High Energy Physics and Nuclear Physics

Volume 17, Number 1

Variational Study of 3-Dimensional $SU(2)$ Lattice Gauge Theory with Independent Plaquette Trial Action

Liu Jinming and Gong Di

(Department of Physics, Zhongshan University, Guangzhou, China)

The phase structure of 3-dimensional $SU(2)$ lattice gauge theory is studied by using variational method with an independent plaquette trial action. The mean-plaquette internal energy $E_p \sim \beta$ curve is smooth, which shows only one confining phase, and is better than that given by independent link trial action.

So far, lattice gauge theories (LGT) [1] have been an essential unperturbed method for studying strong interaction. The phase structure of $SU(2)$ gauge field has also been one of the subjects studied earliest and most extensively. Creutz first applied Monte Carlo method and studied the phase structures of $SU(2)$ gauge field successfully [2]. For analytic calculations of the phase structures, the mean field theory [3,4], variational method [4,5] and variational cumulant expansion method [6-9] have been developed and applied to study the phase structures of lattice gauge theories. In [8,9] the mean plaquette energy for $SU(2)$ gauge field has been calculated up to third and even fourth order corrections by variational cumulant expansion method. The results are in good agreement in strong and weak coupling regions, although in the crossover region, the agreement is still to be improved. The essential point is that the trial action used is the independent link action. In this paper, the mean plaquette internal energy of 3-dimensional $SU(2)$ lattice gauge field is studied by using the variational method with an independent plaquette trial action. We find that the calculation is improved, the internal energy curve is smooth, and there is no phase transition, and the results agree with the predictions of the

---

Supported in part by the National Natural Science Foundation of China and the Foundation of Zhongshan University Advanced Center.

Received on October 4, 1991.

current theories. It is shown that the approximate free energy obtained by the independent plaquette trial action is always lower than that given by the independent link trial action. This shows that using the independent plaquette trial action is a better approximation to the real free energy of the system and to make the internal energy curve easy to approach the real internal energy of the system. In addition, applying the approximate free energy to 3 + 1-dimensional Hamiltonian of lattice gauge theories to calculate variational state mode would be a better approximation, so this method can be applied to calculate the glueball masses [10].

For concreteness we discuss the $SU(2)$ lattice gauge field, in which the standard Wilson action could be shown as follows

$$S = \frac{\beta}{4} \sum_p \mathrm{tr}(U_p + U_p^+), \ \beta = 4/g^2. \tag{1}$$

and the system is described by the partition function

$$\mathrm{Z} = \mathrm{e}^{-W} = \int \mathrm{D}U_l \mathrm{e}^S, \mathrm{D}U_l \equiv \prod_l \mathrm{d}U_l. \tag{2}$$

where W is the free energy of the system, U_l is called link variable is a group element for the link l, U_p is called plaquette variable is the product of four link variables around p plaquette. In the variational formulations, let us introduce a trial action $S_0(z)$ which can be calculated exactly, z is the variational parameter. Let the trial partition function be the following form

$$\mathrm{Z}_0 = \int \mathrm{D}U_l \mathrm{e}^{S_0}, \tag{3}$$

according to the convex inequality

$$W \leqslant W_{\mathrm{eff}}, \ W_{\mathrm{eff}} = -\ln \mathrm{Z}_0 - \langle S - S_0 \rangle_0, \tag{4}$$

where W_{eff} is called approximate free energy and the variational parameter z is determined by the condition of minimizing free energy:

$$\frac{\partial W_{\mathrm{eff}}}{\partial z} = 0, \ \frac{\partial^2 W_{\mathrm{eff}}}{\partial z^2} > 0. \tag{5}$$

The order parameter used in studying the phase structures is the mean plaquette internal energy E_p:

$$E_p = \frac{1}{N_p} \left\langle \sum_p \left[1 - \frac{1}{2\mathrm{tr}1} \mathrm{tr}(U_p + U_p^+) \right] \right\rangle \simeq 1 + \frac{\partial W_{\mathrm{eff}}}{\partial \beta}, \tag{6}$$

where N_p is the total number of plaquette, $W_{\mathrm{eff}} = W_{\mathrm{eff}}/N_p$ is the approximate free energy density. It is possible to discuss the phase structures of the system in accordance with $E_p \sim \beta$ curves.

