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Calculation of Vacuum State for 2+1-Dimensional $SU(3)$ Lattice Gauge Theory

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Using the method of an eigenvalue equation truncated according to the continuum limit, we calculate the vacuum state of 2+1-dimensional $SU(3)$ lattice gauge theory. The results are consistent with those computed through strong-coupling series in the strong coupling region, and display satisfactory scaling behavior in the weak coupling region.

Key words: 2+1-dimensions, lattice gauge theory, vacuum state of gauge field, method of eigenvalue.

1. INTRODUCTION

Non-abelian gauge theory possesses a nontrivial vacuum state, which may lead to confinement. In order to study the low energy physics of hadron, it is necessary to investigate the structure of the vacuum state. Lattice gauge theory (LGT) provides a framework to study non-perturbative aspects of gauge fields from the very fundamental principles. Recently, we developed a method for studying LGT, which is based on Greensite's truncated eigenvalue equation method[2], but used a different way to truncate the equation, that is truncating an eigenvalue equation according to the continuum limit [1]. In this paper, we use this method to calculate the long wavelength vacuum wave function for 2+1 dimensional $SU(3)$ LGT. To verify its effectiveness, we also compute the same quantities using the traditional strong coupling perturbation expansion.

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2. THE METHOD TO TRUNCATE AN EIGENVALUE EQUATION ACCORDING TO THE CONTINUUM LIMIT

It is generally accepted that the vacuum state of gauge field is highly disordered when the scale is larger than the confinement scale. Hence, the long wavelength vacuum state can be approximated by

$$\psi(A) = \exp\left[-\mu \int d^4x \mathrm{tr}(F_{ij})^2\right] \tag{2.1}$$

where A is the gauge potential in the continuous space and time, and F_{ij} the field strength.

On the lattice, the corresponding vacuum wave function is approximately represented by [2]

$$\psi(U) = \exp\left[\mu \sum_p \mathrm{tr}(U_p + U_p^\dagger)\right] \tag{2.2}$$

The results obtained by using Monte Carlo simulations to study the vacuum state of LGT are consistent with Eq. (2.2).

For any calculations on lattice, the lattice spacing a should be taken as going to zero to get its continuum limit. In the continuum limit,
$\psi(U)$ in (2.2) can be expanded as

$$\psi_0(U) \simeq \exp\left[-\mu_0 \int d^2x \mathrm{tr}F^2(x) - \mu_2 \int d^2x \mathrm{tr}(D_i F(x))^2 + \text{high order terms}\right] \tag{2.3}$$

In this paper, we calculate μ_0 and μ_2 to study the long wavelength behavior.
2+1 dimensional LGT is superrenormalizable, the scaling behavior is

$$\mu_0 \cdot e^2 \to \text{const} \tag{2.4a}$$

$$\mu_2 \cdot e^6 \to \text{const} \tag{2.4b}$$

where e is the invariant charge which is related to the dimensionless coupling constant g by $g^2 = e^2 a$.

In the method of eigenvalue equation, the vacuum state is supposed to be

$$|\psi_0\rangle = e^{R(U)}|0\rangle \tag{2.5}$$

where $R(U)$ consists of various Wilson loops, and state $|0>$ is defined as

$$E_l^a|0\rangle = 0 \tag{2.6}$$

Kogut-Susskind Hamiltonian is

$$H = \frac{g^2}{2a}\left[\sum_l E_l^2 - \frac{2}{g^4}\sum_p'(\mathrm{tr}u_p^\dagger + \mathrm{tr}u_p)\right] \tag{2.7}$$

The eigenvalue equation $H|\psi_0> = \varepsilon_0|\psi_0>$ leads to

$$\sum_l ([E_l^a,[E_l^a,R]] + [E_l^a,R][E_l^a,R]) - \frac{2}{g^4}\sum_p (\mathrm{tr}u_p + \mathrm{tr}u_p^\dagger) = \frac{2a}{g^2}\varepsilon_0 \cdots \tag{2.8}$$

