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β Function for $SU(2) \times SU(2)$ Chiral Model on 2-Dimensional Random Triangle Lattice

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The β function behavior of the $SU(2) \times SU(2)$ chiral model on a two-dimensional random triangle lattice has been studied with the Monte Carlo renormalization group method. Its behavior is analogous to that of the 4-dimensional $SU(2)$ gauge theory. However, it is smoother from the strong to the weak coupling region. There is no nonperturbative peak structure. The asymptotic scaling starts from β = 1.6.

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1. INTRODUCTION

The spin systems in d dimensions are similar to the gauge models in $2d$ dimensions [1]. The chiral model is the analogy of the nonabelian gauge theory and it is the most important spin system. Therefore, studying more simple $SU(2)$ $SU(2)$ chiral spin model in two-dimensions is an important method to explore the properties of more complex $SU(N)$ gauge models in four dimensions. Migdal et. al.[2] have shown that the lattice Schwinger-Dyson equations for both theories are similar and have the same property—asymptoticfreedom. Besides this, Migdal has obtained real renormalization recursion relations which are identical for both of them. Therefore, they have the similar β function behavior, the only property which the non-abelian gauge theories do not share with the chiral model is the presence of instantons. Instanton induced effects in four dimensions should be absent in two

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dimensions. So the crossover from the strong to the weak coupling region in the two dimensional chiral-model is smoother than that in four dimensional lattice gauge theories.

The Monte Carlo renormalization group (MCRG) method is a powerful technique for studying the critical properties of spin systems and the continuous limit of gauge systems. From the studies on Ising model, Potts model and ϕ^4 model [3] one can see the advantages of the random triangle lattice over regular lattice for studying the critical behavior and continuous limit because in which the Lorentz invariance and translation invariance are maintained [4]. In this paper we use the Monte Carlo renoramlization group method to study the $SU(2) \times SU(2)$ chiral model in the two dimensional random triangle lattice, and discuss if there exists a phase transition for this model and what the behavior of the β function is and then compare the results obtained with the results from the $SU(2)$ gauge model in four dimensions.

2. THE MODEL AND THE METHOD

Consider the following $SU(2) \times SU(2)$ chiral model with an $SU(2)$ matrix U_i defined on each site i of a random triangle lattice and only the nearest neighbor interaction. The action is

$$S = \sum_{\langle ij \rangle} \left(1 - \frac{1}{2} \mathrm{ReTr} \lambda_{ij} U_i^+ U_j\right). \\ = \sum_i \left[1 - \frac{1}{2} \mathrm{ReTr} \left(U_i^+ \sum_{l_i} \lambda_{i,l_i} U_{l_i}\right)\right] \tag{1}$$

where l_i and λ_{i,l_i} denote respectively the nearest neighbor site of the i-th lattice site and its weight factor. In this paper, we take the equal weight case, i.e. $\lambda_{i,l_i} = 1$

The partition function of the system is

$$z = \int e^{-\beta s} dU. \tag{2}$$

where $\beta = 2N/g^2$. In Monte Carlo simulations, we have parametrized the $SU(2)$ matrix as

$$a = a_0 \mathbf{1} + i\boldsymbol{\sigma} \cdot \boldsymbol{a}, \quad a_0^2 + |\boldsymbol{a}|^2 = 1. \tag{3}$$

where $\sigma^j (j = 1, 2, 3)$ are the three Pauli matrices.

The action can be rewritten as

$$S = \sum_i \left[1 - \frac{1}{2} \mathrm{ReTr}(U_i^+ \gamma_i)\right] \tag{4}$$

where the matrix

$$\gamma_i = \sum_{l_i} U_{l_i}. \tag{5}$$

For different site i, the number of l_i is different, but the average number is six in the 2-dimensional random triangle lattice. Obviously.

$$\gamma_i = \gamma_{i_0} + i\boldsymbol{\sigma} \cdot \boldsymbol{\gamma}_i = K_i R_i. \quad R_i \in SU(2) \tag{6}$$

$$K_i = (\det(\gamma_i))^{1/2}. \tag{7}$$

and the action can be rewritten further as

$$S = \sum_i \left(1 - \frac{1}{2} K_i \mathrm{ReTr} U_i'\right) \tag{8}$$

where

$$U_i' = U_i^+ R_i, \quad U_i' \in SU(2). \tag{9}$$

To raise the rate of the Monte Carlo computation, a mixed heat bath-Metropolis method [5] has been introduced, in which the action is divided in two parts:

$$S(\{U\}) = S_0(\{U\}) + S_1(\{U\}) \tag{10}$$

where $\{U\}$ is a configuration. Let the old configuration be $\{U_0\}$. To generate a new configuration $\{U'\}$ according to the Boltzmann distribution $\exp(-S(\{U'\}))$, one can first generate a $\{U'\}$ according to $\exp(-S_0\{U'\}))$ using the heat bath method, and then accept or reject this $\{U'\}$ according to $\exp(-S_1(\{U'\}))$ using the Metropolis method.

