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Study of Higher Order Cumulant Expansion of $U(1)$ Lattice Gauge Model at Finite Temperature

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The order parameter, Polyakov line $\langle L \rangle$, of the $U(1)$ gauge model on $N_\sigma^3 \times N_\tau (N_\tau = 1)$ lattice by using the cumulant expansion is calculated to the 5-th order. Emphasis is placed on the behavior of the cumulant expansion in the intermediate coupling region. The necessity of higher order expansion is clarified from the connection between the cumulant expansion and the correlation length. The variational parameter in the n-th order calculation is determined by the requirement that corrections of the n-th order expansion to the zeroth order expansion should vanish. The agreement with the Monte Carlo simulation is obtained not only in the weak and strong coupling regions, but also in the intermediate coupling region expect in the very vicinity of the phase transition point.

1. INTRODUCTION

In the studies of the $SU(2)$ and $U(1)$ gauge models at finite temperature by the variational cumulant expansion method [1], the calculated order parameter, Palyakov line $\langle L \rangle$, is in a better agreement with the Monte Carlo (MC) simulations than the mean field result is, but the obtained value of the deconfinement phase transition point for the $U(1)$ model, $\beta_c = 0.33$, is still apart from the MC

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estimation, $\beta_c \doteq 0.41$ [2]. Increasing the order of the expansion does not change the position of β_c [1]. A careful analysis shows that the position of β_c is entirely determined by the position of the first branch nontrivial solution of the variational condition for the free energy in the first order approximation, F_1, with respect to the variational parameter K,

$$\frac{\partial F_1}{\partial K} = 0 \tag{1}$$

The trivial solution of Eq. (1) corresponds to the strong coupling solution $K = 0$, consequently, $\langle L \rangle = 0$, while the nontrivial solution $K = 0$ leads to $\langle L \rangle = 0$. So the key point is the choice of variational parameters. On the other hand, the convergency of the cumulant expansion is also directly related to the choice of the variational parameters.

There are three possible choices of parameters:

1) solve the variational condition of Eq. (1) [3];

2) make the cumulant expansion of the Schwinger-Dyson equation, then solve approximate equations in the same orders in the expansion [4];

3) search the "accumulation point" by the scanning method [5,6].

The calculated results by using choice (1) are not satisfactory in the intermediate coupling region. The problem of choice (2) is that in some cases the S-D equation does not have solution [5]. Choice (3) is based on the parameter scanning. It can provide a comprehensive information about the parameter dependence of calculated quantities. Then, one can choose the parameters with which the calculated results converge in the fastest way. There were various ways to determine variational parameters in the scanning method:

(a) demand the corrections of successive orders in the expansion to be zero (i.e., find out the intersection point of scanning curves in the n-th and $(n-1)$-th orders [5], respectively);

(b) demand the n-th order correction with respect to those in the zeroth order to be zero (i.e., find out the intersection point of scanning curves in the n-th and zeroth orders [5], respectively);

(c) determine the intersection point of scanning curves of all or all even orders (the so-called "accumulation point").

In (a) and (b), the calculations have been done only to the 4-th order, and the significant deviations from the MC result have been observed in the intermediate coupling region. While in (c), by using the accumulation point method, a satisfactory agreement has been obtained in the weak coupling region, but in the intermediate coupling region the accumulation point "disappears". Thus Kerler et al. in their recent work [6] only discussed the problem in the weak coupling region by using the method where the parameters were determined by the accumulation point and declared that the cumulant expansion method is not applicable in the intermediate coupling region.

In our opinion, Kerler's conclusion does not provide sufficient evidence. The unapplicable place is only the method where the parameters are determined by the accumulation point. The calculation of this paper shows that this is one of the advantages of the cumulant expansion method that makes the analytic calculation possible in the intermediate coupling region.

To demonstrate this point, in this paper the Polyakov line $\langle L \rangle$ of the $U(1)$ pure gauge model at finite temperature with $N_\tau = 1$ is calculated, as an example, to the 5-th order expansion. This is motivated by the following considerations: there are MC results to be compared with. The choice of the optimized parameter values by scanning is different for different models and physical quantities. Therefore, it is necessary to study the parameter dependence of the cumulant expansion for different models and physical quantities by scanning. The $\langle L \rangle$ of the $U(1)$ model has not yet been studied by using the cumulant expansion via scanning except in [2]. Moreover, choosing $N_\tau = 1$ would make the calculation much simpler.