The simplest trial action S_0 used in lattice gauge theories is the independent link trial action

$$S_0 = \frac{z}{4} \sum_l \mathrm{tr}(U_l + U_l^+). \tag{7}$$

If we change the link variables U_l to plaquette variables U_p, it should be multiplied by Jacobian when element of the integral $\mathrm{D}U_l$ is changed to element of the integral $\mathrm{D}U_p$ and this Jacobian has been

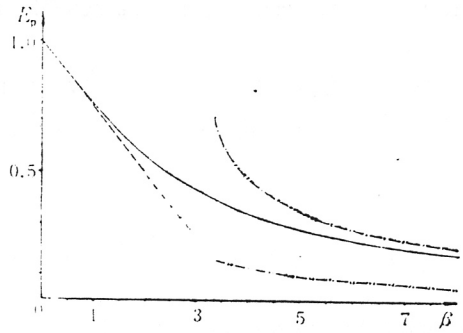

Fig. 1

The mean plaquette internal energy E_p for the 3-dimensional $SU(2)$ LGT as a function of β. ——— : the result from S_1; - - - : the result from strong coupling limit; -·- : the result from S_0; -···- : the result from S_0 with variational cumulant expansion approach up to second order.

proven equal to the product of lattice Bianchi identity. Because of a lattice Bianchi identity appears for each lattice cube, in 4-dimensional space-time it is the product of the lattice Bianchi identities of the four kind lattice cubes. Unfortunately, in this case, the Bianchi identity cannot be solved at present. But in 3-dimensional space-time, it contains only one lattice cube, and the lattice Bianchi identity can be solved to give the partition function in terms of the integral of plaquette variables. We can discuss 3-dimension $SU(2)$ lattice gauge field as follows. Let 0, 1, 2 stand for time t, x and y directions respectively, the independent plaquette variables on μ (μ, = 0, 1, 2) plan can be denoted by $U_{01}(txy)$, $U_{02}(txy)$ and $U_{12}(txy)$, respectively. After integrating the Bianchi identity, we obtain [11]

$$Z = \int DU_{01}DU_{02}DU_{12}(t_0)e^S, \tag{8}$$

$$\begin{aligned} S &= \frac{\beta}{4}\sum_p \mathrm{tr}(U_p + U_p^+) = \frac{\beta}{4}\sum_{txy}\mathrm{tr}(U_{01} + U_{02} + U_{12} + U_{01}^+ + U_{02}^+ + U_{12}^+) \\ &= \frac{\beta}{2}\sum_{txy}\mathrm{tr}(U_{01}(txy) + U_{02}(txy)) + \frac{\beta}{2}\sum_{xy}\mathrm{tr}U_{12}(t_0xy) \\ &\quad + \frac{\beta}{2}\sum_{t\neq t_0}{}'\sum_{xy}\mathrm{tr}U_{12}(txy), \end{aligned} \tag{9}$$

where

$$DU_{01} = \prod_{txy} dU_{01}(txy),\ DU_{02} = \prod_{txy} dU_{02}(txy),\ DU_{12}(t_0) = \prod_{x,y} dU_{12}(t_0xy),$$

and in obtaining the last expression of action, we have used the property of $SU(2)$ group: $\mathrm{tr}(U + U^+) = 2\,\mathrm{tr}U$.

The "path gauge" is used as link variables U_0, U_1, U_2 change to plaquette variables U_{01}, U_{02}, U_{12}. Usually we choose the maximum point $(t_0x_0y_0)$ to be the origin of the fixed "path gauge", and define

$$U_0(txy) = U_2(t_0xy) = U_1(t_0xy_0) = 1, \tag{10}$$

Therefore the new independent variables turn into the plaquette variables $U_{01}(txy)$, $U_{02}(txy)$ and $U_{12}(t_0xy)$. However as $t \neq t_0$ space-like plaquette variable $U_{12}(txy)$ is not the independent variable and $U_{12}(txy)$ should be expressed by independent plaquette variables as indicated above, thus we have [11]

$$U_{12}(txy) = V_{01}^{+}(txy)V_{12}(t_0xy)V_{02}^{+}(tx+1y)V_{12}^{+}(t_0xy+1) \cdot V_{01}(txy+1)V_{02}(txy), \tag{11}$$

where

$$V_{01}^{+}(txy) = \prod_{t'=t}^{t_0-1} U_{01}^{+}(t'xy), \quad V_{02}^{+}(txy) = \prod_{t'=t}^{t_0-1} U_{02}^{+}(t'xy), \quad V_{12}(t_0xy) = \prod_{y'=y}^{y_0-1} U_{12}(t_0xy)$$

