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Analytic Study of $SU(3)$ Lattice Gauge Theory

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The variational-cumulant expansion method is generalized to the case of the lattice $SU(3)$ Wilson model. The plaquette energy as an order parameter is calculated to the 2nd order expansion. The result shows that the 1st order phase transition is not observed in the four-dimension case, which agrees with the result from the Monte Carlo calculation, but is clearly seen in the five-dimension case. This method can be used to study the problems in the lattice gauge theory with the $SU(3)$ gauge group.

On the supercubic lattice the action of the pure gauge $SU(3)$ Wilson model can be written as

$$S = \frac{\beta}{6} \sum_{p} (\mathrm{tr} U_p + \mathrm{tr} U_p^{\dagger}). \tag{1}$$

where $U_p = \prod_{l \in P} U_l$ an ordered product of $U_l \in SU(3)$ defined on the four links around a plaquette P, and P runs over all plaquettes on the lattice. According to the variational-cumulant expansion method[1], the trial action is taken to be

$$S_0 = \frac{1}{2} J \sum_{l} (\mathrm{tr} U_l + \mathrm{tr} U_l^{\dagger}). \tag{2}$$

where l runs over all links, and J is a variational parameter

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which is determined by the variational condition, i.e., minimizing the free energy in the lowest order approximation.

On the analogy of Ref.[1], by expanding the cumulant to the 2nd order approximation, the free energy per link, F, and the average plaquette energy E_p, as the order parameter, can be expressed as

$$F = F_0 - \frac{1}{N_l}\sum_{n=1}^{2}\frac{1}{n!}\left[\sum_{l=0}^{n-1}(-J)^l C_n^l \frac{\partial^l}{\partial J^l}\langle S^{n-l}\rangle_c + (-J)^l \frac{\partial^n}{\partial J^n}\ln Z_0\right], \tag{3}$$

and

$$E_p = \frac{1}{3}\langle \mathrm{tr}U_p\rangle = \frac{1}{N_p}\sum_{n=1}^{2}\frac{1}{n!}\frac{\partial}{\partial\beta}\left[\sum_{l=0}^{n-1}(-J)^l C_n^l \frac{\partial^l}{\partial J^l}\langle S^{n-l}\rangle_c\right]. \tag{4}$$

with

$$Z_0 = e^{-N_l F_0} = \int [dU] e^{S_0} \equiv z_0^{N_l}, \tag{5}$$

where $[dU]$ is the invariant measure of the group integral, $N_l = Md$ and $N_p = 1/2\,Md(d\text{-}1)$ are the total number of links and plaquettes, respectively, M and d are the number of sites and the dimension, respectively, and $<\cdots>_c$ is the cumulant average with the partition function Z_0, which can be systematically calculated by evaluating the statistical averages $<\cdots>_0$[1].

In the case of the $SU(2)$ group, because $\mathrm{tr}U = \mathrm{tr}U^+$, consequently $\mathrm{tr}U_p = \mathrm{tr}U_p^+$, it is not necessary to distinquish the different orientations of plaquettes. This, together with the fact that the single link integral z_0 is a good analytical function, makes the problem relatively simple. But in the case of the $SU(3)$ gauge group, we have to consider the orientations of plaquettes and properly deal with the calculation of z_0. As is known, the $SU(3)$ single-link integral is

$$z_0 = \int [dU] e^{\frac{1}{2}J\mathrm{tr}(U+U^+)}. \tag{6}$$

According to Ref.[2], we can express it as a contour integral [3] or as an infinite quadruple summation series[4]. However, these expressions are not convenient for the treatment in the variational-cumulant expansion. The proper way is in terms of the Weyl parametrization of group elements[2], directly expressing the group integral in formula (6) as an ordinary double integral

$$z_0 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} [d\phi_1 d\phi_2] e^{J[\cos\phi_1+\cos\phi_2+\cos(\phi_1+\phi_2)]}. \tag{7}$$

where

$$[d\phi_1 d\phi_2] = d\phi_1 d\phi_2 \left[\sin\frac{\phi_1-\phi_2}{2}\sin\frac{2\phi_1+\phi_2}{2}\sin\frac{\phi_1+2\phi_2}{2}\right]^2. \tag{8}$$

and the normalization constant which does not affect the physics is ignored. Because integrating over one of the variables in Eq.(7) would leave an infinite series as the integrand in the second integral[5], perform the standard Gaussian multi-integration to directly evaluate z_0 to the desired accuracy for the fixed values of J. Simiarly, we can write the following functions which will be used in the

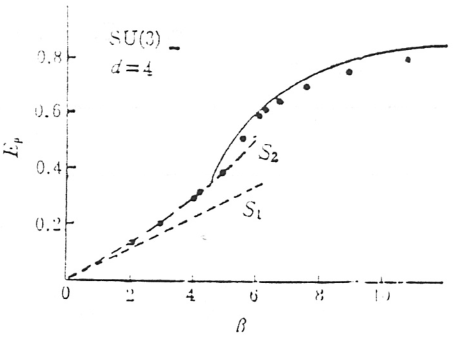


FIG. 1

sebsequent calculation, in the forms of double integrals and evaluate them. It is easy to prove that $\langle \mathrm{tr}U \rangle_0 = \langle \mathrm{tr}U^+ \rangle_0$, $\langle \mathrm{tr}U^2 \rangle_0 = \langle \mathrm{tr}U^{+2} \rangle_0$, etc.

