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Hamiltonian Formula of WZNW Theory Under Chevalley Bases

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The Hamiltonian canonical formula of two-dimensional WZNW field theory based on arbitrary semi-simple Lie algebras is given under Chevalley bases. The Poisson brackets of conserved chiral currents are calculated, which are the classical Kac-Moody current algebras.

Two-dimensional Wess-Zamino-Novikov-Witten (WZNW) model is one of the most fundamental conformably invariant soluble field theories and can be regarded as a prominent Lagrangian realization of the Kac-Moody current algebra and the chiral Virasoro algebra. The study on the WZNW theory has drawn increasing attention, and various progresses have been made in this field.

One of the most important progresses is that E. Witten established the Hamiltonian formula of the WZNW theory in his famous paper [2], which laid a foundation for the canonical quantizations of the so-called the ganged WZNW theories [3,8]. However, Witten's formula was given under the natural bases of some Lie algebra. It was inconvenient to apply this formula directly to the canonical quantization of the constrained WZNW theories proposed by J. Balog *et al.* [4,5,9,10]. In order to overcome this difficulty, we now rearranged the Hamiltonian formula of the WZNW theory under Chevalley bases.

At first, we introduce some signs and formulas used in this paper. Let $\mathscr{G}$ denote the Lie algebra of a semi-simple Lie group G, and Φ and Δ the set of roots and simple roots, respectively, with respect to a given Cartan subalgebra. Chevalley bases of $\mathscr{G}$ are defined as

$$[H_i, H_j] = 0, \ (i,j = 1,2,\cdots,\text{rank}\,\mathscr{G}\}, \tag{1a}$$

$$[H_i, E_\alpha] = K_{\alpha i} E_\alpha, \tag{1b}$$

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$$[E_\alpha, E_\beta] = \sum_{ij} K_{ij}^{-1} K_{j\alpha} H_i \delta_{\alpha+\beta,0} + N_{\alpha,\beta} E_{\alpha+\beta}, \tag{1c}$$

where $\alpha, \beta \in \Phi$, $\alpha_i, \beta_i \in \Delta$, $K_{ab} = 2a \cdot b/b^2$, $\sum_b K_{ab}^{-1} K_{bc} = \delta_{ac}$. $N_{\alpha,\beta}$ is a constant (its calculation rules are listed in Appendix A. It is noticed that the Serre's relations have been involved in Eq.(1). The Killing bilinear form of $\mathscr{G}$ in terms of the Chevalley bases is

$$\mathrm{Tr}(H_i H_j) = \frac{2}{\alpha_i^2} K_{ij}, \quad \mathrm{Tr}(E_\alpha E_\beta) = \frac{2}{\alpha^2} \delta_{\alpha+\beta,0},$$
$$\mathrm{Tr}(H_i E_\alpha) = 0. \tag{2}$$

The two-dimensional WZNW model is a field system with the following action:

$$S(g) = \frac{\kappa}{2} \int_{S_2} d^2x \mathrm{Tr}(\partial_\mu g g^{-1} \partial^\mu g g^{-1}) - \frac{\kappa}{3} \int_{B_3} d^3x \varepsilon_{ijk} \mathrm{Tr}(\partial_i g g^{-1} \partial_j g g^{-1} \partial_k g g^{-1}), \tag{3}$$

where κ is the coupling constant, $\mathscr{G} \in G$ a group-valued field and B_3 a three-dimensional manifold whose boundary is Minkowski space-time S_2. Now we choose a set of parameters $\theta^a = \theta^a(x)$ $(1 \le a \le \dim G)$ on the group manifold G:

$$g = g(x) = g(\theta(x)) \in G.$$

and introduce an anti-symmetric tensor $\lambda_{ab}(\theta)$ and a non-singular matrix $\Omega(\theta)$ on G as follows:

$$\partial_a g g^{-1} = \frac{\partial g}{\partial \theta^a} g^{-1} = \sum_{i=1}^{\mathrm{rank}\mathscr{G}} H_i \Omega_a^i(\theta) + \sum_{\alpha\in\Phi} E_\alpha \Omega_a^{-\alpha}(\theta), \tag{4a}$$

$$\mathrm{Tr}(\partial_a g g^{-1}[\partial_b g g^{-1}, \partial_c g g^{-1}]) = \partial_c \lambda_{ab}(\theta) + \partial_a \lambda_{bc}(\theta) + \partial_b \lambda_{ca}(\theta), \tag{4b}$$