In the next section, the model is described. In the third section, the calculation of $\langle L \rangle$ is presented and a discussion is given in the fourth section.

2. $U(1)$ LATTICE GAUGE MODEL AT FINITE TEMPERATURE

On the $N_\sigma^3 \times N_\tau$ ($N_\tau = 1$) lattice, we take the action of the $U(1)$ gauge model as

$$S = \beta \sum_{P_\sigma} \cos\Theta_{P_\sigma} + \beta \sum_{P_\tau} \cos\Theta_{P_\tau}. \tag{2}$$

where $\Theta_P = \theta_\mu(x) + \theta_\nu(x+\hat{\mu}) - \theta_\mu(x+\hat{\nu}) - \theta_\nu(x)$ comes from the product of $U_\mu(x) = e^{i\theta_\mu(x)} \in U(1)$, four links around a plaquette. P_σ and P_τ represent the space-like plaquette and the plaquette with two timelike links, respectively. In the finite temperature theory, N_τ takes a finite value, and the periodical condition is imposed up on the time direction. The temperature is defined as $T = 1/N_\tau a$ (a denotes the lattice spacing). Then $N_\tau = 1$ corresponds to the case of high temperature.

The trial action is introduced as follows,

$$S_0 = J \sum_\sigma \cos\theta_\sigma + K \sum_\tau \cos\theta_\tau, \tag{3}$$

where σ and τ represent the space link and time link, respectively. For the auxiliary system with the action S_0, the partition function Z_0 and the free energy per site F_0 can be written as

$$e^{-F_0 M} = Z_0 = \int_{-\pi}^{\pi} \prod_i \frac{d\theta_i}{\alpha\pi} e^{S_0} = \omega_1^{N_s} \cdot \xi_1^{N_t}, \tag{4}$$

where M is the total number of sites, $N_s = (D-1)M$, $N_t = M$, and D is the dimension.

$$\omega_n = \frac{I_n(J)}{I_0(J)}, \qquad \xi_n = \frac{I_n(K)}{I_0(K)}, \tag{5}$$

where I_i is the modified Bessel function in the i-th order. J and K are variational parameters to be determined, and one should choose them properly so that the cumulant expansion can provide the best convergent result.

The order parameter, Polyakov line of the system, is a closed loop along the τ direction due to the periodical condition. In the case of $N_\tau = 1$

$$L \equiv L(x) = \cos\theta_\tau(x), \tag{6}$$

The n-th order approximation in the variational cumulant expansion [1] can be written as

$$\langle L \rangle \cong \langle L \rangle_n = \langle L \rangle_0 + \Delta\langle L \rangle_n, \tag{7}$$

and

$$\Delta\langle L \rangle_n = \sum_{i=1}^{n} \frac{1}{i!} \langle L(S - S_0)^i \rangle_c. \tag{8}$$

Here $\langle \cdots \rangle_0$ and $\langle \cdots \rangle_c$ are the statistical average and cumulant average in an auxiliary system with the action S_0, respectively. In the case of $N_\tau = 1$, all the connected diagrams composed by P_σ and P_τ are decoupled [2]. This reflects the well-known fact that there is no interaction between the space Wilson loop and the Polyakov line at high temperature in the $U(1)$ theory [8]. Therefore, the expansion of $\langle L \rangle$ in Eq. (7) does not contain P_σ and depends on parameters K and β only. In fact, the free energy at $J = 0$ is lower than that at $J = 0$ [2]. Then, we can set $J = 0$ in Eq. (3) to calculate $\langle L \rangle$.

