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A Rigorous Calculation for the Radiator in e^+e^- Processes

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Using the precise expression of electron, positron distribution functions we obtained previously, we get the rigorous analytical expression of the radiator in e^+e^- collision processes. This series expression converges rapidly to the precise result required. The comparison with existing approximate expression is also given. Our result benefits the precise calculation for the radiative corrections in e^+e^- collision processes.

To obtain the correct physical results, one must make the radiative corrections for the data in e^+e^- processes. Usually, this correction is not very small and depends sensitively on the procedure which is used for the correction. Therefore, it becomes a theoretical problem to make the correction as precise as possible. In 1985, Kuraev and Fadin [1] proposed the structure function method for calculating the radiative correction in e^+e^- processes, improving the traditional method which calculates the Feynman diagrams order by order. Their method has become very popularly.

In this procedure, we introduce the distribution functions of electron, positron $D_e(x,S)$, $D_{\bar{e}}(x,S)$. This is the probability distribution for finding an electron or positron of transverse momentum fraction x at the energy scale $\sqrt{S}$ after the emission of photons from them, $x = p/E$. S is its energy scale. The cross section for the e^-e^- collision process (by single photon annihilation) can be expressed as:

$$\sigma(S) = \int_0^1 dx_1 dx_2 D_e(x_1, S) D_{\bar{e}}(x_2, S) \sigma_B(x_1 x_2 S). \tag{1}$$

where $\sigma_B(x_1 x_2 S)$ is the cross section where the electron and positron collide in the initial colliding point with the total energy squared $x_1 x_2 S$ in the center-of-mass system.

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Furthermore, by introducing a new variable x, $\sigma(S)$ can be rewritten as:

$$\sigma(S) = \int_0^1 \mathrm{d}x F(x, S)\sigma_B((1-x)S). \tag{2}$$

where

$$x_1 x_2 = 1 - x, \tag{3}$$

and the radiator $F(x,S)$ is defined as

$$F(x, S) = \int_{1-x}^1 \frac{\mathrm{d}z}{z} D_e(z,S) D_{\bar{e}}\left(\frac{1-x}{z}, S\right). \tag{4}$$

Equation (2) is the universal expression of the cross section for the e^+e^- collision processes by single photon annihilation. In the literature, functions D_e, $D_{\bar{e}}$ are given by exponential formulas. Inevitably, there is always a certain amount of leeway in this procedure. Therefore, the radiator and the radiative correction calculated from this distribution function contains some errors.

In order to calculate the precise radiative correction for e^+e^- processes, we calculate the electron and positron distribution functions as precisely as possible [2]. This is an analytic series expression which can converge rapidly with the required high precision. Therefore, it is possible to calculate the radiator analytically. We will give this result in our paper.

The electron and positron distribution functions are given in [2] and consist of non-singlet and singlet components. The latter is smaller than the former by several orders of magnitude, even at a very high energy scale (e.g., a few tens of TeV). Therefore, it is sufficiently precise to consider only the non-singlet component:

$$D_e(x,S) = e^{(\frac{3}{4}-C)2F}\left(\ln\frac{1}{x}\right)^{2F-1}\sum_{k=0}^{\infty} G_k(F)\frac{\left(\ln\frac{1}{x}\right)^k}{\Gamma(2F+k)}. \tag{5}$$

where

$$F = -\frac{3}{2}\ln\left(1 - \frac{\alpha}{3\pi}\ln\frac{S}{m_e^2}\right), \tag{6}$$

$$C = 0.5772156649\cdots \text{ Euler constant.}$$

The coefficients are

$$G_0 = 1,$$

$$G_1 = -F,$$

$$G_2 = 2F\frac{3(2F)+14}{24},$$

$$G_3 = -(2F)\frac{(2F)^2+14(2F)+24}{48},$$

$$G_4 = 2F\frac{15(2F)^3+420(2F)^2+2420(2F)+2832}{5760},$$

$$G_5 = -(2F)\frac{3(2F)^4+140(2F)^3+1700(2F)^2+6192(2F)+5760}{11520} \tag{7}$$

$$\cdots\cdots$$

In order to get the radiator by completing integral (4), we expand electron distribution function (5) as the power function of $(1 - x)$:

$$D_e(x,S) = e^{(\frac{3}{4}-C)2F}\{C_0(1-x)^{2F-1} + C_1(1-x)^{2F} + C_2(1-x)^{2F+1} + C_3(1-x)^{2F+2} + C_4(1-x)^{2F+3} + \cdots\}. \quad (8)$$

where

$$C_0 = \frac{1}{\Gamma(2F)},$$

$$C_1 = \frac{1}{\Gamma(2F+1)}(2F^2 - 2F),$$

$$C_2 = \frac{1}{\Gamma(2F+2)}\left(2F^4 - \frac{4}{3}F^3 - F^2 + F\right),$$

$$C_3 = \frac{1}{\Gamma(2F+3)}\frac{4}{3}(F^6 + F^5 - F^4 + F^2),$$

$$C_4 = \frac{1}{\Gamma(2F+4)}\frac{1}{6}\left(4F^8 + 16F^7 + \frac{52}{3}F^6 + \frac{12}{5}F^5 + 7F^4 + 18F^3 + 9F^2\right).$$

$$\cdots\cdots$$

Then, we make the following considerations as illustrated in [1].

