High Energy Physics and Nuclear Physics

Volume 15, Number 1

The Fluid Theory of Self-Sustaining Magnetically Confined Electron Clouds (I) Fluid Equations

Yu Qingchang

Institute of High Energy Physics, the Chinese Academy of Sciences, Beijing, China

In this paper the axisymmetrical self-sustaining magnetically confined electron clouds are studied by means of the fluid theory. The electron clouds are maintained by the Penning discharge. The property of the electron motion in the clouds can be described with the fluid equations containing the continuity equation, the momentum equation, the energy equation, the Poisson equation and the heat transfer equation. The diffusion and escape of electrons and the energy transport in the electron clouds are discussed.

1. INTRODUCTION

A collection of electrons confined in a certain device by electromagnetic field is called a magnetically confined electron cloud. It is a type of non-neutral plasma, which can be studied by the methods used in plasma physics. As the electron cloud is worth studying in both theory and application, it intrigues many physicists [1,2].

The simplest confinement device consists of three cylindrical electrodes with the same diameter. The electrodes on both sides are grounded and a positive voltage is applied across the central electrode and the ground. The whole device is in an axial magnetic field and high vacuum (Fig.1). The electron cloud is confined within the central electrode, where the electric field is basically radial due to the space-charge effect of the electron cloud. While moving along the spirals around magnetic lines of force, the electrons in the cloud drift around the symmetrical axis under the influence of the both electric and magnetic fields.

Supported by the National Natural Science Foundation of China.
Received on November 27, 1989.

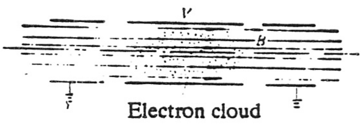

Fig. 1
The simplest magnetic confinement device for the electron cloud.

Moreover, the electrons in the cloud diffuse outwards because of the collision between the electrons and atoms of residual gas. Some electrons will be lost on the walls of the confinement device. The loss of electrons can be compensated by several methods. A usual method is to ionize the gas atoms by a self-sustaining Penning discharge in the device [3,4]. When the number of the electrons generated with the ionization equals the number of the lost electrons, the electron cloud remains stable. Such a electron cloud is called a self-sustaining magnetically confined electron cloud.

This paper studies the self-sustaining magnetically confined electron clouds by means of the fluid theory of the motion of electrons. We assume that the clouds are in the paraxial region, and the annular clouds are not discussed.

2. THE FLUID EQUATIONS OF THE MOTION OF ELECTRONS

The fluid equations of the motion of electrons, including the ionization process, were derived by Ilic [5] as the follows:

$$\nabla \cdot (n\boldsymbol{v}) = Zn, \tag{1}$$

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{e}{m}(\nabla U - \boldsymbol{v} \times \boldsymbol{B}) - \frac{1}{mn}\nabla(nkT) - \nu'\boldsymbol{v}, \tag{2}$$

$$\boldsymbol{v} \cdot \nabla\left(\frac{1}{2}mv^2 + \frac{5}{2}kT\right) = e\boldsymbol{v} \cdot \nabla U - Z\left(\frac{1}{2}mv^2 + \frac{5}{2}kT + E_i\right) - \frac{3m\nu}{M}kT. \tag{3}$$

Equations (1), (2) and (3) are the continuity equation, the momentum equation and the energy equation respectively. In the equations e, m, n, T and $\boldsymbol{v}$ are the electron charge (absolute value), the electron mass, the electron density, the electron temperature and the average velocity of electrons, respectively, k the Boltzmann's constant, U the electric potential, $\boldsymbol{B}$ the magnetic induction, ν the effective momentum transfer frequency for electron-atom collisions, Z the ionization frequency of atoms, $\nu' = \nu + Z$, M and E_i are the mass and ionization energy of the atoms, respectively.

For describing the motion of the magnetically confined electron clouds with the above mentioned equations, a few comments are useful.

