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On the Exchange Algebra of Periodic WZNW Model

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Using the free field representation, we calculate the r-matrix and the classical exchange algebra for WZNW model under the periodic boundary condition, and discuss the quantum counterparts of the exchange algebra as well as the performance of the r-matrix and the classical exchange algebra under Hamiltonian reductions.

1. INTRODUCTION

Recently, the conformal field theory, the quantum groups and their relations have been studied extensively. As an important kind of symmetry in two dimensional conformal invariant theories, quantum groups are receiving more and more attention. In particular, quantum group symmetries are found in both WZNW and Toda theories, which are the most extensively studied as prototypes of conformal field theories.

The classical analogue of the quantum group symmetry is described by the exchange algebra characterized by the classical r-matrix. In many cases the exchange algebra is the quadratic Sklyanin algebra or the Poisson-Lie algebra. The Sklyanin algebra is even regarded as a more fundamental mathematical structure than the usual quantum group.

For WZNW model, the classical exchange algebras and r-matrices are well studied. Blok first derived the r-matrix and the exchange algebra of $SL(2, R)$ WZNW model under a special boundary condition (the initial-final boundary condition), using the free field representations [1]. Later, Alekseev and Shatashvilli expanded his work to an arbitrary monodromy matrix, in which the concrete boundary condition dependence of the classical r-matrix and exchange algebra is eliminated [2]. Furthermore, according to the chiral decomposition of the WZNW field, Balog *et al.* also derived some different r-matrices, which satisfy the so-called modified classical Yang-Baxter equation and were proved to correspond to nonstandard quantum groups [3,4].

In the present work, we shall study the classical exchange algebra and r-matrix of WZNW model under a periodic boundary condition. The periodic condition corresponds to the case where the monodromy matrix is a unit one. Under this condition, the r-matrix given by Alekseev *et al.* has

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a singular performance. Using the free field representation, we derive the nonsingular r-matrix and the corresponding exchange algebra in this case, and prove that in order to make this algebra covariant under nonlocal chiral transformations, the transformation matrices must satisfy the Sklyanin relation. We also discuss briefly the quantum case and the problem of Hamiltonian reduction. It is noteworthy that our r-matrix coincides exactly with that obtained by Gervais [5] by using the discrete method under the same boundary condition. The Hamiltonian reduction keeps this r-matrix invariant.

2. THE FREE FIELD REPRESENTATION OF $SL(2,R)$ WZNW MODEL AND THE EXCHANGE ALGEBRA UNDER THE PERIODIC BOUNDARY CONDITION

Let us consider the WZNW model on the noncompact Lie group $SL(2, R)$. We have the following chiral decomposition of the WZNW field $g(z, \bar{z})$:

$$g(z,\bar{z}) = g_R(z)\cdot g_L(\bar{z}). \tag{1}$$

Consider the positive part $g_R(z)$ for $g(z, \bar{z})$. Since $SL(2,R)$ is noncompact, $g_R(z)$ can be decomposed in the Gauss way:

$$g_R(z) = \begin{pmatrix} 1 & 0 \\ \chi & 1 \end{pmatrix}\begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix}\begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}, \tag{2}$$

where χ, ψ and φ all are scalar functions of z. The right Kac-Moody current is then written as

$$J(z) \equiv \frac{\kappa}{2}\partial g_R g_R^{-1} = \frac{\kappa}{2}\begin{pmatrix} \partial\varphi - \chi\omega & \omega \\ \partial\chi + 2\chi\partial\varphi - \chi^2\omega & -\partial\varphi + \chi\omega \end{pmatrix}, \tag{3}$$

where

$$\omega(z) = \partial\psi(z)e^{+2\varphi(z)}. \tag{4}$$

Using the Kac-Moody current relation of $J(z)$ we can easily write the Poisson brackets for the fields χ, ω and φ as

$$\begin{gathered}\{\varphi(x),\varphi(y)\} = \frac{1}{\kappa}\,\mathrm{sign}(x-y),\quad \{\omega(x),\chi(y)\} = \frac{4}{\kappa}\delta(x-y),\\ \{\varphi(x),\chi(y)\} = \{\varphi(x),\omega(y)\} = \{\chi(x),\chi(y)\} = \{\omega(x),\omega(y)\} = 0.\end{gathered} \tag{5}$$

Now, we assume that the field $g_R(z)$ satisfies the periodic condition

$$g_R(z) = g_R(z+2\pi), \tag{6}$$

namely,

$$g_R(z)g_R^{-1}(z+2\pi) = M = 1. \tag{7}$$

M is called the monodromy matrix of the model. In Ref. [2], the r-matrix is singular at $M = 1$. In what follows we shall give a nonsingular r-matrix at the same condition.