3. CALCULATION OF POLYAKOV LINE

All contributed terms in Eq. (8) can be evaluated analytically [2],

$$
\begin{aligned}
&\langle L\rangle_0 = \xi_1, \quad \Delta\langle L\rangle_1 = \beta r\xi_1\xi'_1 - K\xi'_1,\\
&\Delta\langle L\rangle_2 = \Delta\langle L\rangle_1 + \frac{1}{2}\{\beta^2 r[\frac{1}{2}(\frac{1}{2}\xi_2^2 - \xi_1^4)' + R_0(\xi_1^2 - \xi'_1)'] - \beta rK(\xi_1^2)'' + K^2\xi''_1\},\\
&\Delta\langle L\rangle_3 = \Delta\langle L\rangle_2 + \frac{1}{3!}\{\beta^3 r[\frac{1}{2}(\frac{1}{4}\xi_3^2 - \frac{3}{4}\xi_1^2 - \frac{3}{2}\xi_1^2\xi_2^2 + 2\xi_1^6) + 3R_0\cdot\frac{1}{2}\xi_1(\frac{1}{2}\xi_2^2 - \xi_1^4)\\
&\qquad + 6R_0(\xi_1^2\xi'^2_1) + R_0R_1(\xi_1^3\xi''_1)]' - 3\beta^2 rK[\frac{1}{2}(\frac{1}{2}\xi_2^2 - \xi_1^4) + R_0(\xi_1^2 - \xi'_1)]''\\
&\qquad + 3\beta K^2 r(\xi_1\xi'_1)'' - K^3\xi'''_1\}.\\
&\qquad \cdots\cdots
\end{aligned}
\tag{9}
$$

where $r = 2(D-1)$, and $R_i = r - 1 - i$. The derivatives are taken with respect to K. The expressions of 4-th and 5-th order contributions are too lengthy to be presented here.

The emphasis of the study is focused on the intermediate coupling region. To choose the values of the variational parameter K properly, the calculation of $\langle L\rangle_i$ is performed by scanning the K value with β fixed. The typical scanning patters up to the 5-th order expansion at $\beta = 0.55$ are presented in Fig. 1. Indeed, the "accumulation point" cannot be seen here. In fact, the appearance of the "accumulation point" is the most ideal case where the cumulant expansion converges. In this case, as long as the calculation is performed up to the certain order, e.g., the 4-th order, a very good value K_c (i.e., the "accumulation point" value) which makes the expansion converges can be found. However, if the expansion converges gradually, it is possible that the optimized parameter values vary gradually with the increasing orders of the expansion. Although in the early scanning work, the intersection point of the n-th and zeroth order scanning curves or the n-th and $(n$-1)-th order scanning curves was used to determine the parameters, unfortunately, the investigation in the intermediate coupling region was

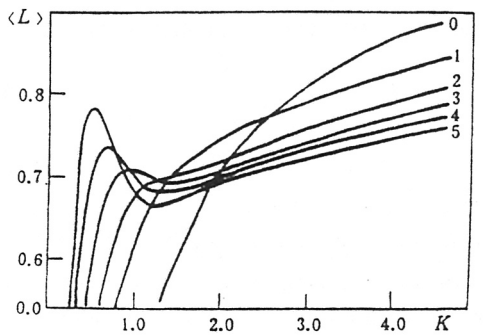

Fig.1
The scanning pattern of $\langle L\rangle_i$-K at β=0.55, i=0-5.

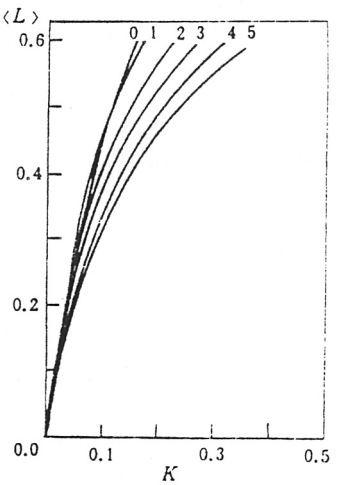

Fig. 2
The scanning pattern of $\langle L\rangle_i - K$ at β=0.38, i=0-5.

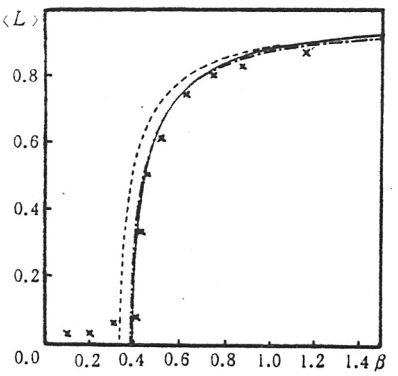

Fig. 3
The relation between $\langle L\rangle$ and β. The solid curve is $\langle L\rangle_5$. The dash-dotted line is $\langle L\rangle_4$, the dashed line is the mean field result, and the crosses are the MC results.

not continued, when the deviation between the calculated values and the MC results were observed, because it was not realized that the order in the calculation was not high enough.