Although the action S, which has been changed, is still a standard Wilson action, in the integra of Eq. (8), the first two terms in S decouple (see Eq. (a)). The coupling between plaquettes are all collected and are represented by the last term in Eq. (9) which is sum of the space-like plaquettes of $t \neq t_0$. The form of the partition function in Eq. (8) has a complicated effective action and cannot be calculated exactly at present. However, if we take only the sum of the independent plaquette variables to be a new trial action S_1, and

$$S_1 = \frac{z}{2}\sum_{txy} \mathrm{tr}(U_{01}+U_{02}) + \frac{z}{2}\sum_{xy} \mathrm{tr}U_{12}(t_0xy), \tag{12}$$

where z is the variational parameter, which is determined by the condition of minimizing free energy. S_1 is called independent plaquette trial action. Then the trial partition function Z_1 can be calculated exactly. In fact, the trial action S_1 is the truncate proximation for action S of the system, this means that the first two terms are taken only and the last term of coupling between plaquettes is neglected completely in Eq. (9). This method is analogous to an approximation in independent link trial action S_0 in which the sum of every independent link is taken only and the coupling between every link are neglected completely. The calculated results point that as a trial action S_1 is better than S_0, here the S_1 is based on the independent plaquette variables and the S_0 is based on the independent link variables. So the value of Z_1

$$Z_1 = \int DU_{01}DU_{02}DU_{12}(t_0)e^{S_1} = z_0^{N_F}, \tag{13}$$

$$z_0 = \int dU e^{\frac{z}{2}\mathrm{tr}U} = \frac{2I_1(z)}{z}, \tag{14}$$

where $N_F \approx 2N_0N_1N_2$ is the total number of independent plaquette variables, N_0, N_1 and N_2 are the total sites of lattices in t, x and y directions respectively. Let $I_1(z)$ is Bessel function of imaginary argument. Approximate free energy (the calculating process is demonstrated in appendix)

$$W_{\mathrm{eff}}^{(1)} = -N_F \ln z_0 - N_F(\beta - z)y_2 - N_1N_2\beta\frac{y_2^5}{(1-y_2^4)}, \tag{15}$$

where $y_2 = I_2(z)/I_1(z)$. The last term of Eq. (15) can be neglected as the number of lattices in every directions all approach infinity. By using the condition of minimizing free energy Eq. (5), we have variational parameter $z = \beta$. Then the approximate free energy density $W_{\mathrm{eff}}^{(1)}$ and order parameter $E_P^{(1)}$ can be represented respectively

$$W_{\mathrm{eff}}^{(1)} = -\ln z_0(\beta), \tag{16}$$

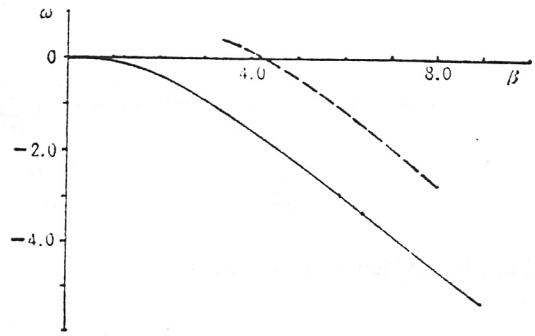

Fig. 2

The approximate free energy density W_{eff} for the 3-dimensional $SU(2)$ LGT as a function β. ——— : the result from the independent plaquette trial action S_1, - - - : the result from the independent link trial action S_0.

$$E_p^{(1)} = 1 - y_2(\beta). \tag{17}$$

For lack of the results of 3-dimensional $SU(2)$ group calculated by Monte Carlo, it is possible to compare the calculated results by using independent plaquette trial action S_1 with the calculated results by using independent link trial action S_0. The comparison is shown in Figs. 1 and 2 respectively.