1) Add the contribution from virtual electron loop [1,3]. Its lowest contribution is on the order of $(\alpha/\pi)^2$.

$$\left(\frac{\alpha}{\pi}\right)^2\left[-\frac{1}{36}L^3 + \frac{15}{72}L^2 + \left(\frac{1}{18}\pi^2 - \frac{265}{216}\right)L + \text{constant}\right],$$

$$L = \ln\frac{S}{m_e^2}$$

The contribution from $(\alpha/\pi)^2L$ term and below are less then 10^{-5}. It is precise enough to consider only the first two terms. Therefore, we set C_0 in the expansion (8) as

$$C_0 = \frac{1}{\Gamma(2F)}\left[1 - \frac{\beta^2}{288}(2L - 15)\right]. \quad (9)$$

$$\beta \equiv \frac{2\alpha}{\pi}\ln\frac{S}{m_e^2} = \frac{2\alpha}{\pi}L. \quad (10)$$

2) Comparing the cross section calculated from this distribution function with that by the soft photon approximation [1, 4], we make the substitution $L \to L - 1$.

The following integral formula is useful to complete the integral $F(x,S)$ [5].

$$\int_{1-x}^{1}\frac{dz}{z}(1-z)^a\left(1-\frac{1-x}{z}\right)^b = x^{a+b+1}B(a+1,b+1)E(b+1,a+1,a+b+2,x)$$

$$= x^{a+b+1}\sum_{n=0}^{\infty}\frac{\Gamma(b+1+n)\Gamma(a+1+n)}{\Gamma(a+b+2+n)}\frac{x^n}{n!}. \quad (11)$$

where $B(x,y)$ and $F(a,b,c,x)$ are beta function and hyper-geometric function, respectively. Finally, we obtain the expression of the radiator:

$$F(x,S) = e^{(\frac{3}{4}-C)4F}\{a_0x^{4F-1} + a_1x^{4F} + a_2x^{4F+1} + a_3x^{4F+2} + a_4x^{4F+3} + \cdots\}.$$

where

$$a_0 = \frac{(1-a)^2}{\Gamma(4F)},$$

$$a_1 = \frac{1}{\Gamma(4F+1)}(8F^2 - 4F + 4aF - 12aF^2 + 4a^2F^2),$$

$$a_2 = \frac{1}{\Gamma(4F+2)}\left(2F - 4F^2 - \frac{32}{3}F^3 + 32F^4 - 2aF + 10aF^2 + \cdots\right),$$

$$a_3 = \frac{1}{\Gamma(4F+3)}\left\{\frac{16}{3}F^2(1 - 4F^2 + 8F^3 + 16F^4) - a(-8F^2 - 24F^4 - 104F^5 - 72F^6) + 16a^2F^2(4 + 24F + 52F^2 + 48F^3 + 16F^4)\right\},$$

$$a_4 = \frac{1}{\Gamma(4F+4)}\left\{\frac{4}{9}\left(\frac{27}{2}F^2 + 54F^3 + 42F^4 + \frac{144}{5}F^5 + 416F^6 + 768F^7 + 384F^8\right) - a\left(11F^2 + \frac{154}{3}F^3 + 93F^4 + \frac{3652}{15}F^5 + \frac{4204}{9}F^6 + 384F^7 + 108F^8\right) + a^2\left(6F^2 + 44F^3 + \frac{386}{3}F^4 + 192F^5 + \frac{464}{3}F^6 + 64F^7 + \frac{32}{3}F^8\right)\right\}.$$

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This series expression is a rigorous analytical result. Figure 1 shows the curves to each order in the expansion. We note that this series converges rapidly. The relative error between the sum from the first three terms and the saturated values is less than 4×10^{-4}, when $x < 0.3$. This error is less than 3×10^{-3}, when $x < 0.5$.

We also find that the contribution concentrates in the small x region. If we discuss the resonance or the threshold behaviors in e^+e^- collision processes, we will pay more attention to the behavior of $F(x,S)$ in small x region, because the main contribution comes from this region (small x corresponds to $x_1x_2 = 1$). In the smaller x region, this expansion converges more rapidly. For example, only the first three terms approach to the saturated value, when $x < 0.1$.

There is also the "fixed coupling approximation" used in the literature, i.e., to solve the electron distribution function with

$$\alpha(t) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}t} \simeq \alpha.$$

leading to the following substitution in the distribution function $D(x,S)$:

$$F = -\frac{3}{2}\ln\left(1 - \frac{\alpha}{3\pi}t\right) \to \frac{\alpha}{2\pi}t = \frac{\beta}{4}.$$

The variation of $F(x,S)$ deduced by this approximation is less than $4 \times 10^{-3}(E_b = 2\text{ GeV})$-$8 \times 10^{-3}(E_b = 20\text{ GeV})$, which is smaller than the allowable error (e.g., the allowable error is less than 1%). Therefore, one can accept such a substitution.