In the study of MCRG, we have used the standard method of Ref. [6] to obtain $\Delta\beta(\beta)$ function of the model. The correlation functions under consideration are:

$$\Gamma_1 = \frac{1}{2} \sum_{\langle ij \rangle} \mathrm{Tr} U_i^+ U_j \quad \text{(sum is for nearest neighbor)}$$

$$\Gamma_2 = \frac{1}{2} \sum_{\langle i,j \rangle} \mathrm{Tr} U_i^+ U_j \quad \text{(sum is for second neighbor)} \tag{11}$$

we can get the curves of the correlation functions depending on β for k and $k - 1$ blocked steps, starting from a large lattice and a small lattice respectively. The β vs. β curve is obtained from the matching condition

$$\Gamma(\beta, L)_{(k)} = \Gamma(\beta - \Delta\beta(\beta), L/b)_{(k-1)} \tag{12}$$

and interpolation. In Eq.(12), L and b are the size of the large lattice and the scale factor respectively. Using the method described in Ref. [7], one can obtain the β function from the $\beta(\beta)$ function.

To reduce the computer time, it is desirable that the fixed point can be arrived as quickly as possible through a few blocked steps. To this aim, two improved methods have been used. One of them is to use an improved action which is chosen as close to the given RT as possible [8]. The other is to use an improved renormalization group transformation method, with which one can choose a suitable RG transformation whose RT and fixed point are close to the standard action [9]. In this paper, we use the latter. In the asymptotically-free theories one might use the perturbation theory to find an improved block transformation.

For large β, these correlation functions can be evaluated by perturbation theory, on the tree diagram level, this leads to the following matching condition:

$$1 - \frac{\alpha}{\beta} + o\left(\frac{1}{\beta^2}\right) = 1 - \frac{\alpha'}{\beta'} + o\left(\frac{1}{\beta'^2}\right) \tag{13}$$

and hence,

$$\Delta\beta \equiv \beta' - \beta = \frac{\alpha - \alpha'}{\alpha}\beta + O(1). \tag{14}$$

Therefore minimizing $(\alpha\text{-}\alpha')/\alpha$ to made $\beta = 0$ is the matching condition in the tree diagram level improvement. To this aim, the following block transformation is consider: the probability that the l-th block spin takes the value μ_l is

$$\exp(p\mathrm{Tr}\mu_l^{+}U_{lb}), \quad U_{lb} = \sum_{i\in bl} U_i \Big/ \Big|\sum_{i\in bl} U_i\Big| \quad (b \text{ denotes block}) \tag{15}$$

where the matrix $\mu_l \in SU(2)$, p is a free parameter. $p \to \infty$ corresponds to the majority rule. We can use perturbation theory to find the optimal value of p which usually takes $p = c\beta$. From the matching condition of the correlation functions

$$\Gamma(\beta, p, L)_{(k)} = \Gamma(\beta', p, L/b)_{(k-1)} \tag{16}$$

and the results in perturbation theory for large β[1], we get $c = 4$.

3. THE RESULTS

In MCRG study, to diminish the effect of the finite lattice size, we construct two 2-dimensional random triangle lattices with linear dimension L and L/b respectively. First, we put 320 lattice sites on the 2-dimensional plane randomly, then construct the random triangle lattice according to Ref. [4] and define the nearest neighbor and the second neighbor relationship between the lattice sites. A pair of the nearest neighbor lattice sites is blocked to form a block lattice site. Using the above-mentioned improved block method, these 160 sites can be used as the sites of the unblocked small lattice and the once blocked large lattice. Repeating the above procedure successively, we can get the lattice constructed by 80 sites, it will be used as the twice blocked large lattice and the once blocked small lattice .... Clearly, it ensures that the k time blocked large lattice and the $(k-1)$ time blocked small one will be identical in their geometric structure. As the numbers of the block lattice sites decrease by half, the area of the simplex-triangle increases by a factor of 2 after each block

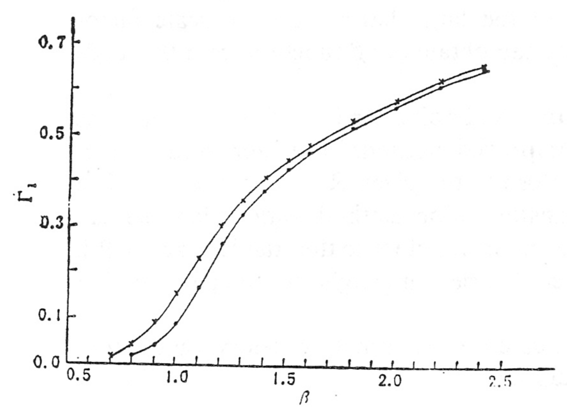

FIGURE 1 The nearest neighbor correlation functions Γ_i for 4 and 3 RG steps on large and small random lattices respectively.