In Fig. 1, the function relation of $E_p^{(1)} \sim \beta$ and $E_p^{(0)}$, $E_{p,2}^{(0)} \sim \beta$ is presented, where $E_p^{(0)}$ and $E_{p,2}^{(0)}$ represent the internal energy obtained using the independent link trial action S_0 by variational method and by variational cumulant expansion approach up to the second order respectively. From Fig. 1 we can see that $E_{p,2}^{(0)} \sim \beta$ curve in much improved $E_p^{(0)} \sim \beta$ curve, where the leap is shrunk obviously around $\beta_c = 3.35$, although the leap still be here. If it had been calculated to higher order approximation, the curve of internal energy would have approached smoother, and [8] points out that the behavior in crossover region can go from bad to worse while calculated to fourth order approximation, so independent link trial action makes it difficult to obtain satisfying results as in lower dimensional space-time. However, by using independent plaquette trial action S_1 we not only obtained smooth internal energy curve but also demonstrated the non-phase transition characteristics. It also has given correct strong coupling limit in strong coupling region. Because it only calculated to the lowest order approximation in this paper, the contribution of higher order terms should be considered further as the more the β changes, the more correlation effect between plaquettes increases.

In Fig. 2, the approximate free energy density $W_{\text{eff}}^{(1)}$ and $W_{\text{eff}}^{(0)} \sim \beta$ have been drawn, where $W_{\text{eff}}^{(1)}$ and $W_{\text{eff}}^{(0)}$ are obtained by using actions S_1 and S_0 respectively. $W_{\text{eff}}^{(1)}$ is always lower than $W_{\text{eff}}^{(0)}$ in Fig. 2. This phenomenon shows that $W_{\text{eff}}^{(1)}$ approach to real free energy density of system more than $W_{\text{eff}}^{(0)}$. This is the better reason to use the independent plaquette trial action S_1. It also shows that this method can be used to calculated the glueball masses [10].

APPENDIX

The Calculation of Approximate Free Energy $W_{\text{eff}}^{(1)}$ = (15)

The approximate free energy $W_{\text{eff}}^{(1)}$ which is given by using independent plaquette trial action S_1 can be expressed as follows:

$$W_{\text{eff}}^{(1)} = -\ln Z_1 - \langle S - S_1 \rangle_1, \tag{A.1}$$

where

$$Z_1\langle S - S_1\rangle_1 = \int DU_{01}DU_{02}DU_{12}(t_0)e^{S_1}\left\{\frac{(\beta - z)}{2}\left[\sum_{txy}\mathrm{tr}(U_{01} + U_{02}) + \sum_{xy}\mathrm{tr}U_{12}(t_0)\right] + \frac{\beta}{2}\sum_{\substack{t,x,y \\ t\neq t_0}}{}'\mathrm{tr}U_{12}(txy)\right\}. \tag{A.2}$$

the above integral can be divided into two terms:

$$\text{The first term: } = \int DU_{01}DU_{02}DU_{12}(t_0)e^{S_1}\frac{(\beta - z)}{2}\left[\sum_{txy}\mathrm{tr}(U_{01} + U_{02}) + \sum_{xy}\mathrm{tr}U_{12}(t_0)\right]$$

$$= (\beta - z)N_F z_0^{N_F}\left(\frac{1}{z_0}\frac{\partial z_0}{\partial z}\right) = (\beta - z)N_F z_0^{N_F} y_2, \tag{A.3}$$

where $y_2 = I_2(z)/I_1(z)$,

$$\text{The second term: } = \int DU_{01}DU_{02}DU_{12}(t_0)e^{S_1}\frac{\beta}{2}\sum_{\substack{txy \\ t\neq t_0}}\mathrm{tr}U_{12}(txy)$$

$$= \sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\sum_{T=1}^{N_0}\int DU_{01}DU_{12}DU_{12}(t_0)\frac{\beta}{2}e^{S_1}\mathrm{tr}U_{12}(t_0 - T, N_1 - n_1, N_2 - n_2), \tag{A.4}$$