By the fields φ, ψ and χ, the boundary condition (6) can be written as

$$\varphi(z) = \varphi(z+2\pi),\psi(z) = \psi(z+2\pi),\chi(z) = \chi(z+2\pi). \tag{8}$$

Obviously, there is also

$$\omega(z) = \omega(z + 2\pi). \tag{9}$$

According to Eq.(4), if we assume that the integration of the periodic function $\omega e^{-2\varphi}$ on the interval $2 \sim 2\pi$ is vanishing, we have

$$\psi(z) = \int_0^z \omega e^{-2\varphi} dz' = -\int_z^{2\pi} \omega e^{-2\varphi} dz'. \tag{10}$$

Thus, using the Poisson brackets (5), we can easily obtain the following exchange algebra (for simplicity we write $g(z)$ instead of $g_R(z)$):

$$\begin{aligned} \{g(z) \underset{,}{\otimes} g(z')\} &= g(z) \otimes g(z') r \operatorname{sign}(z - z'), \\ \{g(z) \underset{,}{\otimes} g^{-1}(z')\} &= -(g(z) \otimes 1) r (1 \otimes g^{-1}(z')) \operatorname{sign}(z - z'), \\ \{g^{-1}(z) \underset{,}{\otimes} g(z')\} &= -(1 \otimes g(z')) r (g^{-1}(z) \otimes 1) \operatorname{sign}(z - z'), \\ \{g^{-1}(z) \underset{,}{\otimes} g^{-1}(z')\} &= r g^{-1}(z) \otimes g^{-1}(z') \operatorname{sign}(z - z'), \end{aligned} \tag{11}$$

where the r-matrix has the simple form

$$r = \frac{1}{\kappa} \sigma_3 \otimes \sigma_3, \tag{12}$$

where σ_3 is the Pauli matrix. Since r is diagonal, it trivially solves the classical Yang-Baxter equation.

3. POISSON-LIE STRUCTURE

As is well known, WZNW model has the left-right Kac-Moody symmetries. In addition, it also has the quantum group symmetry. At the classical level, this symmetry is described by the covariance of the exchange algebra under the action of Poisson-Lie group.

Transform the field $g(z)$ as follows:

$$g(z) \rightarrow g(z) \cdot h(z) \equiv g_h(z), \tag{13}$$

where $h(z)$ is a matrix function of z independent of χ, ω and φ. If the exchange algebra (11) is covariant under this transformation, i.e.,

$$\begin{aligned} \{g_h(z) \underset{,}{\otimes} g_h(z')\} &= g_h(z) \otimes g_h(z') r \operatorname{sign}(z - z'), \\ \{g_h(z) \underset{,}{\otimes} g_h^{-1}(z')\} &= -(g_h(z) \otimes 1) r (1 \otimes g_h^{-1}(z')) \operatorname{sign}(z - z'), \\ \{g_h^{-1}(z) \underset{,}{\otimes} g_h(z')\} &= -(1 \otimes g_h(z')) r (g_h^{-1}(z) \otimes 1) \operatorname{sign}(z - z'), \\ \{g_h^{-1}(z) \underset{,}{\otimes} g_h^{-1}(z')\} &= r g_h^{-1}(z) \otimes g_h^{-1}(z') \operatorname{sign}(z - z'), \end{aligned} \tag{14}$$

then $h(z)$ and $h^{-1}(z)$ have to satisfy the nontrivial Poisson brackets

$$\begin{aligned} \{h(z) \underset{,}{\otimes} h(z')\} &= -[r, h(z) \otimes h(z')] \operatorname{sign}(z - z'), \\ \{h(z) \underset{,}{\otimes} h^{-1}(z')\} &= -((h(z) \otimes 1) r (1 \otimes h^{-1}(z')) \\ &\quad - (1 \otimes h^{-1}(z')) r (h(z) \otimes 1)) \operatorname{sign}(z - z'), \end{aligned}$$

$$\{h^{-1}(z) \otimes h(z')\} = -((1 \otimes h(z'))r(h^{-1}(z) \otimes 1)$$
$$- (h^{-1}(z) \otimes 1)r(1 \otimes h(z')))\mathrm{sign}(z - z'),$$
$$\{h^{-1}(z) \otimes h^{-1}(z')\} = [r, h^{-1}(z) \otimes h^{-1}(z')]\mathrm{sign}(z - z'), \tag{15}$$

where the first and the fourth equations are precisely the well-known Sklyanin algebra relations, which show that the $h(z)$ in Eq.(13) does not take value on an usual $SL(2, R)$ group, but on a Poisson-Lie group. The second and the third equations give the Poisson-Lie product between reciprocal elements of the Poisson-Lie group.

Now let us discuss a special property of the transformation (13).

According to Eq.(1), the full WZNW field is the product of two chiral parts thereof,

$$g(z, \bar{z})_{ij} = \sum_{\alpha} g(z)_{i\alpha} g(\bar{z})_{\alpha j}. \tag{16}$$

When we transform $g(z)$ as in Eq.(13), and at the same time, transform the field $g(\bar{z})$ as

$$g(\bar{z}) \rightarrow h(\bar{z}) g(\bar{z}) \tag{17}$$

the exchange algebra is covariant. Substituting (13) and (17) in the form of matrix elements into (16), we find that the Poisson-Lie group acts on the dumb index α. In the same way, the corresponding quantum group (if any) also acts on the dumb index. This result is very similar to that obtained by Faddeev [6] in his discussion of the quantum exchange matrix of SU(2) WZNW field.