In this way, the action $S(g)$ can be expressed as a surface integral

$$S(g) = \int_{S_2} d^2x \mathscr{L}(x),$$

$$\mathscr{L}(x) = \mathscr{L}(\theta^a, \dot\theta^a, \theta'^a) = \frac{\kappa}{2}\Big[\sum_{ij} \frac{2}{\alpha_i^2} K_{ij} \Omega_a^i \Omega_b^j$$
$$+ \sum_{\alpha\in\Phi} \frac{2}{\alpha^2} \Omega_a^\alpha \Omega_b^{-\alpha}\Big](\dot\theta^a \dot\theta^b - \theta'^a \theta'^b) - \kappa \lambda_{ab} \dot\theta^a \theta'^b, \tag{5}$$

where $\dot\theta^a = \partial\theta^a(x)/\partial x^0$, $\theta'^a = \partial\theta^a(x)/\partial x^1$ are convention. $\mathscr{L}(x)$ is the Lagrangian density of the WZNW field, and the term with $\lambda_{ab}(\theta)$ represents the contribution of the topological term in action (3). Thus, Euler-Lagrangian equation of the system is expressed as

$$0 = \frac{\partial \mathscr{L}}{\partial \theta^c} - \partial_0\left(\frac{\partial \mathscr{L}}{\partial \dot\theta^c}\right) - \partial_1\left(\frac{\partial \mathscr{L}}{\partial \theta'^c}\right)$$

$$= -\kappa\Big[\sum_{ij} \frac{2}{\alpha_i^2} K_{ij} (\partial_a \Omega_b^j - \partial_b \Omega_a^j) \Omega_c^i + \sum_{\alpha\in\Phi} \frac{2}{\alpha^2} (\partial_a \Omega_b^{-\alpha} - \partial_b \Omega_a^{-\alpha}) \Omega_c^\alpha\Big] \cdot \dot\theta^a \theta'^b$$

$$- \kappa \sum_{ij} \frac{2}{\alpha_i^2} K_{ij} (\partial_b \Omega_a^i) \Omega_c^j (\dot{\theta}^a \dot{\theta}^b - \theta'^a \theta'^b)$$

$$- \kappa \sum_{\alpha \in \Phi} \frac{2}{\alpha^2} (\partial_b \Omega_a^{-\alpha}) \Omega_c^{\alpha} (\dot{\theta}^a \dot{\theta}^b - \theta'^a \theta'^b) \tag{6}$$

$$- \kappa \left[\sum_{ij} \frac{2}{\alpha_i^2} K_{ij} \Omega_a^i \Omega_c^j + \sum_{\alpha \in \Phi} \frac{2}{\alpha^2} \Omega_a^{\alpha} \Omega_c^{-\alpha} \right] (\ddot{\theta}^a - \theta''^a),$$

$$(1 \leqslant c \leqslant \dim G).$$

We use formula (B.4) (see Appendix B) in the derivation for Eq.(6). Due to (B.5), Eq.(6) can also be expressed equivalently as ($\partial_- = \partial_0 - \partial_1$),

$$\partial_- \mathscr{J}(H_i, x) = 0, \quad \partial_- \mathscr{J}(E_\alpha, x) = 0, \tag{7}$$

$$\begin{cases} \mathscr{J}(H_i, x) = \kappa(\dot{\theta}^a + \theta'^a) \sum_j \frac{2}{\alpha_j^2} K_{ji} \Omega_a^j, \quad (i = 1, 2, \cdots, \text{rank } \mathscr{G}) \\ \mathscr{J}(E_\alpha, x) = \kappa(\dot{\theta}^a + \theta'^a) \frac{2}{\alpha^2} \Omega_a^{\alpha}. \quad (\alpha \in \Phi) \end{cases} \tag{8}$$

These equations describe the laws of conservation of left chiral currents.

Introducing a matrix $L_{AB} = \mathrm{Tr}(AgBg^{-1})$ (where both A and B are the Chevalley bases for $\mathscr{G}$, respectively), we can rewrite the Lagrangian (5) as follows:

$$\begin{aligned} \mathscr{L}(x) = \frac{\kappa}{2} \Big\{ & \sum_{ij} \Omega_a^i \Omega_b^j \left[\sum_{kl} \frac{\alpha_l^2}{2} K_{kl}^{-1} L_{il} L_{jk} + \sum_{\beta \in \Phi} \frac{\beta^2}{2} L_{i\beta} L_{j,-\beta} \right] \\ & + 2 \sum_j \sum_{\alpha \in \Phi} \Omega_a^j \Omega_b^{-\alpha} \left[\sum_{kl} \frac{\alpha_l^2}{2} K_{kl}^{-1} L_{jk} L_{\alpha l} + \sum_{\beta \in \Phi} \frac{\beta^2}{2} L_{\alpha\beta} L_{j,-\beta} \right] \\ & + \sum_{\alpha \in \Phi} \sum_{\gamma \in \Phi} \Omega_a^{-\alpha} \Omega_b^{-\gamma} \left[\sum_{kl} \frac{\alpha_l^2}{2} K_{kl}^{-1} L_{\gamma k} L_{\alpha l} + \sum_{\beta \in \Phi} \frac{\beta^2}{2} L_{\alpha\beta} L_{\gamma,-\beta} \right] \Big\} \\ & \cdot (\dot{\theta}^a \dot{\theta}^b - \theta'^a \theta'^b) - \kappa \lambda_{ab} \dot{\theta}^a \theta'^b, \end{aligned} \tag{5$'$}$$