where for convenience let the lattice spacing $a = 1$, and $t_0 = N_0$, $t = t_0 - T$, $x = N_1 - n_1$, $y = N_2 - n_2$, please notice that in integrand, $U_{12}(t_0 - T, N_1 - n_1, N_2 - n_2)$ is not the independent variables and it should substitute Eq.(11) into Eq. (A.4) while integrating. One of terms in integrand posses the form

$$\begin{aligned}
&\frac{\beta}{2}\ e^{S_1}\mathrm{tr}U_{12}(t_0 - T, N_1 - n_1, N_2 - n_2) \\
&= \frac{\beta}{2}\ e^{S_1}U_{01}^{+i_0i_1}(t_0 - T, N_1 - n_1, N_2 - n_2)U_{01}^{+i_1i_2}(t_0 - T + 1, N_1 - n_1, N_2 - n_2) \\
&\quad \cdots\cdots U_{01}^{+i_{T-1}i_T}(t_0 - 1, N_1 - n_1 N_2 - n_2)U_{12}^{i_Ti_1}(t_0 N_1 - n_1 N_2 - n_2)\cdot \\
&\quad U_{12}^{i_1i_2}(t_0, N_1 - n_1, N_2 - n_2 + 1)U_{12}^{+i_{n_2-1}i_1}(t_0, N_1 - n_1, N_2 - n_2 + 1) \\
&\quad U_{12}^{i_2i_3}(t_0 N_1 - n_1, N_2 - n_2 + 2)\ U_{12}^{+i_{n_2-2}i_{n_2-1}}(t_0, N_1 - n_1, N_2 - n_2 + 2) \\
&\quad \vdots \qquad\qquad\qquad \vdots \\
&\quad U_{12}^{i_{n2-2}i_{n2-1}}(t_0 N_1 - n_1 N_2 - 2)\ U_{12}^{+i_1i_2}(t_0 N_1 - n_1 N_2 - 2) \\
&\quad U_{12}^{i_{n2-1}i_{n2}}(t_0 N_1 - n_1 N_2 - 1)\ U_{12}^{+k_Ti_1}(t_0 N_1 - n_1, N_2 - 1) \\
&U_{02}^{+i^{n2}k_1}(t_0 - T, N_1 - n_1 + 1, N_2 - n_2)U_{02}^{+k_1k_2}(t_0 - T + 1, N_1 - n_1 + 1, N_2 - n_2) \\
&\quad \cdots\cdots U_{02}^{+k_{T-1}k_T}(t_0 - 1, N_1 - n_1 + 1, N_2 - n_2) \\
&\quad \cdot U_{01}^{m_1m_2}(t_0 - 1, N_1 - n_1, N_2 - n_2 + 1)\cdots\cdots U_{01}^{m_Tn_1}(t_0 - T, N_1 - n_1 N_2 - n_2 \\
&\quad + 1)U_{02}^{n_1n_2}(t_0 - 1, N_1 - n_1 N_2 - n_2)\cdots\cdots U_{02}^{n_Ti_0}(t_0 - T, N_1 - n_1 N_2 - n_2)
\end{aligned} \tag{A.5}$$

where there is a total of $(2N_2 + 4T - 1)$ independent plaquette variables to multiply one by one. For easy integrating we have arranged space-like plaquette variable U_{12} and U_{12}^+ with same argument in the same row. Because of the arguments of $4T$ time-like plaquette variables U_{01}, U_{01}^+, U_{02}, U_{02}^+ and the

arguments of one space-like plaquette variables $U_{12}(t_0, N_1 - n_1, N_2 - n_2)$ are different from each other, the following formula of integration of $SU(2)$ group can be applied

$$\int \mathrm{d}U e^{\frac{z}{2}\mathrm{tr}U} U_{ij} = \int \mathrm{d}U e^{\frac{z}{2}\mathrm{tr}U} U_{ij}^{+} = V\delta_{ij}, \quad V = \frac{2I_2(z)}{z}, \tag{A.6}$$

