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Exact Calculation of String Tension in Lattice Schwinger Model

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The string tension of the infinite string in the lattice Schwinger model is precisely calculated with the Hamiltonian which has the solvable exact ground state. The result shows that in the lattice Schwinger model, one can obtain the linear confinement potential and no phase transition occurs when a approaches zero. This result coincides with that obtained in the continuum theory.

1. INTRODUCTION

As is known to all, in the strong coupling limit in the lattice gauge theory, one can prove that quarks are confined. Does any deconfinement phase transition occur when a approaches zero? This is one of the basic problems in the lattice gauge theory. Although physicists have done a lot of investigations, they still have not obtained reliable results yet, because (1) They only have those results from the Monte Carlo calculations and do not have the exact analytical ones; (2) They have not studied the theory with fermions.

On the other hand, it is still a mystery how the "doubling problem" of Naive fermions in the free fermion case affects the lattice gauge theory with fermions.

In this paper, we will precisely calculate the string tension in the lattice Schwinger model by using the Hamiltonian which has the solvable ground state[1] in both the Native and Susskind schemes. The result shows that no deconfinement phase transition occurs when $a \rightarrow 0$. This result is in accord with that in the continuum theory[2]. Moreover, by using the Naive fermion theory, one can obtain correct physics at least in some aspects. This investigation is just an example of studying the lattice gauge theory with fermions in a strict analytical manner.

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In Section 2, the model and string tension are given. The detailed calculation is shown in Section 3. Finally, in Section 4, the results are discussed.

2. MODELS AND STRING TENSION

The Hamiltonian with the solvable ground state in the Naive lattice Schwinger model is[1]

$$H = \frac{1}{2} e^2 a \sum_x e^{-CR_1} E(x) e^{2CR_1} E(x) e^{-CR_1} \tag{2.1}$$

where

$$R_1 = \sum_{\substack{x \\ k=\pm 1}} \bar{\psi}(x) \gamma_k U(x, k) \psi(x+k), \tag{2.2}$$

and C satisfies the equation

$$-2c + 3\int_0^C dC' I_0(-4C') - 2CI_0(-4C) = \frac{1}{(ae)^2}, \tag{2.3}$$

with $I_0(z)$ being the Bessel function of zero order. By choosing the representation

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{2.4}$$

and denoting

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta^+(x) \end{pmatrix}, \qquad \psi^+(x) = (\xi^+(x) \quad \eta(x)), \tag{2.5}$$

we obtain

$$R_1 = -\sum_{\substack{x \\ k=\pm 1}} (\xi^+(x) i^k U(x, k) \xi(x+k) - \eta(x) i^k U(x, k) \eta^+(x+k)). \tag{2.6}$$

Let us define $|0\rangle$ in the following way:

$$E(x)|0\rangle = 0, \\ \xi(x)|0\rangle = 0, \quad \eta(x)|0\rangle = 0, \tag{2.7}$$

Then, the simplest ground state is

$$|\Omega\rangle = e^{CR_1}|0\rangle. \tag{2.8}$$

We denote the n-string state $|M_n\rangle$ in which a pair of the quark and anti-quark are connected by a n-string of the gauge field as

$$|M_n\rangle = e^{CR_1} M_n^+ |0\rangle,$$

$$M_n^+ = \sum_{\Gamma = \pm n}^{x} \zeta^+(x) i^{\Gamma} U(x, \Gamma) \eta^+(x + \Gamma), \tag{2.9}$$

$$U(x, \pm n) = \prod_{i=1}^{n-1} U(x \pm i, \pm 1).$$

Since the energy of the ground state is zero, the string tension of the infinite string is

$$\alpha = \lim_{n \to \infty} \frac{1}{na} \frac{\langle M_n | H | M_n \rangle}{\langle M_n | M_n \rangle}, \tag{2.10}$$

where the lattice spacing a is arbitrarily given.

In the same way, for Susskind fermions[3], if we write the Hamiltonian with the solvable ground state as[1]

$$H_s = \frac{1}{2} e^2 a \sum_{x} e^{-CR_{s1}} E(x) e^{2CR_{s1}} E(x) e^{-CR_{s1}}, \tag{2.11}$$

with

$$R_{s1} = \sum_{k=\pm 1}^{x} (\xi^+(2x) i^k U(2x, k) \eta^+(2x + k) + \eta(2x + 1) i^k U(2x + 1, k) \xi(2x + 1 + k)). \tag{2.12}$$

where $\xi^+(x)$ and $\xi(x)$ are defined on even sites, $\eta^+(x)$ and $\eta(x)$ on odd sites and C satisfies Eq.(2.3). It is not too difficult to write out the ground state $|\Omega_s\rangle$, the n-string state $|M_n^s\rangle$, and the string tension of the infinite string

$$\alpha_s = \lim_{n \to \infty} \frac{1}{na} \frac{\langle M_n^s | H_s | M_n^s \rangle}{\langle M_n^s | M_n^s \rangle}, \tag{2.13}$$

where n takes odd number.

The calculation in the following section shows

$$\alpha = \alpha_s = \frac{1}{2} e^2, \tag{2.14}$$

which is in accord with that in the continuum theory[2].