From this expression, we obtain the Euler-Lagrange equation, and the conservation law of right chiral currents, ($\partial_+ = \partial_0 + \partial_1$)·

$$\partial_+ \tilde{\mathscr{J}}(H_i, x) = 0, \quad \partial_+ \tilde{\mathscr{J}}(E_\alpha, x) = 0, \tag{9}$$

$$\begin{cases} \tilde{\mathscr{J}}(H_i, x) = -\kappa(\dot{\theta}^a - \theta'^a) \left[\sum_j \Omega_a^j L_{ji} + \sum_{\beta \in \Phi} \Omega_a^{-\beta} L_{\beta i} \right], \\ \tilde{\mathscr{J}}(E_\alpha, x) = -\kappa(\dot{\theta}^a - \theta'^a) \left[\sum_j \Omega_a^j L_{j\alpha} + \sum_{\beta \in \Phi} \Omega_a^{-\beta} L_{\beta\alpha} \right], \end{cases} \tag{10}$$

$$(i = 1, 2, \cdots, \text{rank}\mathscr{G}, \ \alpha \in \Phi).$$

It is noteworthy that Eqs.(9) and (7) are in fact equivalent to each other, which is a well-known fact.

Our next task is to translate the above Lagrangian formula of WZNW theory into the Hamiltonian canonical formula. The concrete method is as follows. We denote by θ^a ($1 \leq a \leq \dim G$) the canonical coordinates on the group manifold G and define their conjugate canonical momenta π_a and the fundamental Poisson brackets as follows:

$$\pi_a = \frac{\partial \mathscr{L}}{\partial \theta^a} = \kappa \left[\sum_{ij} \frac{2}{\alpha_i^2} K_{ij} \Omega_a^i \Omega_b^j + \sum_{\alpha \in \Phi} \frac{2}{\alpha^2} \Omega_a^\alpha \Omega_b^{-\alpha} \right] \dot{\theta}^b - \kappa \lambda_{ab} \theta'^b, \tag{11}$$

$$\{\theta^a(x), \pi_b(y)\} = \delta_b^a \delta(x_1 - y_1), \quad (1 \leqslant a, b \leqslant \dim G). \tag{12}$$

In addition, we define the canonical Hamiltonian density of the system in terms of Legendrean transformation:

$$\begin{aligned} \mathscr{H} &\equiv \pi_a \dot{\theta}^a - \mathscr{L} \\ &= \frac{\kappa}{2} \left[\sum_{ij} \frac{2}{\alpha_i^2} K_{ij} \Omega_a^i \Omega_b^j + \sum_{\alpha \in \Phi} \frac{2}{\alpha^2} \Omega_a^\alpha \Omega_b^{-\alpha} \right] (\dot{\theta}^a \dot{\theta}^b + \theta'^a \theta'^b) \\ &= \frac{1}{4\kappa} \sum_{ij} \frac{\alpha_j^2}{2} K_{ij}^{-1} [\mathscr{J}(H_i, x)\mathscr{J}(H_j, x) + \tilde{\mathscr{J}}(H_i, x)\tilde{\mathscr{J}}(H_j, x)] \\ &+ \frac{1}{4\kappa} \sum_{\alpha \in \Phi} \frac{\alpha^2}{2} [\mathscr{J}(E_\alpha, x)\mathscr{J}(E_{-\alpha}, x) + \tilde{\mathscr{J}}(E_\alpha, x)\tilde{\mathscr{J}}(E_{-\alpha}, x)], \end{aligned} \tag{13}$$

This expression is obviously the Sugawara construction of the energy density and can also be written as an expression independent of the bases of the Lie algebra $\mathscr{H}$:

$$\mathscr{H} = \frac{1}{4\kappa} \mathrm{Tr}[\mathscr{J}^2(x) + \tilde{\mathscr{J}}^2(x)], \tag{14}$$

$$\mathscr{J}(x) = \sum_{ij} \frac{\alpha_j^2}{2} K_{ij}^{-1} H_i \mathscr{J}(H_j, x) + \sum_{\alpha \in \Phi} \frac{\alpha^2}{2} E_{-\alpha} \mathscr{J}(E_\alpha, x), \tag{15a}$$

$$\tilde{\mathscr{J}}(x) = \sum_{ij} \frac{\alpha_j^2}{2} K_{ij}^{-1} H_i \tilde{\mathscr{J}}(H_j, \alpha) + \sum_{\alpha \in \Phi} \frac{\alpha^2}{2} E_{-\alpha} \tilde{\mathscr{J}}(E_\alpha, x). \tag{15b}$$