1. Given the space-charge effect of the electron cloud, the Poisson equation should also be included in the fluid equations,

$$\nabla^2 U = en/\varepsilon_0 \tag{4}$$

where ε_0 is the permittivity of vacuum. The influence of the current in the electron cloud on the magnetic field is small so that $\boldsymbol{B}$ is known.

2. The electron-electron collision and heat transfer are not considered in the fluid equations. The former can be ignored because it hardly leads to the diffusion of electrons, and the latter will be discussed in Section 5.

3. The last term in Eq.(3) is so small that it can be ignored.

For an axisymmetrical system, the use of a cylindrical coordinate system (r, φ, z) is appropriate. Setting $v_\varphi = \omega r$, the fluid equations can be rewritten as

$$\frac{1}{r}\frac{\partial}{\partial r}(rnv_r) + \frac{\partial}{\partial z}(nv_z) = Zn,$$

$$rv_r\frac{\partial \omega}{\partial r} + rv_z\frac{\partial \omega}{\partial z} = \frac{e}{m}(v_rB_z - v_zB_r) - 2v_r\omega - \nu'\omega r,$$

$$v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} = \frac{e}{m}\left(\frac{\partial U}{\partial r} - \omega rB_z\right) + \omega^2 r - \frac{1}{mn}\frac{\partial}{\partial r}(nkT) - \nu' v_r,$$

$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = \frac{e}{m}\left(\frac{\partial U}{\partial z} + \omega rB_r\right) - \frac{1}{mn}\frac{\partial}{\partial z}(nkT) - \nu' v_z,$$

$$v_r\frac{\partial}{\partial r}\left(\frac{1}{2}mv^2 + \frac{5}{2}kT\right) + v_z\frac{\partial}{\partial z}\left(\frac{1}{2}mv^2 + \frac{5}{2}kT\right)$$
$$= ev_r\frac{\partial U}{\partial r} + ev_z\frac{\partial U}{\partial z} - Z\left(\frac{1}{2}mv^2 + \frac{5}{2}kT + E_i\right),$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{\partial^2 U}{\partial z^2} = \frac{en}{\varepsilon_0}. \tag{5}$$

where ω, v_r and v_z are called the drift angular velocity, the radial and axial diffusion velocity, respectively.

3. EQUILIBRIUM STATE AND NEAR-EQUILIBRIUM STATE

Assuming there were no collision, i. e., $\nu = 0$, $Z = 0$, the electron cloud would gradually reach a equilibrium state, then v_r and v_z would vanish, and T would be constant. For the axisymmetrical condition, ω is also constant due to the conservation of the canonical angular momentum [6,7].

For a equilibrium state, Eqs.(1) and (3) are meaningless because both sides of them equal zero. The remaining equations are

$$\frac{e}{m}(\nabla U - \omega r\boldsymbol{e}_\varphi \times \boldsymbol{B}) + \omega^2 r\boldsymbol{e}_r - \frac{kT}{mn}\nabla n = 0, \tag{6}$$

$$\nabla^2 U = en/\omega_0 \tag{7}$$

where e_r and e_φ are the radial and angular unit vectors. We introduce the modified potential [8]

$$\phi = U - \omega rA_\varphi + \frac{m\omega^2 r^2}{2e}, \tag{8}$$

where A is the vector potential of the magnetic field, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Substituting Eq.(8) into Eq.(6) gives

$$e\nabla\psi = kT/n \,\nabla n. \tag{9}$$

The zero of the modified potential can be selected as arbitrarily as the electric potential. Now select $\psi = 0$ where n is in its maximum value n_0, apparently n can be expressed as

$$n = n_0 \exp\,(e\psi/kT) \tag{10}$$

Substituting Eq.(10) into Eq.(7) yields

$$\nabla^2\psi = \frac{en_0}{\varepsilon_0}\exp\left(\frac{e\psi}{kT}\right) - 2\omega\left(B_z - \frac{m\omega}{e}\right). \tag{11}$$

For the external electromagnetic field, n_0, ω and T are known, Eq.(11) can be solved for n and U at any point [8]. But the theory of equilibrium state does not show the value of n_0, ω and T.