$$\omega_1 \equiv \frac{1}{3}\frac{\partial}{\partial J}\ln z_0, \qquad \omega_{21} \equiv \langle \mathrm{tr}U \cdot \mathrm{tr}U^+ \rangle_0,$$
$$\omega_{22} \equiv \langle \mathrm{tr}U^2 \rangle_0, \qquad \omega_{23} \equiv \langle (\mathrm{tr}U)^2 \rangle_0. \tag{9}$$

Using two square frame diagrams with different orientations to express $\mathrm{tr}U_p$ and $\mathrm{tr}U_p^+$, respectively, we have

$$\langle S^n \rangle_C = \left(\frac{\beta}{6}\right)^n \left\langle \left[\sum_p \left(\boxed{\rightarrow} + \boxed{\leftarrow}\right)\right]^n \right\rangle_C. \tag{10}$$

According to the properties of the cumulant expansion[1], only the connected diagrams contribute to the $\langle \cdots \rangle_c$. Thus, in the practical calculation, we only need the statistical averages of connected diagrams. As a result, we can obtain corresponding cumulant averages from the statistical averages of the same order diagrams and the cumulant or statistical averages of lower order diagrams. In Table 1, we list the inequivalent diagrams to the second order expansion and their statistical averages. Consequently we obtain

$$F = F_0 - \frac{d-1}{2}\sum_{n=1}^{2}\frac{1}{n!}\left[\sum_{l=0}^{n-1}(-J)^l C_n^l \left(\frac{\beta}{6}\right)^{n-l}\frac{\partial^l}{\partial J^l}\sum_i \alpha_{n-l,i}\langle D_{n-l,i}\rangle_C \right.$$
$$\left. + (-J)^n \frac{\partial^n}{\partial J^n}\ln Z_0\right], \tag{11}$$

and

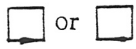
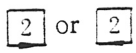
$$E_p = \sum_{n=1}^{2}\frac{1}{n!}\left[\sum_{l=0}^{n-1}(-J)^l C_n^l \left(\frac{n-l}{6}\right)\left(\frac{\beta}{6}\right)^{n-l-1}\frac{\partial^l}{\partial J^l}\sum_i \alpha_{n-l,i}\langle D_{n-l,i}\rangle_C\right]. \tag{12}$$

Furthermore, we can determine the variational parameter $J(\beta)$ as a function of β by using the similar method in the $SU(2)$ case[1]. There are two branches of solutions: $J = 0$ and $J = J(\beta) \neq 0$. When $J = 0$, the functions in Eq.(9) take the following values

$$\omega_1 = 0, \quad \omega_{21} = 1, \quad \omega_{22} = 0, \quad \omega_{23} = 0. \tag{13}$$

Substituting them into Eq.(12), we obtain the strong coupling expansion result, $E_p = 1/18\,\beta$. As in the $SU(2)$ case, the $J = 0$ solution in the cumulant expansion

Table 1

n	i	$D_{n,i}$	$\alpha_{n,i}$	$\langle D_{n,i}\rangle_0$
1	1		2	$3\omega_1^4$
2	1		2	$\frac{1}{8^3}\left\{\left[\omega_{22}^2+\omega_{23}^2-\frac{2}{3}\omega_{22}\omega_{23}\right]\left[\frac{10}{9}(\omega_{22}^2+\omega_{23}^2)-\frac{4}{3}\omega_{22}\omega_{23}\right] + \left[2\omega_{22}\omega_{23}-\frac{1}{3}(\omega_{22}^2+\omega_{23}^2)\right]\left[\frac{20}{9}\omega_{22}\omega_{23}-\frac{2}{3}(\omega_{22}^2+\omega_{23}^2)\right]\right\}$
	2		2	$1+\frac{1}{8^3}(\omega_{21}-1)^4$
	3		$8(2d-3)$	$\omega_1^6\omega_{21}$
	4		$8(2d-3)$	$\omega_1^6\omega_{23}$

contains the strong coupling expansion result. Therefore, in the ß < 6 region, we can directly use the strong coupling expansion result[6]. We plot all the results in Fig.1. In this figure, the dashed curves S_1 and S_2 represent the first and second order strong coupling expansion results respectively, and in the $\beta \gtrsim 6$ region, we use the $J = J(\beta) \neq 0$ solution with the lower free energy. The solid curve and solid dots represent the cumulant expansion result to the second order and the Monte Carlo result[7], respectively. From this figure, one can see that no first order phase transition exists. We can expect that similar to the $SU(2)$ case, by including the corrections of higher orders in the calculation, the theoretical curve will smoothly approach the Monte Carlo result.

Moreover, we perform the calculation in the five dimension case. It is clear that there exists a phase transition of the first order at $\beta_c = 3.72$.

In summary, the variational-cumulant expansion method is generalized to the case of the $SU(3)$ gauge group. The various problems with $SU(3)$ gauge group on the lattice might be treated by using this method.

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