3. CALCULATION OF STRING TENSION

As an example let us prove Eq.(2.14) in the Naive fermion theory. For an arbitrarily given a, $\sum_{k=0}^{\infty} \langle 0 | M_n \frac{1}{(2k)!} (2CR_1)^{2k} M_n^+ | 0 \rangle$ is uniformly convergent for all n. Therefore, according to Eq.(2.9), we have

$$
\begin{aligned}
\lim_{n\to\infty}\langle M_n|M_n\rangle &= \lim_{n\to\infty}\langle 0|M_n e^{2CR_1}M_n^+|0\rangle \\
&= \lim_{n\to\infty}\lim_{2k\to\infty}\sum_{k=0}^{k}\langle 0|M_n\frac{1}{(2k)!}(2CR_1)^{2k}M_n^+|0\rangle \\
&= \lim_{2k\to\infty}\lim_{\substack{n\to\infty\\ n>2k}}\sum_{k=0}^{k}\langle 0|M_n\frac{1}{(2k)!}(2C)^{2k}R_1^{2k}M_n^+|0\rangle,
\end{aligned}
\tag{3.1}
$$

In the third step, we have considered the zero contribution from $\langle 0|M_nR_1^{2k}M_n^+|0\rangle=\sum_x 2$. In the last step the constraint $n > 2k$ means that only the configuration shown in Fig.1 contributes to $\lim_{n\to\infty} <M_n\,|\,M_n>$.In Fig.1, the left and right parts represent the contraction paths of $\xi^-(x)$ and $\eta^+(x)$, respectively. From this figure, it is easy to understand that

$$
\langle 0|M_nR_1^{2k}M_n^+|0\rangle = \sum_x 2\cdot 2^{2k}\frac{(2k)!}{k!k!},\quad \text{when } n>2k, \tag{3.2}
$$

where $(2k)!/(k!k!)$ is the total number of the closed paths with $2k$ links. Consequently, we can obtain

$$
\begin{aligned}
\lim_{n\to\infty}\langle M_n|M_n\rangle &= \sum_{\substack{k=0\\x}}^{\infty} 2\frac{1}{k!k!}(4C)^{2k} \\
&= \sum_x 2I_0(-8C).
\end{aligned}
\tag{3.3}
$$

Similarly, by using Eqs.(2.1) and (2.9), we get

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{na}\langle M_n|H|M_n\rangle \\
&= \frac{1}{2}e^2a\lim_{n\to\infty}\frac{-1}{na}\langle 0|[M_n,E]e^{2CR_1}[M_n^+,E]|0\rangle \\
&= \frac{1}{2}e^2\lim_{2k\to\infty}\lim_{\substack{n\to\infty\\ n>2k}}\sum_{k=0}^{k}\frac{-1}{n}\left\langle 0\left|[M_n,E]\frac{1}{(2k)!}(2CR_1)^{2k}[M_n^+,E]\right|0\right\rangle.
\end{aligned}
\tag{3.4}
$$

For convenience, in this equation, we have omitted the space index of $E(x)$ and the summation over x. Obviously, only the configuration shown in Fig.1 contributes to $\lim_{n\to\infty}\langle /na <M_n\,|\,H\,|\,M_n>$ By considering the effect of E, we have

$$
\begin{aligned}
\sum_x 2(n-2k)2^{2k}\frac{(2k)!}{k!k!} &\leqslant \langle 0|[M_n,E]R_1^{2k}[M_n^+,E]|0\rangle \\
&\leqslant \sum_x 2n\cdot 2^{2k}\frac{(2k)!}{k!k!}.
\end{aligned}
\tag{3.5}
$$

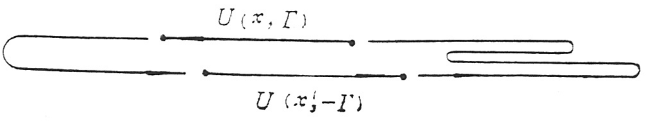

FIG.1

Since

$$\lim_{2k\to\infty}\lim_{\substack{n\to\infty\\ n>2k}}\frac{-1}{n}\sum_{k=0}^{k}2\cdot 2k\frac{1}{k!k!}(4C)^{2k}$$

$$=\lim_{n\to\infty}\frac{1}{2n}C\frac{d}{dC}I_0(-8C)=0, \tag{3.6}$$

we further obtain

$$\lim_{n\to\infty}\frac{1}{na}\langle M_n|H|M_n\rangle=\frac{1}{2}e^2\sum_x 2I_0(-8C), \tag{3.7}$$

namely,

$$\alpha=\frac{1}{2}e^2. \tag{3.8}$$

Similarly,

$$\alpha_s=\frac{1}{2}e^2. \tag{3.9}$$

4. DISCUSSION

(1) For an arbitrary a, the string tension of the infinite string is $1/2\,e^2$. This indicates that, in the lattice Schwinger model, quarks are confined by the linear potential.

(2) When $a\to 0$, no deconfinement phase transition occurs. This is in accord with that in the continuum theory[2].

(3) Our results are valid for both Naive fermions and Susskind fermions. This means that the "doubling problem" of the Naive fermion spectrum does not always dominate the lattice gauge theory.

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