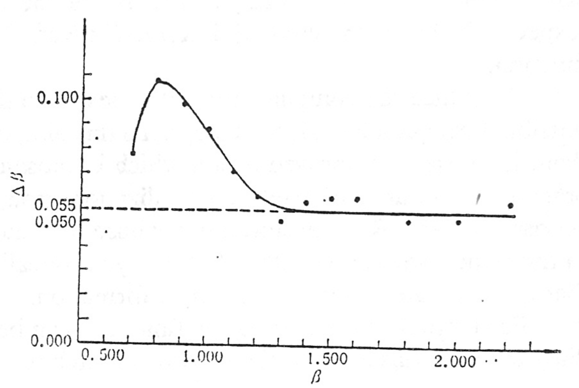

FIGURE 2 The curve of $\Delta\beta(\beta)$ vs. β.

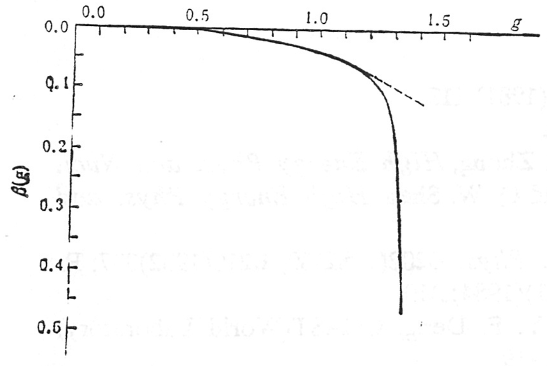

FIG.3 The behavior of the β function *vs.* g. The dash line shows the result of perturbation theory

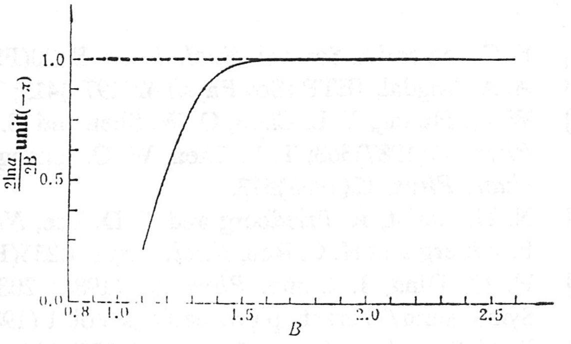

FIGURE 4 The slope of the string tension of $SU(2) \times SU(2)$ chiral model (in units of $-1/8b_0$)

transformation, hence, the block scale factor b is equal to $\sqrt{2}$.

We have performed 1,800 sweeps of Monte Carlo iteration for the large and small lattices respectively. Using the technique in Ref. [5], the acceptable rate is near to 90%. Discarding 200 sweeps of iteration to arrive thermal equilibrium, we measure the average value of the correlation functions. The curves of the nearest neighbor correlation function Γ_1 depending on β are shown in Fig.1. One can see that the two curves do not cross each other and there is no chance for $\Delta\beta = 0$. Therefore there is no phase transition point for the 2-dimensional $SU(2)\times SU(2)$ chiral model, which is consistent with the property of the $SU(2)$ gauge model in four dimensions. From these two curves and the linear interpolation, the curve of $\Delta\beta(\beta)$ vs. β can be obtained and is shown in Fig.2. From Fig.2, one can see that $\Delta\beta$ is near 0.055 when $\beta > 1.3$. This is consistent with the result obtained by perturbation theory after discarding high order terms, i.e. $\beta = \ln b/2\pi = (1/2\pi)\ln\sqrt{2} \approx 0.055$ [9]. To get the β function of this model, we fit the $\Delta\beta(\beta)$ *vs.* β curve by using the method in Ref.[7], and get

$$\Delta\beta(\beta) = 1.28(\beta - 0.90)e^{-13.75(\beta-0.90)} + 0.055 \tag{17}$$

and

$$\beta(g) = -b_0 g^3 \prod_{i=2}^{\infty}\left(1 - \left.\frac{d\Delta\beta(\beta)}{d\beta}\right|_{g_i}\right). \quad b_0 = \frac{1}{8\pi}. \tag{18}$$

The behavior of the $\beta(g)$ function is shown in Fig. 3. From this figure we find that the behavior of the $\beta(g)$ function is similar to that of the 4-dimensional $SU(2)$ gauge model, but it is smoother than the latter in transiting from strong coupling to weak coupling. There is no sharp peak in the β function of the $SU(2)$ gauge model, which is also consistent with the prediction of the theory.

The derivative $\partial\ln a/\partial\beta$ is related to the β function by

$$\frac{\partial \ln a}{\partial\beta} = \frac{g^3}{8\beta(g)} = -\frac{1}{8b_0}\left(\prod_{i=2}^{\infty}\left(1 - \left.\frac{d\Delta\beta(\beta)}{d\beta}\right|_{g_i}\right)\right)^{-1}. \tag{19}$$

Thus, we can obtain the slope of the string tension, and is plotted in Fig.4. These results suggest that the asymptotic scaling starts only from $\beta = 1.6$

This paper is performed on an M-340 computer.

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