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Construction of Creation-Annihilation Operators for Hydrogen Atom and Related Potentials with the Concepts of Supersymmetric Quantum Mechanics

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Using new creation-annihilation operators constructed in this work, we are able to determine the eigenvalues and eigenfunctions of the hydrogen by the algebraic method, which is analogous to the well-known method for the harmonic oscillator.

1. CONSTRUCTION OF CREATION-ANNIHILATION OPERATORS

First, let us study the radial Schrödinger operator

$$H = -\frac{1}{r}\frac{d^2 r}{dr^2} + V_-(r). \tag{1}$$

In the above equation, we assume the existence of the ground-state with the eigenvalue E_0 and eigenfunction

$$\varphi_0(r) \equiv \frac{1}{r}\exp(-w). \tag{2}$$

where $w = -is/\hbar$, s is the action of particles and $s = -E_t + p \cdot r$. Consequently

$$\varphi_0(r) \equiv \frac{1}{r}\exp(-w) = \frac{1}{r}e^{is/\hbar} = \frac{1}{r}e^{-i\omega t + ik\cdot r} = \frac{1}{r}e^{-iEt/\hbar + ip\cdot r/\hbar};$$

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Meanwhile, we use E_n to represent the eigenvalue of H. According to the supersymmetry quantum mechanics [1], we can represent $H_- \approx H - E_0$ as

$$H_- = A^+ A, \tag{3}$$

where

$$\left.\begin{aligned} A &= \varphi_0 \frac{\mathrm{d}}{\mathrm{d}r} \varphi_0^{-1} \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r + w', \\ A^+ &= -\varphi_0^{-1} \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r} r^2 \varphi_0 \equiv -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r + w'; \end{aligned}\right\} \tag{4}$$

and the prime $'$ indicates the derivative. It is noteworthy that A^+ is the adjugate of A with respect to the scalar product, which corresponds to the following radial equation:

$$\langle \varphi, \psi \rangle \approx \int_0^\infty r^2 \varphi^*(r)\psi(r)\mathrm{d}r. \tag{5}$$

In order to calculate Eq.(5), we should notice that

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^+ \neq -\frac{\mathrm{d}}{\mathrm{d}r}, \text{ but } \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r\right)^+ = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r; \tag{6}$$

By constructing the new creation-annihilation operator, we find that the minimal eigenvalue is equal to zero. According to Eq.(4), the explicit factorization for Eq.(3) is equivalent to the revelation of the model ground-state.

The essential idea of the supersymmetry quantum mechanics is to define a new operator through the following transformation:

$$H_+ \approx AA^+ \text{ (i.e., Darboux-Crum transformation)} \tag{7}$$

where H_+ is called the supersymmetry partner of H_-, and H_-, H_+ can be written as

$$H_\mp = -\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2} r + (w')^2 \mp w''. \tag{8}$$

Although the spectrum of H_+ is equal to that of H_-, H_- does not have a minimal eigenvalue of zero. Moreover, the n-th renormalized eigenfunction $\varphi_{(+),n}$ of H_+ is correlated to the $(n+1)$-th renormalized eigenfunction $\varphi_{(-),n+1}$ H_- through the following equations:

$$\left.\begin{aligned} \varphi_{(-),n+1} &= (E_{n+1} - E_0)^{-\frac{1}{2}} A^+ \varphi_{(+),n}, \\ \varphi_{(+),n} &= (E_{n+1} - E_0)^{-\frac{1}{2}} A \varphi_{(-),n+1}; \end{aligned}\right\} \tag{9}$$

Obviously, both $\varphi_{(-),n+1}$ and $\varphi_{(+),n}$ correspond to the eigenvalue E_{n+1}.