According to Eq. (13), we need calculate the Poisson brackets between chiral currents first when we want to derive Hamilton's canonical equations. The expressions of the components of the conservative chiral currents in canonical variables are (see Appendix B):

$$\mathscr{J}(H_i, x) = \omega^{ai} \pi_a + \kappa \omega^{ai} \lambda_{ab} \theta'^b + \kappa \sum_j \frac{2}{\alpha_j^2} K_{ji} \Omega_a^j \theta'^a, \tag{16a}$$

$$\mathscr{J}(E_\alpha, x) = \omega^{a\alpha} \pi_a + \kappa \omega^{a\alpha} \lambda_{ab} \theta'^b + \kappa \frac{2}{\alpha^2} \Omega_a^\alpha \theta'^a, \tag{16b}$$

$$\begin{aligned} \tilde{\mathscr{J}}(H_i, x) = &- \sum_j L_{ji} \left[\sum_l \frac{\alpha_l^2}{2} K_{jl}^{-1} (\omega^{al} \pi_a + \kappa \omega^{al} \lambda_{ab} \theta'^b) - \kappa \Omega_a^j \theta'^a \right] \\ &- \sum_{\beta \in \Phi} L_{-\beta, i} \left[\frac{\beta^2}{2} (\omega^{a\beta} \pi_a + \kappa \omega^{a\beta} \lambda_{ab} \theta'^b) - \kappa \Omega_a^\beta \theta'^a \right], \end{aligned} \tag{16c}$$

$$\tilde{\mathscr{J}}(E_\alpha, x) = - \sum_j L_{j\alpha} \left[\sum_l \frac{\alpha_l^2}{2} K_{jl}^{-1} (\omega^{al} \pi_a + \kappa \omega^{al} \lambda_{ab} \theta'^b) - \kappa \Omega_a^j \theta'^a \right]$$

$$-\sum_{\beta\in\Phi} L_{-\beta,\alpha}\left[\frac{\beta^2}{2}(\omega^{a\beta}\pi_a+\kappa\omega^{a\beta}\lambda_{ab}\theta'^b)-\kappa\Omega_a^{+\beta}\theta'^a\right] \tag{16d}$$

$$i=1,2,\cdots,\mathrm{rank}\mathscr{G},\quad \alpha\in\Phi.$$

where the matrix ω is the inverse of Ω ($\omega\Omega = 1$). Combining (16) with (12), we can easily calculate all the Poisson brackets between the conservative chiral currents. For example,

$$\{\mathscr{J}(H_i,x),\mathscr{J}(H_j,y)\}=\{\omega^{ai}\pi_a,\omega^{cj}\pi_c\}+\{\omega^{ai}\pi_a,\kappa\omega^{cj}\lambda_{cd}\theta'^d\}$$
$$+\left\{\omega^{ai}\pi_a,\kappa\sum_l\frac{2}{\alpha_l^2}K_{lj}\Omega_c^l\theta'^c\right\}$$
$$+\{\kappa\omega^{ai}\lambda_{ab}\theta'^b,\omega^{cj}\pi_c\}$$
$$+\left\{\kappa\sum_k\frac{2}{\alpha_k^2}K_{ki}\Omega_a^k\theta'^a,\omega^{cj}\pi_c\right\},$$

By using (B.2), (B.5) and (B.6) in Appendix B, we obtain

$$\{\omega^{ai}\pi_a,\omega^{cj}\pi_c\}=[(\partial_c\omega^{ai})\omega^{cj}\pi_a-(\partial_a\omega^{cj})\omega^{ai}\pi_c]\delta(x_1-y_1)$$
$$=\omega^{ai}\omega^{bj}\pi_e\left[\sum_l\omega^{el}(\partial_a\Omega_b^l-\partial_b\Omega_a^l)\right.$$
$$\left.+\sum_{\beta\in\Phi}\omega^{e\beta}(\partial_a\Omega_b^{-\beta}-\partial_b\Omega_a^{-\beta})\right]\delta(x_1-y_1)=0,$$