using the above formula the above part contributes a factor of

$$V^{4T+1}\delta_{m_1 i_1}\delta i_{n_2} k_T. \tag{A.7}$$

From the other formula of integration of $SU(2)$ group

$$\int \mathrm{d}U e^{\frac{z}{2}\mathrm{tr}U} U_{ij}U_{\alpha\beta}^{+} = A\delta_{ij}\delta_{\alpha\beta} + B\delta_{i\beta}\delta_{j\alpha},$$
$$A = \frac{2I_3(z)}{z}, \quad B = \frac{4I_2(z)}{z^2}, \quad A + 2B = z_0 = \frac{2I_1(z)}{z}, \tag{A.8}$$

we can obtain $(n_2 - 1)$ factors $U_{12}U_{12}^{+}$ which has same arguments respectively. This part can contribute another factor as follows

$$(A\delta_{i_1 i_2}\delta_{l n_2-1 m_1} + B\delta_{i_1 m_1}\delta_{i_2 l n_2-1})(A\delta_{i_2 i_3}\delta_{l n_2-2 l n_2-1} + B\delta_{i_2 l n_2-1}\delta_{i_3 l n_2-2})\cdots\cdots$$
$$\cdots\cdots(A\delta_{i n_2-1 i n_2}\delta_{k_T l_1} + B\delta_{i n_2-1 l_1}\delta_{i n_2 k_T}). \tag{A.9}$$

Please note that

$$\delta_{m_1 i_1}(A\delta_{i_1 i_2}\delta_{l n_2-1 m_1} + B\delta_{i_1 m_1}\delta_{i_2 l n_2-1}) = A\delta_{i_2, l n_2-1} + 2B\delta_{i_2, l n_2-1} = z_0\delta_{i_2, l n_2-1},$$
$$\delta_{i_2 l n_2-1}(A\delta_{i_2 i_3}\delta_{l n_2-2 l n_2-1} + B\delta_{i_2 l n_2-1}\delta_{i_3 l n_2-2}) = A\delta_{i_3, l n_2-2} + 2B\delta_{i_3, l n_2-2} = z_0\delta_{i_3, l n_2-2}.$$

After multiplying Eq. (A.7) by Eq. (A.9) we can obtain the results as follows

$$\beta z_0^{N_F} y_2^{4T+1}, \tag{A.10}$$

where equality $V/z_0 = y_2$ has been applied. To repeat the same procedure for the other terms in Eq. (A.4) and to sum up all terms in Eq. (A.4), at last we have

$$\text{The second term: } = \beta N_1 N_2 z_0^{N_F} \frac{y_2^5}{(1 - y_2^4)}. \tag{A.11}$$

To sum up Eqs. (A.3) and (A.11), obviously $W_{\mathrm{eff}}^{(1)} = (15)$.

REFERENCES

[1] K. Wilson, *Phys. Rev.*, **D10** (1974), p. 2445.

[2] M. Creutz, *Phys. Rev. Lett.*, **43** (1979), p. 553; **45** (1980), p. 313; *Phys. Rev.*, **D21** (1980), p. 2308.

[3] R. Balian, J. Drouffe and C. Itzykson, *Phys. Rev.*, **D10** (1974), p. 3376; **D11** (1975), p. 2089, p. 2104.

[4] E. Brezin and J. Drouffe, *Nucl. Phys.*, **B200** (1982), p. 93.

[5] Zheng Xiti, Chung-I Tan and Chen Tian-lun, *Phys. Rev.*, **D26** (1982), p. 2843.

[6] T. C. Hsien, X. H. He and Y. S. Song, *Phys. Lett.*, **153B** (1985), p. 417.
[7] X. T. Zheng, T. L. Chen and D. C. Xian, *Phys. Lett.*, **154B** (1985), p. 166.

[8] X. T. Zheng, Z. G. Tan and J. Wang, *Nucl. Phys.*, **B287** (1987), p. 171.
[9] Wu Jiming and Zhao Peiying, *High Energy Phys. Nucl. and Phys.* (in Chinese), **10** (1986), p. 497.
[10] Guo Shuohong, Zheng Weihong and Liu Jinming, *Phys. Rev.*, **D38** (1988), p. 2591.
[11] G. Batrouni, *Nucl. Phys.*, **B208** (1982), p. 467.