An actual self-sustaining magnetically confined electron cloud is not in a equilibrium state because the electron-atom collisions in the cloud are not negligible. But it can be thought to be near a equilibrium state, or in a near-equilibrium state, if the diffusion velocity of electrons and the variance of ω and T are small enough.

4. THE DIFFUSION OF ELECTIONS

If the diffusion velocity of electrons in the electron cloud is small, the second-order terms of v_r and v_z in the fluid equations can be ignored. Then Eq.(2) can be rewritten as

$$\boldsymbol{F} = m\boldsymbol{v} \times \boldsymbol{\Omega} + m\nu'\boldsymbol{v} \tag{12}$$

where $\Omega = (e/m)\boldsymbol{B}$, $\boldsymbol{F}$ is the sum of all the terms except the magnetic field term and the collision term:

$$\left.\begin{aligned} F_r &= e\frac{\partial U}{\partial r} - \frac{1}{n}\frac{\partial}{\partial r}(nkT) + m\omega^2 r, \\ F_\varphi &= -2mv_r\omega - mrv_r\frac{\partial\omega}{\partial r} - mrv_z\frac{\partial\omega}{\partial z}, \\ F_z &= e\frac{\partial U}{\partial z} - \frac{1}{n}\frac{\partial}{\partial z}(nkT). \end{aligned}\right\} \tag{13}$$

We analyze $\boldsymbol{F}$ and $\boldsymbol{v}$ into the angular components F_φ, v_φ, the components along the magnetic lines of force F_L, v_L, and the components perpendicular to them F_N, v_N. Apparently

$$\begin{aligned} F_L &= m\nu' v_L \\ F_N &= m\omega r\Omega + m\nu' v_N, \\ F_\varphi &= m\nu'\omega r - mv_N\Omega. \end{aligned} \tag{14}$$

Eliminating ω gives

$$v_L = \frac{F_L}{m\nu'}, \quad v_N = \frac{\nu'}{m(\Omega^2 + \nu'^2)}\left(F_N - \frac{\Omega}{\nu'}F_\varphi\right). \tag{15}$$

The relations between F_L, F_N and F_r, F_z are

$$F_L = F_r \sin\theta + F_z \cos\theta, F_N = F_r \cos\theta - F_r \sin\theta, \theta = \text{ararctan}(B_r/B_z). \quad (16)$$

The diffusion of electrons parallel or perpendicular to the magnetic lines of force are called the longitudinal or transverse diffusion. The transverse diffusion of the magnetically confined electrons needs an aid of the collision. In the self-sustaining magnetically confined electron cloud ν is far smaller than Ω (usually $\nu/\Omega < 10^{-6}$) so that the transverse diffusion is much more difficult than the longitudinal one. On the other hand, in the longitudinal diffusion a smaller F_L can lead to a large v_L. Therefore F_L must be very small in order to avoid excessively high diffusion velocity and keep a electron cloud stable, i.e.,

$$F_L = e(\partial U/\partial L) - 1/n\ \partial/\partial L(nkT) + m\omega^2\sin\theta \approx 0. \quad (17)$$

where $\partial/\partial L$ denotes the derivative along the magnetic lines of force.

5. THE ESCAPE OF ELECTRONS

Some electrons are lost on the walls of the confinement device due to the diffusion. For the convenience of discussion, a closed boundary surrounding the confinement region is drawn along the inner surfaces of the walls, and we assume that any escaping electrons would not return to the confinement region.