Let us compare our results (to the 5th order) with existing exponential formulas.

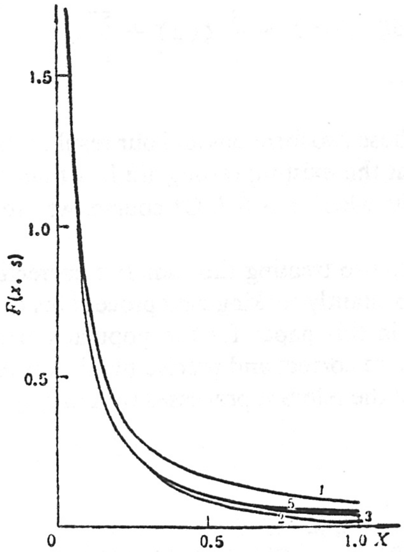

Fig. 1

The behavior of $F(x,S)$. Curves 1, 2, 3, and 5 show the behavior to the 1st, 2nd, 3rd, and 5th orders (the contribution from the first four orders is between curves 3 and 5).

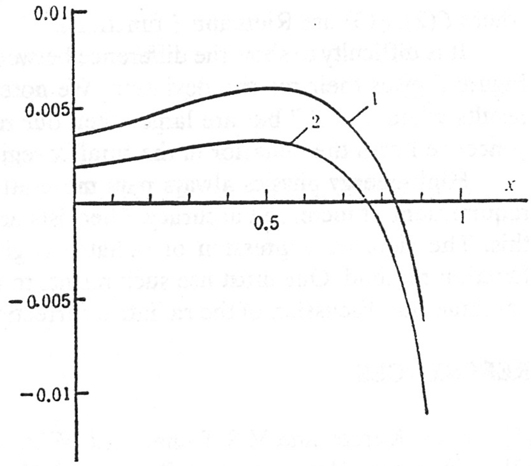

Fig. 2

The relative errors between rigorous results and the approximate results. Curve 1 corresponds to the Kuraev and Fadin result, and curve 2 corresponds to the Nicrosini and Trentadue result.

1) Kuraev and Fadin [1] give:

$$F(x,S) = \beta x^{\beta-1}\left[1+\frac{3}{4}\beta+\frac{\alpha}{\pi}\left(\frac{\pi^2}{3}-\frac{1}{2}\right)-\frac{\beta^2}{24}\left(\frac{1}{3}\ln\frac{S}{m_e^2}+2\pi^2-\frac{37}{4}\right)\right]$$
$$-\beta\left(1-\frac{x}{2}\right)$$
$$+\frac{1}{8}\beta^2\left[4(2-x)\ln\frac{1}{x}-\frac{1+3(1-x)^2}{x}\ln(1-x)-6+x\right].$$

2) Nicrosini and Trentadue [6] give:

$$F(x,\ S) = \Delta(S)\beta x^{\beta-1}-\frac{1}{2}\beta(2-x)$$
$$+\frac{1}{8}\beta^2\left\{(2-x)[3\ln(1-x)-4\ln x]-4\,\frac{\ln(1-x)}{x}+x-6\right\}.$$

where

$$\beta = \frac{2\alpha}{\pi}\left(\ln\frac{S}{m_e^2}-1\right),$$

$$\Delta(S) = 1+\frac{\alpha}{\pi}\left[\frac{3}{2}\ln\frac{S}{m_e^2}+2(\zeta(2)-1)\right]$$
$$+\left(\frac{\alpha}{\pi}\right)^2\left\{\left[\frac{9}{8}-2\zeta(2)\right]\left(\ln\frac{S}{m_e^2}\right)^2\right.$$
$$+\left[3\zeta(3)+\frac{11}{2}\zeta(2)-\frac{45}{16}\right]\ln\frac{S}{m_e^2}$$

$$+\left[-\frac{6}{5}\zeta^2(2)-\frac{9}{2}\zeta(3)-6\zeta(2)\ln 2+\frac{3}{8}\zeta(2)+\frac{57}{12}\right]\bigg\}.$$

where $\xi(2)$, $\xi(3)$ are Riemann ξ functions.

It is difficulty to show the difference between these two formulas and our result, shown in Fig. 1. Figure 2 gives their relative deviation. We note that the existing results are less than our rigorous results when $x < 0.7$ but are larger then our results when $x > 0.7$. Of course, we are still mainly concerned with the behavior in the small x region.

High-energy physics always pays more attention to treating the radiative correction with the requirement of increased accuracy. Theorists are constantly seeking new procedures to accomplish this. The rigorous expression of radiator is given in this paper for the popularly used structure function method. One must use such results to achieve correct and precise physical results. We will continue our discussion of the radiative correction of the relevant processes with this rigorous result.

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