Owing to $\{\theta'^a(x),\pi_b(y)\}=\delta_b^a\delta'(x_1-y_1)$, we have

$$\{\omega^{ai}\pi_a,\kappa\omega^{cj}\lambda_{cd}\theta'^d\}+\{\kappa\omega^{ai}\lambda_{ab}\theta'^b,\omega^{cj}\pi_c\}$$
$$=\kappa\omega^{ai}\left[-(\partial_a\omega^{cj})\lambda_{cd}\theta'^d-\omega^{cj}(\partial_a\lambda_{cd})\theta'^d+\frac{\partial}{\partial x'}(\omega^{cj}\lambda_{ca})\right]\delta(x_1-y_1)$$
$$+\kappa\omega^{cj}[(\partial_c\omega^{ai})\lambda_{ab}\theta'^b+\omega^{ai}(\partial_c\lambda_{ab})\theta'^b]\delta(x_1-y_1)+\kappa\omega^{ai}\lambda_{ab}\frac{\partial}{\partial x'}\omega^{bj}\delta(x_1-y_1)$$
$$=-\kappa\omega^{ai}\omega^{bj}\theta'^c[\partial_a\lambda_{bc}+\partial_b\lambda_{ca}+\partial_c\lambda_{ab}]\delta(x_1-y_1)$$
$$+\kappa\lambda_{bc}\theta'^c[\omega^{ai}(\partial_a\omega^{bj})-\omega^{aj}(\partial_a\omega^{bi})]\delta(x_1-y_1)$$
$$=\kappa\lambda_{bc}\theta'^c\omega^{ai}\omega^{dj}\left[\sum_l\omega^{bl}(\partial_a\Omega_d^l-\partial_d\Omega_a^l)+\sum_{\beta\in\Phi}\omega^{b\beta}(\partial_a\Omega_d^{-\beta}-\partial_d\Omega_a^{-\beta})\right]=0,$$

$$\left\{\omega^{ai}\pi_a,\kappa\sum_l\frac{l}{\alpha_l^2}K_{lj}\Omega_c^l\theta'^c\right\}=\omega^{ai}\kappa\sum_l\frac{2}{\alpha_l^2}K_{lj}\{\pi_a,\Omega_c^l\theta'^c\}$$
$$=\kappa\omega^{ai}\sum_l\frac{2}{\alpha_l^2}K_{lj}[-\partial_a\Omega_c^l\theta'^c\delta(x_1-y_1)+\partial_1\Omega_a^l\delta(x_1-y_1)+\Omega_a^l\delta'(x_1-y_1)]$$
$$=\kappa\omega^{ai}\sum_l\frac{2}{\alpha_l^2}K_{lj}[(\partial_c\Omega_a^l-\partial_a\Omega_c^l)\theta'^c\delta(x_1-y_1)+\Omega_a^l\delta'(x_1-y_1)]$$
$$=\frac{2\kappa}{\alpha_i^2}K_{ij}\delta'(x_1-y_1),$$

$$\therefore \quad \left\{\kappa \sum_{k} \frac{2}{\alpha_k^2} K_{ki} \Omega_a^k \theta'^a,\ \omega^{ci} \pi_0\right\} = \frac{2\kappa}{\alpha_i^2} K_{ij} \delta'(x_1 - y_1),$$

In the calculation of the above Poisson brackets, we have used the formulas listed in Appendices without explanation. From those results we obtain

$$\{\mathscr{J}(H_i, x), \mathscr{J}(H_j, y)\} = \frac{4\kappa}{\alpha_i^2} K_{ij} \delta'(x_1 - y_1), \tag{17a}$$

In the same way, we have

$$\{\mathscr{J}(H_i, x), \mathscr{J}(E_\beta, y)\} = K_{\beta i} \mathscr{J}(E_\beta, x) \delta(x_1 - y_1), \tag{17b}$$

$$\{\mathscr{J}(E_\alpha, x), \mathscr{J}(E_\beta, y)\} = \delta_{\alpha+\beta,0} \left[\sum_{ij} K_{ij}^{-1} K_{j\alpha} \mathscr{J}(H_i, x) \delta(x_1 - y_1) + \frac{4\kappa}{\alpha^2} \delta'(x_1 - y_1)\right] + N_{\alpha,\beta} \mathscr{J}(E_{\alpha+\beta}, x) \delta(x_1 - y_1), \tag{17c}$$

$$\{\tilde{\mathscr{J}}(H_i, x), \tilde{\mathscr{J}}(H_j, y)\} = -\frac{4\kappa}{\alpha_i^2} K_{ij} \delta'(x_1 - y_1), \tag{18a}$$

$$\{\tilde{\mathscr{J}}(H_i, x), \tilde{\mathscr{J}}(E_\beta, y)\} = K_{\beta i} \tilde{\mathscr{J}}(E_\beta, x) \delta(x_1 - y_1), \tag{18b}$$

$$\{\tilde{\mathscr{J}}(E_\alpha, x), \tilde{\mathscr{J}}(E_\beta, y)\} = \delta_{\alpha+\beta,0} \left[\sum_{ij} K_{ij}^{-1} K_{j\alpha} \tilde{\mathscr{J}}(H_i, x) \delta(x_1 - y_1) - \frac{4\kappa}{\alpha^2} \delta'(x_1 - y_1)\right] + N_{\alpha,\beta} \tilde{\mathscr{J}}(E_{\alpha+\beta}, x) \delta(x_1 - y_1), \tag{18c}$$