Let us calculate the escaping electron flow density j near a certain point on the boundary. The angular motion of electrons needs not to be considered for it does not contribute to j. As the diffusion velocity is small, it is assumed that the velocity of each individual electron w approximately follows the Maxwellian distribution,

$$f(\boldsymbol{w}) = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{mw^2}{2kT}\right). \quad (18)$$

When the boundary is perpendicular to the magnetic lines of force, the electrons escape along the longitudinal direction, and

$$j = 2\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta \int_0^{\infty} w^3 f(\boldsymbol{w})dw = n\sqrt{\frac{kT}{2\pi m}}, \quad (19)$$

where θ is the included angle between $\boldsymbol{w}$ and $\boldsymbol{B}$. When the boundary is parallel to the magnetic lines of force, the electrons escape along the transverse direction. Then the electrons pass across the boundary with the aid of the collision. In the collision, the guiding centers of electrons move outwards, and their average displacement in the direction perpendicular to the magnetic lines of force in an unit time is $\frac{\nu}{\Omega^2}\overline{\boldsymbol{\Omega} \times \boldsymbol{w}}$ [9], hence

$$j = \frac{n\nu}{\Omega}\overline{w\sin\theta} = \frac{2\pi\nu}{\Omega}\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta \int_0^{\infty} \omega^3 f(\boldsymbol{w})d\theta = \frac{n\nu}{2\Omega}\sqrt{\frac{\pi kT}{2m}}. \quad (20)$$

Eqs. (19) and (20) show that the transverse escape is not as important as the longitudinal one and usually can be ignored. When the boundary is with an angle of α to the magnetic lines of force,

$$j = n\sqrt{\frac{kT}{2\pi m}}\sin\alpha. \tag{21}$$

Assuming the total electron number in the cloud is N, according to the balance relationship between the escaping electrons and the electrons generated with the ionization,

$$N\bar{Z} = \oiint jdA. \tag{22}$$

where $\bar{Z}$ is the average ionization frequency, dA is the areal element and the integral is over the closed boundary. Defining $V_e = N/n_0$ as the effective volume of the electron cloud and $\sigma = j/n_0$ as the leakage factor near a point on the boundary, Eq.(22) can be rewritten as

$$V_e = \frac{1}{\bar{Z}}\oiint \sigma dA. \tag{23}$$

If the electron cloud is in a near-equilibrium state, we have

$$\sigma = \sqrt{\frac{kT}{2\pi m}}\exp\left(\frac{e\phi}{kT}\right)\sin\alpha. \tag{24}$$

6. THE ENERGY TRANSPORT IN ELECTRON CLOUD

The heat transfer term was ignored in Ilic's equations. Adding the term is necessary for the study of the energy transport in the electron cloud. Then Eq.(3) becomes

$$\boldsymbol{v}\cdot\nabla\left(\frac{1}{2}mv^2+\frac{5}{2}kT\right) = e\boldsymbol{v}\cdot\nabla U - \frac{1}{n}\nabla\cdot\boldsymbol{q} - Z\left(\frac{1}{2}mv^2+\frac{5}{2}kT+E_i\right). \tag{25}$$

where $\boldsymbol{q}$ is the heat flow density which can be calculated through the heat transfer equation

$$\frac{5nkT}{2m}\nabla(kT) + \frac{e}{m}\boldsymbol{q}\times\boldsymbol{B} + \nu\boldsymbol{q} = 0 \tag{26}$$

In an axisymmetrical system, Eq.(26) can be rewritten as

$$\frac{5nkT}{2m}\frac{\partial}{\partial r}(kT) + \frac{e}{m}q_\varphi B_z + \nu q_r = 0,$$

$$\frac{5nkT}{2m}\frac{\partial}{\partial z}(kT) - \frac{e}{m}q_\varphi B_r + \nu q_z = 0,$$

$$\frac{e}{m}(q_z B_r - q_r B_z) + \nu q_\varphi = 0. \tag{27}$$

According to the analogy between Eqs.(26) and (12) we can derive

$$q_L = -\frac{5nkT}{2mv}\frac{\partial}{\partial L}(kT),$$

$$q_N = -\frac{5nvkT}{2m(\Omega^2+v^2)}\frac{\partial}{\partial N}(kT). \tag{28}$$

where q_L and $\partial/\partial L(kT)$ are the components of q and $\nabla(kT)$ along the magnetic lines of force, q_N and $\partial/\partial N(kT)$ are the components of q and $\nabla(kT)$ perpendicular to the magnetic lines of force. Since ν is small, $\partial/\partial L(kT)$ has to be kept very small in order to avoid an excessively high q_L.