4. QUANTUM EXCHANGE MATRIX

It is important to derive the quantum exchange relations in order to study the quantum group symmetry of the model. In general, there are two different ways for deriving the quantum exchange algebra. One of them is to quantize the model exactly and derive the quantum exchange matrix out of the operator product algebra of the quantum field. The other is simply to obtain the quantum exchange matrix by using the correspondence principle of the already known classical exchange algebra, and regard the classical r-matrix as the semiclassical limit of the quantum R-matrix. In the following we shall use the latter approach to give the quantum exchange relation under the present condition.

Notice that in the semiclassical limit, the quantum R-matrix and the classical r-matrix is related to the following equation:

$$R(z - z') = 1 + \hbar r \mathrm{sign}(z - z') + O(\hbar^2), \hbar \rightarrow 0, \tag{18}$$

we can obtain the quantum exchange relations

$$g_2(z')g_1(z) = g_1(z)g_2(z')R(z' - z),$$
$$g_2^{-1}(z')g_1(z) = g_1(z)R(z - z')g_2^{-1}(z'),$$
$$g_1^{-1}(z')g_2(z) = g_2(z)R(z - z')g_1^{-1}(z'),$$
$$g_2^{-1}(z')g_1^{-1}(z) = R(z' - z)g_1^{-1}(z)g_2^{-1}(z'), \tag{19}$$

where

$$g_1(z) = g(z) \otimes 1, \quad g_2(z) = 1 \otimes g(z),$$

The quadratic Poisson-Lie algebra (15) is also quantized corresponding to the quantization of the exchange algebra (11). The result is

$$R(z-z')h_1(z)h_2(z')=h_2(z')h_1(z)R(z-z'),$$
$$h_1(z)R(z-z')h_2^{-1}(z')=h_2^{-1}(z')R(z'-z)h_1(z),$$
$$h_1^{-1}(z)R(z-z')h_2(z')=h_2(z')R(z'-z)h_1^{-1}(z).$$
$$R(z'-z)h_1^{-1}(z)h_2^{-1}(z')=h_2^{-1}(z')h_1^{-1}(z)R(z'-z). \tag{20}$$

It is easy to see that $R(z-z')$ solves the quantum Yang-Baxter equation.

5. HAMILTONIAN REDUCTION

The definition of the Kac-Moody current J can be written in the form of the linear differential equation

$$\left(\partial-\frac{2}{\kappa}J\right)g=0, \tag{21}$$

where J acts as a Lax connection of an integrable system. Now it is well known that when appropriate constraints are imposed on J, the WZNW model can be reduced to the Toda model, and then Eq.(21) will become one member of the linear system of Toda model.

In our situation, the required constraints are

$$\chi=0,\quad \omega=1. \tag{22}$$

According to Eq. (5), this set of constraints is of second class. Imposing such constraints will not affect the result of the exchange algebra. Therefore, considering the constraints (22), we can also say that Eq.(11) is the exchange algebra of Toda model.

After imposing the constraints, Eq.(21) becomes

$$\left\{\partial-\frac{2}{\kappa}\begin{pmatrix}\partial\varphi & 1\\ 0 & -\partial\varphi\end{pmatrix}\right\}g(z)=0. \tag{23}$$

Define

$$\phi=\partial\varphi$$

and Eq.(23) becomes one of the linearized equations of mKdV equation:

$$\left[\partial-\frac{2}{\kappa}\begin{pmatrix}\phi & 1\\ 0 & -\phi\end{pmatrix}\right]g=0. \tag{24}$$

Finally, by a local transformation (Miura transformation) which does not change the Poisson bracket, Eq.(24) can be transformed into one of the linearized equations of KdV system,

$$\left[\partial-\frac{2}{\kappa}\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}\right]\tilde{g}=0, \tag{25}$$

$$u=\partial\phi+\phi^2.$$

So the r-matrix in this paper can also be regarded as the r-matrix of KdV system. Actually, our r-matrix coincides with that obtained by Gervais [5] for KdV field in terms of a completely different method.

6. CONCLUSION

Using the free field representation, we derive the classical and quantum exchange algebras for periodic WZNW field, and discuss the performance of the classical exchange algebra under Hamiltonian reduction. The main results are

1. The periodic WZNW field has a simple nonsingular r-matrix;
2. The exchange algebra of WZNW field transforms covariantly under the action of a Poisson-Lie group;
3. After quantization, the exchange algebra will give rise to the quantum Yang-Baxter relation, and possibly result in a quantum group symmetry;
4. The classical exchange algebra is invariant in the reduction from WZNW to Toda model.

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