Then we define a type of specific operators. Construct an operator $H(_-\alpha)$ which depends on the aggregate $\alpha \equiv (\alpha_1, \ldots, \alpha_s)$ so that the supersymmetry partner $H_+(\alpha)$ has the same form as $H(_-\alpha)$ and obeys the following relation for the different parameter $\alpha^{(1)}$:

$$H_+(\alpha) = H_-(\alpha^{(1)}) + c(\alpha^{(1)}). \tag{10}$$

where $c(\alpha^{(1)})$ is a constant. According to some well-defined methods, $\alpha^{(1)}$ depends on α

$$\alpha^{(1)} = f(\alpha), \tag{11}$$

Then, we construct the following sequence of operators [3]:

$$H^{(n)} \approx H_{-}(\alpha^{(n)}) + \sum_{k=0}^{n} c(\alpha^{(k)}), \tag{12}$$

where

$$\left.\begin{aligned} &\alpha^{(n)} \approx f(\alpha^{(n-1)}) = f^{n}(\alpha) \equiv f \circ \cdots \circ f(\alpha); \\ &\alpha^{(0)} = \alpha, \quad c(\alpha^{(0)}) = E_0; \end{aligned}\right\} \tag{13}$$

We find that

$$\begin{aligned} H^{(n+1)} &= H_{-}(\alpha^{(n+1)}) + c(\alpha^{(n+1)}) + \sum_{k=0}^{n} c(\alpha^{(k)}) \\ &= H_{+}(\alpha^{(n)}) + \sum_{k=0}^{n} c(\alpha^{(k)}), \end{aligned} \tag{14}$$

This implies that the spectrum of $H^{(n+1)}$ is the same as $H^{(n)}$'s, but does not have the minimal eigenvalue of zero. By using iteration method, we find out that the 0-th eigenvalue of $H^{(n+1)}$ is equal to the first eigenvalue of $H^{(n)}$, the $(n+1)$-th eigenvalue of $H^{(0)}$, and H. Since the minimal eigenvalues of $H(\alpha^{(j)})$ is always zero, this means [2]

$$E_n = \sum_{k=0}^{n} c(\alpha^{(k)}), \tag{15}$$

Finally, we calculate the eigenfunctions. Let $\varphi_{(-),n}$ and $\varphi_{(+),n}$ be the renormalized eigenfunctions of $H(\alpha)$ and $H_{+}(\alpha)$, respectively. Eq.(10) gives

$$\varphi^{(\alpha)}_{(+),n} = \varphi^{(f(\alpha))}_{(-),n}, \tag{16}$$

solving this equation together with Eq.(9), we obtain

$$\begin{aligned} \varphi^{(\alpha)}_{(-),n+1} &= (E_{n+1} - E_0)^{-\frac{1}{2}} A^{+}_{\alpha} \varphi^{(\alpha)}_{(+),n}, \\ &= (E_{n+1} - E_0)^{-\frac{1}{2}} A^{+}_{\alpha} \varphi^{(f(\alpha))}_{(-),n}; \end{aligned} \tag{17}$$

If we further define an isometric operator U through the basis vector

$$U \varphi^{(\alpha)}_{(-),n} \approx \varphi^{(f(\alpha))}_{(-),n}, \tag{18}$$

we can rewrite Eq.(17) as

$$\varphi^{(\alpha)}_{(-),n+1} = (E_{n+1} - E_0)^{-\frac{1}{2}} A^{+}_{\alpha} U \varphi^{(\alpha)}_{(-),n}, \tag{19}$$

Thus,

$$\varphi^{(\alpha)}_{(-),n} = \nu_n (A^{+}_{\alpha} U)^n \varphi^{(\alpha)}_{(-),0}, \tag{20}$$

where

$$\nu_n = \prod_{k=1}^{n} (E_k - E_0)^{-\frac{1}{2}}, \tag{21}$$

is renormalized. Eq.(20) defines a creation operator a^+ for the original hydrogen atom problem. Using the same method, we can construct an annihilation-operator a. They are

$$a^+ \equiv A_a^+ U, \quad a \equiv U^{-1} A_a; \tag{22}$$

In order to calculate the eigenfunction explicitly, we can rewrite Eq.(20) as

$$\begin{aligned} \varphi_{(-),n}^{(\alpha)} &= \nu_n A_a^+ U \cdots A_a^+ U \varphi_{(-),0}^{(\alpha)} \\ &= \nu_n A_a^+ A_{f(\alpha)}^+ A_{f^2(\alpha)}^+ \cdots A_{f^{n-1}(\alpha)}^+ \varphi_{(-),0}^{(f^n(\alpha))}. \end{aligned} \tag{23}$$

This indicates that once the ground-state wavefunction $\varphi_{(-),0}^{(f^n(\alpha))}$ is obtained, we can determine all eigenfunctions in terms of the set of parameters $f^n(\alpha)$, $n = 0, 1, \ldots$.