In the intermediate transition region, as we know, the correlation length is increasing and is reaching a very large or even infinite value when the system is approaching the phase transition point. The essential of the cumulant expansion is an expansion according to the correlation. The l-th order correction to $\langle L\rangle_0$ is just the nonzero contribution from l plaquettes or links which are connected with L. The plaquettes which are l-links (la) apart or more from L do not contribute to $\langle L\rangle$ in the (l-1)-th order expansion, because the farthest distance which the connected diagram can reach is $(l\text{-}1)a$. The longest correlation length which can be described in this order is $(l-1)a$. Therefore, in the intermediate coupling region, when the system approaches to the phase transition point, the order of the cumulant expansion should be increased correspondingly, then the behavior of the system might be well described. The curves in Fig. 1 give us a hint that one can use the following condition to determine the optimized parameters, which lead to the convergency, in the successive orders: In the n-th order approximation,

$$\langle L\rangle \cong \langle L\rangle_n = \langle L\rangle_0|_{K=\bar{K}_n,\beta} \tag{10}$$

Then $\bar{K}_n$ is the solution of the equation

$$\Delta\langle L\rangle_n = 0. \tag{11}$$

Obviously, $\bar{K}_n$ is the intersection point of the n-th and zeroth order scanning curves. The meaning of Eq. (10) is clear. For each fixed β, at the point $\bar{K}_n$ we take $\langle L\rangle_0$ in the auxiliary system as the n-th order approximation of $\langle L\rangle$ in the original system. At this point, the total correction from the first to the n-th order cumulant expansion is zero. If $\bar{K}_n$ approaches to a fixed point when $n \to \infty$, it is an exact accumulation point. Meanwhile, when n is finite, we can get an asymptotically approximate result only.

The typical scanning pattern of $\langle L\rangle_i$ with respect to K at $\beta = 0.38$ in the strong coupling region is presented in Fig. 2. All $\langle L\rangle_i$ (i=1-5) decrease monotonically to $\langle L\rangle_i = 0$ at $K = 0$. The nonzero

intersection points of $\langle L\rangle_0$ and $\langle L\rangle_i$ exist in the lower orders ($i \leq 3$) only, and $\langle L\rangle_0$ does not intersect $\langle L\rangle_4$ except at $K = 0$. Thus, to the 4-th order expansion, the parameter can only take the value $K = 0$ which corresponds to $\langle L\rangle = 0$. When β is increased to a certain value, the nonzero intersection point of $\langle L\rangle_0$ and $\langle L\rangle_4$ appears, but there is no intersection point of $\langle L\rangle_0$ and $\langle L\rangle_5$ except at $K = 0$. When β is increased further, $\langle L\rangle_0$ begins to intersect $\langle L\rangle_5$ at $\beta_c = 0.39$, which is the phase transition point from $\langle L\rangle = 0$ to $\langle L\rangle = 0$ in the 5-th order approximation. This result greatly improves the previous result $\beta_c = 0.33$ [2].

Equations (10) and (11) can also be used to determine parameters in the weak coupling region. Analogue to the other models, scanning curves of different orders gather in a small region (as an approximate "accumulation point"). $\overline{K}_n$ locates in this interval, and the expansion shows a good convergent behaviors. Therefore, by using Eqs. (10) and (11) and changing β, we finally get the $\langle L\rangle - \beta$ relation. This is presented in Fig. 3, where the calculated results of the 4-th and 5-th orders, the mean field result and the MC simulations [7] are all plotted for comparison. In the weak coupling region, the result of the 4-th order is slightly better than that of the 5-th order in comparison with MC results. This is in accordance with the observation of [6] that in the weak coupling region, the expansion of the even order shows a better convergent behavior. However, in the intermediate coupling region, the higher order the expansion goes, the better the result that can be obtained. This is the best result ever calculated by the analytical methods in the intermediate region.

4. DISCUSSION

In this paper, the applicability of the variational cumulant expansion method in the intermediate coupling region has been emphasized. The necessity of the higher order calculations is explained by the consideration of the correlation length. Although at the phase transition point and in its vicinity, the expansion to a very high order is required in principle, practically, when the calculation goes to a certain order, the phase transition point can be determined accurately in a very narrow interval.

There are other possibilities to determine parameters. They may depend on models and physical quantities concerned, at least, in the finite order approximation. Therefore, further discussion of models is worthwhile as is calculation of more physical quantities.

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