c}\frac{dz}{2\pi i}\int_{0}^{\mu^{2}}\frac{dk^{2}}{4\pi}\frac{1}{z-k^{2}}=\frac{1}{4\pi}\mu^{2}. \tag{8}$$

The calculation of the other two terms in Eq.(7) depends on the forms of $\hat{Q}$ and $\hat{K}$. The fact is that these two terms are finite and have non-trivial contribution to the conformal anomaly.

We have finished the formal representation of regularization, now we discuss it in detail. First, in the case of bosonic string, the covariant action of bosonic string

$$S=-\frac{1}{2}\int d^{2}\sigma\sqrt{g}\,g^{ab}\partial_{a}X^{\rho}\partial_{b}X_{\mu} \tag{9}$$

has the property of local reparametrization invariance and Weyl index invariance. To define the functional integration of a (quantized) string precisely, we perform a Wick rotation to bring the two-dimensional Minkowski an space into Euclidean space and at the same time, introduce the Faddeev-Popov ghosts $\eta(\sigma)=\begin{pmatrix}\eta^{1}(\sigma)\\ \eta^{2}(\sigma)\end{pmatrix}$ and $\xi(\sigma)=\begin{pmatrix}\xi^{1}(\sigma)\\ \xi^{2}(\sigma)\end{pmatrix}$ to fix the gauge to be $g_{12}(\sigma)=0$, $g_{11}(\sigma)-g_{22}(\sigma)=0$. This is, we choose the conformal gauge $g_{ab}(\sigma)=\delta_{ab}\rho(\sigma)$. In this gauge, the functional integration of BRST invariance corresponding to Eq.(9) is

$$Z=\int\prod_{\sigma}[dc][d\tilde{X}^{\mu}][d\xi][d\bar{\eta}]e^{-S'}=\int\prod_{\sigma}[dc]Z[\rho] \tag{10a}$$

$$S'=\frac{1}{2}\int d^{2}\sigma\partial_{a}\frac{\tilde{X}^{\mu}}{\sqrt{\rho}}\partial_{a}\frac{\tilde{X}^{\mu}}{\sqrt{\rho}}+\int d^{2}\sigma\xi\sqrt{\rho}\,\not{\partial}\,\frac{1}{\rho}\bar{\eta} \tag{10b}$$

where $|c| = \sqrt{\rho}$, $\not\partial = \gamma^1\partial_1 + \gamma^3\partial_3$, γ^i, are the Pauli matrices. $\widetilde{X}^\mu = \sqrt{\rho}X^\mu$ and $\tilde{\eta} = \rho\eta$ are independent variables. Action (10b) is invariant under Weyl scale transformation:

$$\rho(\sigma) \to e^{\alpha(\sigma)}\rho(\sigma), \quad \widetilde{X}^\mu(\sigma) \to e^{-\frac{1}{2}\alpha(\sigma)}\widetilde{X}^\mu(\sigma)$$
$$\xi(\sigma) \to e^{-\frac{1}{2}\alpha(\sigma)}\xi(\sigma), \tilde{\eta}(\sigma) \to e^{\alpha(\sigma)}\tilde{\eta}(\sigma) \tag{11}$$

Define vacuum functional $W[\rho]$,

$$W[\rho] = \ln Z[\rho] = -\frac{D}{2}\cdot\frac{1}{2}\operatorname{Tr}\ln(R_0^+R_0) + \operatorname{Tr}\ln R_1 \tag{12}$$

where we have used the symbol (n is integer or half-integer)

$$R_n = \rho(\sigma)^{\frac{n}{2}}\not\partial\rho(\sigma)^{-\frac{n+1}{2}} \tag{13}$$

Operator R_n is non-Hermitian in general, therefore $\det R_n$ (and $\operatorname{Trln}R_n$ correspondingly) should be represented as

$$\det R_n = \sum_i \int d^2\sigma v_{ni}^+(\sigma)R_n u_{ni}(\sigma) \tag{14}$$

here two-component functions $v_{ni}(\sigma)$ and $u_{ni}(\sigma)$ satisfy

$$R_n u_{ni} = \lambda_{ni}v_{ni}, \quad R_n^+ v_{ni} = \lambda_{ni}u_{ni}, \tag{15a}$$

$$R_n^+R_n u_{ni} = \lambda_{ni}^2 u_{ni}, \quad R_nR_n^+ v_{ni} = \lambda_{ni}^2 v_{ni}, \tag{15b}$$