$$\{\mathscr{J}(H_i, x), \tilde{\mathscr{J}}(H_j, y)\} = 0, \tag{19a}$$

$$\{\mathscr{J}(H_i, x), \tilde{\mathscr{J}}(E_\beta, y)\} = 0, \tag{19b}$$

$$\{\mathscr{J}(E_\alpha, x), \tilde{\mathscr{J}}(H_j, y)\} = 0, \tag{19c}$$

$$\{\mathscr{J}(E_\alpha, x), \tilde{\mathscr{J}}(E_\beta, y)\} = 0, \tag{19d}$$

$$(i, j = 1, 2, \cdots, \mathrm{rank}\mathscr{G},\ \alpha, \beta \in \Phi),$$

Eqs.(17) and (18) are the classical Kac-Moody current algebras relations, and (19) shows the chirality of the WZNW theory. As the corollaries of (17—19), we have

$$\{\mathscr{H}(x), \mathscr{J}(H_i, y)\} = \mathscr{J}(H_i, x) \delta'(x_1 - y_1) \tag{20a}$$

$$\{\mathscr{H}(x), \mathscr{J}(E_\alpha, y)\} = \mathscr{J}(E_\alpha, x) \delta'(x_1 - y_1), \tag{20b}$$

$$\{\mathscr{H}(x), \tilde{\mathscr{J}}(H_i, y)\} = -\tilde{\mathscr{J}}(H_i, x) \delta'(x_1 - y_1), \tag{20c}$$

$$\{\mathscr{H}(x), \tilde{\mathscr{J}}(E_\alpha, y)\} = -\tilde{\mathscr{J}}(E_\alpha, x) \delta'(x_1 - y_1), \tag{20d}$$

$$(i = 1, 2, \cdots, \mathrm{rank}\mathscr{G};\ \alpha \in \Phi).$$

The Hamiltonian of the WZNW field is

$$H = \int dx_1 \mathscr{H}(x), \tag{21}$$

and the canonical Hamilton equations of motion for the components of the conservative currents are

$$\partial_{x_0}\mathscr{J}(H_i, x) = \{\mathscr{J}(H_i, x), H\} = \partial_{x_1}\mathscr{J}(H_i, x), \tag{22a}$$

$$\partial_{x_0}\mathscr{J}(E_\alpha, x) = \{\mathscr{J}(E_\alpha, x), H\} = \partial_{x_1}\mathscr{J}(E_\alpha, x), \tag{22b}$$

$$\partial_{x_0}\tilde{\mathscr{J}}(H_i, x) = \{\tilde{\mathscr{J}}(H_i, x), H\} = -\partial_{x_1}\tilde{\mathscr{J}}(H_i, x), \tag{22c}$$

$$\partial_{x_0}\tilde{\mathscr{J}}(E_\alpha, x) = \{\tilde{\mathscr{J}}(E_\alpha, x), H\} = -\partial_{x_1}\tilde{\mathscr{J}}(E_\alpha, x), \tag{22d}$$

These are consistent with the Lagrange's Eqs.(7) and (9).

We have now completed the discussions on the Hamiltonian formula of the WZNW field theory under the chevalley bases.

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APPENDICES

A.

We list the calculation rules of $N_{\alpha,\beta}$ in this appendix. $N_{\alpha,\beta}$ is defined in (1c):

$$[E_\alpha, E_\beta] = \sum_{ij} K_{ij}^{-1} K_{j\alpha} H_i \delta_{\alpha+\beta,0} + N_{\alpha,\beta} E_{\alpha+\beta}, \tag{1c}$$

In general, we assume $H_i^t = H_i$, $E_\alpha^t = E_{-\alpha}$[1] ($i = 1, 2, \ldots,$ rank $\mathscr{G}$; $\alpha \in \Phi$). It is not difficult to prove

$$N_{\alpha,\beta} = -N_{\beta,\alpha}(N_{\alpha,-\alpha} = 0), \tag{A.1}$$

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}, \tag{A.2}$$

$$N_{\alpha,\beta} = \frac{(\alpha+\beta)^2}{\beta^2} N_{-\alpha,\alpha+\beta}, \tag{A.3}$$

B.

We now give some equations satisfied by tensor $\lambda_{ab}(\theta)$ elements of matrices $\Omega(\theta)$ and $L_{AB}(\theta)$.

1. $\lambda_{ab}(\theta)$ is a anti-symmetrical tensor,

$$\lambda_{ab} = -\lambda_{ba}, \tag{B.1}$$

[1] t denotes transpose.