2. APPLY NEWLY CONSTRUCTED OPERATORS TO THE HYDROGEN-ATOM

Now we can apply the creation-annihilation operators constructed in the above section to the hydrogen-atom problem. We apply the above method to the radial-operator of the hydrogen-atom:

$$H = -\frac{1}{r}\frac{d^2}{dr^2} r + \frac{l(l+1)}{r^2} - \frac{2\gamma}{r}, \quad l = 0, 1, \cdots; \tag{24}$$

By the inserting method, we prove that H can be factorized into Eqs.(3) and (4) and

$$\left.\begin{aligned} E_0 &= -\gamma^2/(l+1)^2, \\ w' &= \frac{\gamma}{l+1} - \frac{l+1}{r}; \end{aligned}\right\} \tag{25}$$

If we specify l by α, Eq.(25) is equivalent to the following statement for the ground-state:

$$\varphi_{(-),0}^{(l)} = \frac{1}{r} e^{-w} = \left(\frac{2\gamma}{l+1}\right)^{l+3/2} ([2l+2]!)\gamma^l e^{-\gamma r/(l+1)}, \tag{26}$$

We can also prove that H belongs to the type of operators defined by Eq.(10):

$$\begin{aligned} H_+(l) &= -\frac{1}{r}\frac{d^2}{dr^2} r + \frac{(l+1)(l+2)}{r^2} - \frac{2\gamma}{r} + \frac{\gamma^2}{(l+1)^2} \\ &= H_-(l+1) - \frac{\gamma^2}{(l+2)^2} + \frac{\gamma^2}{(l+1)^2}, \end{aligned} \tag{27}$$

In our opinion,

$$\alpha^{(1)} \equiv f(\alpha) = l + 1, \text{ i.e., } \alpha^{(k)} = l + k \tag{28}$$

thus

$$\left.\begin{aligned} c(\alpha^{(1)}) &= -\frac{\gamma^2}{(l+2)^2} + \frac{\gamma^2}{(l+1)^2} = -\frac{\gamma^2}{(\alpha^{(1)}+1)^2} + \frac{\gamma^2}{(\alpha^{(1)})^2}, \\ c(\alpha^{(k)}) &= -\frac{\gamma^2}{(l+k+1)^2} + \frac{\gamma^2}{(l+k)^2}, \end{aligned}\right\} \tag{29}$$

Inserting Eq.(29) into Eq.(15), we can obtain eigenvalues

$$E_n = \sum_{k=0}^{n} c(\alpha^{(k)}) = -\frac{\gamma^2}{(l+n+1)^2}, \tag{30}$$

Using Eq.(23), we obtain eigenfunctions and

$$\left.\begin{aligned} \varphi_{(-),0}^{(f^n(a))} &= \left(\frac{2\gamma}{l+n+1}\right)^{l+n+\frac{3}{2}} \cdot ([2l+2n+2]!)^{-\frac{1}{2}} \cdot r^{l+n} \cdot e^{-\gamma r/(l+n+1)}, \\ A_{fj(l)}^{+} &= -\frac{1}{r}\frac{d}{dr}r + \frac{\gamma}{l+j+1} - \frac{l+j+1}{r} \end{aligned}\right\} \tag{31}$$

Moreover, we can also show that the usual expression of the eigenfunction including Laguerre-polynomials can be derived.

3. DISCUSSION

The method employed in this paper is different from the other methods used to treat the microscopic quantum system:

(1) According to the supersymmetry quantum mechanics, a series of simplified operators and operator sequences was constructed, so that the minimal eigenvalue is equal to zero. Thus, the eigenvalue and eigenfunction problem can be solved by algebraic methods, such as factorization. Compared with other methods, this is much simpler.

(2) The iteration and insertion methods can be used. Once the related operators and operator sequences are constructed, the computer program for the complicated calculation can be written, and the problem can be solved faster.

(3) This method is similar to the harmonic oscillator method. Thus, we can apply it to the classical field theory and linear quantum field theory. It may make the field theory method more efficient and simpler, and may replace the renormalization method.

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