$\{u_{ni}\}$ forms a complete set, so does $\{v_{ni}\}$. Now we examine the change of $W[\rho]$ under the scale transformation (11). The variations of the operators under the transformation (11) are

$$\delta(R_0^+R_0) = \left\{-\frac{1}{2}\alpha(\sigma), R_0^+R_0\right\}, \quad \delta R_1 = \frac{1}{2}\alpha(\sigma)R_1 - R_1\alpha(\sigma). \tag{16}$$

we find that

$$\delta W[\rho] = \frac{D}{4}\int d^2\sigma\alpha(\sigma)\left[\sum_i u_{0i}^+(\sigma)u_{0i}(\sigma) + \frac{1}{2}\sum_i v_{1i}^+(\sigma)v_{1i}(\sigma) - \sum_i u_{1i}^+(\sigma)u_{1i}(\sigma)\right]. \tag{17}$$

Notice that because of the Weyl transformation of the difference of ghost field η and ξ (see Eqs.(11) and (16)), the coefficient of the second term in the above formula is 1/2. This very factor 1/2 makes it impossible to calculate the contribution of ghost fields to anomaly directly from the index theorem. Let us calculate Eq.(17). As it has been pointed out before all sums in the square parentheses in Eq.(17) are divergent, so we perform regularization with the method introduced above. According to the definition of formula

(13) for operator R_n, we have

$$R_n^+R_n = -\rho^{-1}(\sigma)(\not{k}^2 + \hat{K} + \hat{Q}) \tag{18}$$

$$\hat{K} = i\left(\not{k}\not{\partial} + \not{\partial}\not{k} - \frac{n+1}{2}\not{k}(\not{\partial}\ln\rho) + \frac{n-1}{2}(\not{\partial}\ln\rho)\not{k}\right) \tag{19}$$

$$\hat{Q} = (\not{\partial}\ln\rho)\not{\partial} - \not{\partial}\not{\partial} + \frac{n+1}{2}\frac{n-1}{2}(\not{\partial}\ln\rho)^2 + \frac{n+1}{2}\partial^2\ln\rho \tag{20}$$

The calculation of $A_{n\mathrm{reg}}(\sigma)$ is straight forward for given operators $\hat{K}$ and $\hat{Q}$. The result is

$$A_{n\mathrm{reg}}(\sigma) = \oint_c \frac{dz}{2\pi i}\,\mathrm{Tr}\,\frac{1}{z + R_n^+R_n} = \frac{1}{4\pi}\rho(\sigma)\mu^2 + \frac{3n+1}{3}\frac{1}{4\pi}(-\partial^2\ln\rho(\sigma)) \tag{21}$$

First we look at a by-product of our discussion. It is easy to prove the relation $R_n = R_{-n-1}$. A calculation shows

$$\int_M d^2\sigma\left[\oint_c \frac{dz}{2\pi i}\,\mathrm{Tr}\,\frac{1}{z + R_n^+R_n} - \oint_c \frac{dz}{2\pi i}\,\mathrm{Tr}\,\frac{1}{z + R_nR_n^+}\right]$$
$$= \frac{2n+1}{4\pi}\int_M d^2\sigma(-\partial^2\ln\rho(\sigma)) = (2n+1)\chi(M) \tag{22}$$

in the latter part of the equation, we used $\sqrt{g}\,R = -\partial^2\ln\rho(\sigma)$ and the Gauss-Bonnet Theorem

$$\chi(M) = \frac{1}{4\pi}\int_M d^2\sigma\sqrt{g}\,R.$$

In fact, the left-hand side of Eq. (22) is the difference of the numbers of zero-mode of operators $R_n^+R_n$ and $R_nR_n^+$, for the contributions of non-zero mode of these two operators cancel each other. On the other hand, Eq. (22) gives a representation of the Riemann-Roch Theorem for operator R_n, for it is easy to prove $\mathrm{Ker}R_n^+R_n = \mathrm{Ker}R_n$, $\mathrm{Ker}R_nR_n^+ = \mathrm{Ker}R_n^+$. However, the Riemann-Roch Theorem cannot be directly applied to the problem in discussion. The reason is that the existence of the factor 1/2 mentioned above results in the fact that not only the zero-mode, but also the high-order modes of operators $R_n^+R_n$ and $R_nR_n^+$ give contribution to anomaly. Setting $n = 0, -2$ and 1 in Eq. (21) and substituting them into Eq.(17), we have

$$\delta W[\rho] = \frac{D-26}{12}\frac{1}{4\pi}\int d^2\sigma\alpha(\sigma)(-\partial^2\ln\rho) + \frac{D-2}{4}\frac{\mu^2}{4\pi}\int d^2\sigma\alpha(\sigma)\rho(\sigma) \tag{23}$$

Thus, the conformal anomaly of bosonic string can be represented as

$$\frac{\delta W[\rho]}{\delta\alpha} = \frac{D-26}{12}\frac{1}{4\pi}(-\partial^2\ln\rho) + \frac{D-2}{4}\frac{\mu^2}{4\pi}\rho(\sigma). \tag{24}$$

The second term in the above formula can be absorbed in a term which can be explained as a cosmic constant on two-dimensional surface. Meanwhile, the first term is a non-trivial anomaly. It is proportional to the Eular character after integration with respect to the two-dimensional surface. And therefore it is a topological invariant. The anomaly vanishes under the condition $D = 26$.