2. The definition (4) implies that

$$\partial_a \Omega_b^i - \partial_b \Omega_a^i = \sum_j \sum_{\beta \in \Phi} K_{ij}^{-1} K_{j\beta} \Omega_a^{-\beta} \Omega_b^{\beta}, \tag{B2a}$$

$$\partial_a \Omega_b^{-\alpha} - \partial_b \Omega_a^{-\alpha} = \sum_j K_{\alpha j}(\Omega_a^j \Omega_b^{-\alpha} - \Omega_a^{-\alpha} \Omega_b^j) + \sum_{\beta \in \Phi} \frac{\alpha^2}{(\alpha+\beta)^2} N_{\alpha,\beta} \Omega_a^{-\alpha-\beta} \Omega_b^{\beta}, \tag{B2b}$$

$$\begin{aligned} \partial_c \lambda_{ab} + \partial_a \lambda_{bc} + \partial_b \lambda_{ca} &= \sum_j \sum_{\beta \in \Phi} \frac{2}{\beta^2} K_{\beta j}(\Omega_a^j \Omega_b^{-\beta} \Omega_c^{\beta} + \Omega_a^{\beta} \Omega_b^j \Omega_c^{-\beta} + \Omega_a^{-\beta} \Omega_b^{\beta} \Omega_c^j) \\ &+ \sum_{\gamma,\beta \in \Phi} \frac{l}{(\beta+\gamma)^2} N_{\beta,\gamma} \Omega_a^{-\beta-\gamma} \Omega_b^{\gamma} \Omega_c^{\beta}, \end{aligned} \tag{B.3}$$

or

$$\partial_c \lambda_{ab} + \partial_a \lambda_{bc} + \partial_b \lambda_{ca} = \sum_{ij} \frac{2}{\alpha_i^2} K_{ij}(\partial_a \Omega_b^j - \partial_b \Omega_a^j) \Omega_c^i + \sum_{\beta \in \Phi} \frac{2}{\beta^2} (\partial_a \Omega_b^{-\beta} - \partial_b \Omega_a^{-\beta}) \Omega_c^{\beta}, \tag{B.4}$$

3. $\Omega(\theta)$is a non-singular matrix with its inverse,

$$\begin{cases} \Omega_a^i \omega^{aj} = \delta^{ij}, \\ \Omega_a^i \omega^{a\beta} = 0, \quad i, \ j = 1, 2, \ldots, \ \mathrm{rank}\mathcal{G}. \\ \Omega_a^{\alpha} \omega^{aj} = 0, \quad \alpha, \beta \in \Phi. \\ \Omega_a^{\alpha} \omega^{a\beta} = \delta_{\alpha+\beta,0}, \end{cases} \tag{B.5a}$$

$$\sum_i \omega^{ai} \Omega_b^i + \sum_{\alpha \in \Phi} \omega^{a\alpha} \Omega_b^{-\alpha} = \delta_b^a, \tag{B.5b}$$

then,

$$\partial_b \omega^{ci} = -\omega^{ai} \left[\sum_j \omega^{cj} \partial_b \Omega_a^j + \sum_{\beta \in \Phi} \omega^{c\beta} \partial_b \Omega_a^{-\beta} \right], \tag{B.6a}$$

$$\partial_b \omega^{c\alpha} = -\omega^{a\alpha} \left[\sum_j \omega^{cj} \partial_b \Omega_a^j + \sum_{\beta \in \Phi} \omega^{c\beta} \partial_b \Omega_a^{-\beta} \right], \tag{B.6b}$$

4. Constructing a matrix $L_{AB} = \mathrm{Tr}(AgBg^{-1})$. Its typical elements are

$$\begin{aligned} L_{ij} &= \mathrm{Tr}(H_i g H_j g^{-1}), \quad L_{i\beta} = \mathrm{Tr}(H_i g E_\beta g^{-1}), \\ L_{\alpha j} &= \mathrm{Tr}(E_\alpha g H_j g^{-1}), \quad L_{\alpha\beta} = \mathrm{Tr}(E_\alpha g E_\beta g^{-1}), \end{aligned} \tag{B.7}$$

we have

$$g^{-1} H_i g = \sum_{jl} \frac{\alpha_l^2}{2} K_{jl}^{-1} L_{il} H_j + \sum_{\gamma \in \Phi} \frac{\gamma^2}{2} L_{i\gamma} E_{-\gamma}, \tag{B.8a}$$

$$g^{-1}E_\alpha g = \sum_{jl} \frac{\alpha_l^2}{2} K_{jl}^{-1} L_{\alpha l} H_j + \sum_{\gamma \in \Phi} \frac{\gamma^2}{2} L_{\alpha\gamma} E_{-\gamma}, \tag{B.8b}$$