The energy flow density in the electron cloud is

$$\boldsymbol{S} = n\boldsymbol{v}\left(\frac{1}{2}mv^2+\frac{5}{2}kT\right)+\boldsymbol{q}. \tag{29}$$

Through Eq.(1), Eq.(25) can be rewritten in a simpler form as

$$\nabla\cdot\boldsymbol{S} = en\boldsymbol{v}\cdot\nabla U - ZnE_i. \tag{30}$$

Similar to Eqs.(19)-(21), we can calculate the energy carried by the escaping electrons near a point on the boundary. When the boundary is perpendicular to the magnetic lines of force,

$$S = \frac{1}{2}jm\omega^2r^2 + \pi m\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta d\theta\int_0^\infty w^5 f(w)dw = j\left(\frac{1}{2}m\omega^2r^2+2kT\right).$$

When the boundary is parallel to them,

$$S = \frac{1}{2}jm\omega^2r^2 + \frac{\pi m v}{\Omega}\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta\int_0^\infty w^5 f(w)dw = j\left(\frac{1}{2}m\omega^2r^2+2kT\right).$$

It is clearly independent of the included angle between the magnetic lines of force and the boundary,

$$S = j\left(\frac{1}{2}m\omega^2r^2+2kT\right) = n_0\sigma\left(\frac{1}{2}m\omega^2r^2+2kT\right). \tag{31}$$

The work done by the accelerated electrons in the electric field in the unit time is

$$\oiint jUdA - \iiint ZnUdV$$

where the first integral is over the closed boundary, the second integral is over the confinement region, dA is the areal element and dV is the volume element. For the conservation of energy this work should be equal to the energy consumed in the ionization and taken out by the escaping electrons,

$$en_0\oiint \sigma UdA - e\iiint ZnUdV = n_0\bar{Z}V_eE_i + n_0\oiint\sigma\left(\frac{1}{2}m\omega^2r^2+2kT\right)dA$$

or

$$\oiint \sigma\left(eU - \frac{1}{2}m\omega^2 r^2\right) dA = \bar{Z}V_e(E_i + 2kT_s) + \frac{e}{n_0}\iiint ZnUdV, \tag{32}$$

where T_s is the average temperature of the escaping electrons

$$T_s = \frac{1}{\bar{Z}V_e}\oiint \sigma T dA. \tag{33}$$

We have studied the fluid equations of the motion of electrons in the axisymmetrical self-sustaining magnetically confined electron clouds and discussed the processes of the diffusion and escape of electrons and the energy transport. A numerical method is necessary for further research. In the second part of this paper the property of the magnetic surfaces in the electron cloud and the numerical method will be studied.

ACKNOWLEDGEMENT

The author would like to thank Professor Xu Jianmin, who read the early draft of this paper and made helpful suggestions.

REFERENCES

[1] A. W. Trivelpiece, *Comm. Plasma Phys. Controlled Fusion*, **1** (1972), 57.
[2] R. C. Davidson, *Theory of Nonneutral Plasma*, Benjamin, Reading, MA, 1974.
[3] H. W. Lefevre and R. Booth, IEEE Trans. NS-26 (1979), 3115.
[4] W. Schunrman, *Physica*, **36** (1967), 136.
[5] D. B. Ilic, *J. Appl. Phys.*, **44** (1973), 3993.
[6] T. M. O'Neil and C. F. Driscoll, *Phys. Fluids*, **22** (1979) 266.
[7] Yu Qingchang, *High Energy Phys. and Nucl. Phys.* (in Chinese), **13** (1989), 298.
[8] Yu Qingchang, *Atomic Energy Science and Technology* (in Chinese), **21** (1987), 666.
[9] V. E. Golant *et al.*, *Fundamentals of plasma physics (Trans. from Russian)*, Wiley, New York, 1980.