When $D = 26$, by an infinitesimal Weyl rescaling so that $\delta\rho = e^{\alpha}\rho-\rho = \alpha\rho$, and formula (24) (omit the cosmic constant) can be rewritten as

$$\delta W[\rho] = \delta\left(\frac{D-26}{48\pi}\int d^2\sigma(\partial\ln\rho)^2\right). \tag{25}$$

In this form, the metric conformal factor $\rho(\sigma)$ becomes a dynamical degree of freedom, and therefore, formula (25) defines a Liouville field theory.

Now let us discuss the case of superstring. The superstring theory described in Ref. [6] contains two-dimensional supergravity. Besides reparametrizing the symmetry, the local supersymmetry should also be gauge fixed when we define the functional integration. In superconformal gauge, the Lagrangian with BRST invariance is

$$\mathscr{L} = -\frac{1}{2}\partial_a\frac{\tilde{X}^\mu}{\sqrt{\rho}}\partial_a\frac{\tilde{X}_\mu}{\sqrt{\rho}} + \frac{1}{2}\bar{\tilde{\psi}}^\mu\frac{1}{\sqrt{\rho}}\not{\partial}\frac{1}{\sqrt{\rho}}\tilde{\psi}_\mu - \xi\sqrt{\rho}\not{\partial}\frac{1}{\rho}\tilde{\eta} - \bar{\tilde{\lambda}}\rho^{1/4}\not{\partial}(\rho^{-3/4}\tilde{\zeta}) \tag{26}$$

Analogous to $\tilde{X}^\mu$ and $\tilde{\eta}$, the Fermi coordinate, the bosonic ghost and anti-ghost corresponding to graviton have been redefined as

$$\tilde{\psi}^\mu = \sqrt{\rho}\,\psi^\mu,\quad \tilde{\zeta} = \sqrt{\rho}\,\zeta, \tilde{\lambda} = \lambda,$$

in order to maintain the BRST invariance of the functional integration of Eq. (26). Lagrangian (26) is invariant under Weyl scale transformation:

$$\begin{gathered}\rho \to e^{\alpha}\rho,\quad \tilde{X}^\mu \to e^{-\frac{1}{2}\alpha}\tilde{X}^\mu,\quad \tilde{\psi}^\mu \to e^{-\frac{1}{2}\alpha}\tilde{\psi}^\mu,\\ \tilde{\eta} \to e^{\alpha}\tilde{\eta},\quad \xi \to e^{-\frac{1}{2}\alpha}\xi, \tilde{\zeta} \to e^{3/4\alpha}\tilde{\zeta},\quad \tilde{\lambda} \to e^{-1/4\alpha}\tilde{\lambda},\end{gathered} \tag{27}$$

In comparison to Eq. (12), we define the vacuum functional $W[\rho]$ of formula (26). Under infinitesimal transformation, we get

$$\begin{aligned}\delta W[\rho] = \int d^2\sigma\alpha(\sigma)\Big[&\frac{D}{4}\sum_i u_{0i}^{+}u_{0i} - \frac{D}{4}\sum_i u_{-1/2i}^{+}u_{-1/2i}\\ &+\frac{1}{2}\sum_i v_{1i}^{+}v_{1i} - \sum_i u_{1i}^{+}u_{1i} - \frac{1}{4}\sum_i v_{1/2i}^{+}v_{1/2i}\\ &+\frac{3}{4}\sum_i u_{1/2i}^{+}u_{1/2i}\Big]\end{aligned} \tag{28}$$

Choose n = 0, −1/2, −2, 1, −3/2, and 1/2 in Eq. (21) and substituting them into Eq. (28), we find

$$\frac{\delta W[\rho]}{\delta\alpha} = \frac{D-10}{8}\frac{1}{4\pi}(-\partial^2\ln\rho) \tag{29}$$

It shows that the critical dimension of superstring is $D = 10$. If one wants to study superstring which deviates from its critical dimension, similar to the case of bosonic string, one has to introduce a dynamical field variable to compensate the anomaly. After integration, formula(29) presents a Liouville field theory of superstring.

In conclusion we have applied the improved regularization scheme to the study of conformal anomaly in string theory and found that it is a proper approach. In fact, there is no fundamental difficulty to apply this approach to many other general problems.

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