Because $\mathrm{Tr}(AB) = \mathrm{Tr}(g^{-1}Agg^{-1}Bg)$, $\mathrm{Tr}([A, B]gcg^{-1}) = \mathrm{Tr}(g^{-1}Ag[g^{-1}Bg, c])$ we obtain the following equations from the formulas (B.8):

$$\frac{2}{\alpha_i^2} K_{ij} = \sum_{kl} \frac{\alpha_l^2}{2} K_{kl}^{-1} L_{il} L_{jk} + \sum_{\alpha \in \Phi} \frac{\alpha^2}{2} L_{i\alpha} L_{j,-\alpha}, \tag{B.9a}$$

$$\frac{2}{\alpha^2} \delta_{\alpha+\beta,0} = \sum_{jl} \frac{\alpha_l^2}{2} K_{jl}^{-1} L_{\alpha j} L_{\beta l} + \sum_{\gamma \in \Phi} \frac{\gamma^2}{2} L_{\alpha,-\gamma} L_{\beta\gamma}, \tag{B.9b}$$

$$0 = \sum_{jl} \frac{\alpha_l^2}{2} K_{jl}^{-1} L_{il} L_{\alpha j} + \sum_{\gamma \in \Phi} \frac{\gamma^2}{2} L_{i\gamma} L_{\alpha,-\gamma}, \tag{B.9c}$$

and

$$0 = \sum_{\alpha \in \Phi} \frac{\alpha^2}{2} K_{\alpha l} L_{i\alpha} L_{j,-\alpha}, \tag{B.10a}$$

$$0 = \sum_{kl} \frac{\alpha_l^2}{2} K_{kl}^{-1} K_{\alpha l}(L_{il} L_{jk} - L_{ik} L_{j\alpha}) + \sum_{\beta \in \Phi} \frac{\beta^2}{2} N_{\alpha,\beta} L_{i,-\beta} L_{j,\alpha+\beta}, \tag{B.10b}$$

$$K_{\alpha i} L_{\alpha j} = \sum_{\beta \in \Phi} \frac{\beta^2}{2} K_{\beta j} L_{i,-\beta} L_{\alpha\beta}, \tag{B.10c}$$

$$K_{\alpha i} L_{\alpha\beta} = \sum_{jl} \frac{\alpha_l^2}{2} K_{jl}^{-1} K_{\beta l}(L_{i\beta} L_{\alpha j} - L_{ij} L_{\alpha\beta}) + \sum_{\gamma \in \Phi} \frac{\gamma^l}{2} N_{\gamma,\beta} L_{\alpha,-\gamma} L_{i,\gamma+\beta}) \tag{B.10d}$$

$$\delta_{\alpha+\beta,0} \sum_{kl} K_{kl}^{-1} K_{l\alpha} L_{kj} + N_{\alpha,\beta} L_{\alpha+\beta,j} = \sum_{\gamma \in \Phi} \frac{\gamma^2}{2} K_{\gamma j} L_{\alpha,-\gamma} L_{\beta,\gamma}, \tag{B.10e}$$

$$\delta_{\alpha+\beta,0} \sum_{kl} K_{kl}^{-1} K_{l\alpha} L_{k\gamma} + N_{\alpha,\beta} L_{\alpha+\beta,\gamma} = \sum_{jl} \frac{\alpha_l^2}{2} K_{jl}^{-1} K_{\gamma l}(L_{\alpha\gamma} L_{\beta j} - L_{\alpha j} L_{\beta\gamma}) + \sum_{\sigma \in \Phi} \frac{\sigma^2}{2} N_{\sigma,\gamma} L_{\alpha,\sigma+\gamma} L_{\beta,-\sigma}. \tag{B.10f}$$

5. On account of $\partial_a L_{AB} = \mathrm{Tr}([A, \partial_a g g^{-1}]gBg^{-1})$, we find

$$\partial_a L_{jk} = \sum_{\beta \in \Phi} K_{\beta j} \Omega_a^{-\beta} L_{\beta k}, \tag{B.11a}$$

$$\partial_a L_{j\alpha} = \sum_{\beta \in \Phi} K_{\beta j} \Omega_a^{-\beta} L_{\beta\alpha}, \tag{B.11b}$$

$$\partial_a L_{\alpha j} = -\sum_{i} K_{\alpha i} \Omega_a^{i} L_{\alpha j} + \sum_{kl} K_{kl}^{-1} K_{l\alpha} \Omega_a^{\alpha} L_{kj} + \sum_{\gamma \in \Phi} N_{\alpha,\gamma} \Omega_a^{-\gamma} L_{\alpha+\gamma,j}, \tag{B.11c}$$

$$\partial_a L_{\alpha\beta} = -\sum_{i} K_{\alpha i} \Omega_a^{i} L_{\alpha\beta} + \sum_{kl} K_{kl}^{-1} K_{l\alpha} \Omega_a^{\alpha} L_{k\beta} + \sum_{\gamma \in \Phi} N_{\alpha,\gamma} \Omega_a^{-\gamma} L_{\alpha+\gamma,\beta}. \tag{B.11d}$$

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