Defining the order of a graph composed of Wilson loops to be the number of the plaquettes contained by the graph, R can be expanded in order of graphs.

$$R = R_1 + R_2 + R_3 + \cdots \tag{2.9}$$

The vacuum should possesses the same symmetry as that of H, so we take R_1 as

$R_1 = C_0$ () $\equiv C_0 \Sigma (\mathrm{tr}\, u_p + \mathrm{tr}\, u_p^-)$ (2.10)

Generally

$$[E_l^a, [E_l^a, R_n]] \in R_n + \text{low order terms} \tag{2.11a}$$

$$[E_l^a, R_n][E_l^a, R_{n'}] \in R_{n+n'} + \text{low order terms} \tag{2.11b}$$

So, $[E_l^a, R_n][E_l^a, R_{n'}]$ may produce higher order terms. In fact, the high order terms we choose are just produced by this way. The coefficients of each term in R can be fixed by solving Eq. (2.8). Obviously, it is impossible to compute Eq. (2.8) to any high order. We have to truncate it. One of the method is to truncate it according to the continuum limit [1]. Let us briefly review the method, now.

For simplicity, suppose $R = \mu \cdot$  and $\mu a^2 = \mu_p$, then $R = (\mu_p/a^2)$  . In the continuum limit. e^R should be of the form of Eq. (2.3). Therefore graph

$$\mu_p \xrightarrow{a \to 0} \text{const} \tag{2.12}$$

It is easy to prove

$$\begin{aligned} [E,R][E,R] &= \mu^2 \left[E, \square\right]\left[E, \square\right] \xrightarrow{a \to 0} \frac{\mu_p^2}{a^4} \cdot \frac{\text{const}}{a^2} \int \mathrm{d}^2x \cdot a^6 \mathrm{tr}(D_i F)^2 + \cdots \\ &= \mu_p^2 \cdot \text{const} \int \mathrm{d}^2x \mathrm{tr}(D_i F)^2 + O(a^2) \end{aligned} \tag{2.13}$$

Hence, if all terms produced by $\left[E, \square\right]\left[E, \square\right]$ are preserved in the calculation, $\left[E, \square\right] \cdot \left[E, \square\right]$ possesses definite scaling behavior. On the contrary, if some terms are discarded [5], then

$$\begin{aligned} &\frac{\mu_p^2}{a^4}\left[E, \square\right]\left[E, \square\right] \xrightarrow{a \to 0} \frac{\mu_p^2}{a^4} \frac{\text{const}_1}{a^2} \int \mathrm{d}_x^2 \cdot a^4 \mathrm{tr} F^2 \\ &\quad + \frac{\mu_p^2}{a^4} \cdot \frac{\text{const}_2}{a^2} \cdot \int \mathrm{d}^2x \cdot a^6 \mathrm{tr}(D_i F)^2 + \cdots \end{aligned} \tag{2.14}$$

The right side of Eq. (2.14) is not finite. In Ref. [1], it was proved that for general R_i, we have

$$[E, R_i][E, R_k] \xrightarrow{a \to 0} a^4 \cdot \int \mathrm{d}^2x \mathrm{tr}(D_i F)^2 + \cdots \tag{2.15}$$

where R_j, R_k are two arbitrary graphs. So, if we preserve all terms produced by $[E,R_n][E,R_{n'}]$, each term on the left side of Eq. (2.8) possesses adaptative scaling behavior, hence also ε_0.

To conclude, if Eq. (2.8) is truncated according to the continuum limit, and R is expanded to Mth order, the truncated eigenvalue equation is

$$\sum_{n=1}^{M}\sum_{l,a}[E_l^a,[E_l^a,R_n]]+\sum_{n+n'\leqslant M}\sum_{l,a}[E_l^a,R_n][E_l^a,R_{n'}] - \frac{2}{g^4}(\mathrm{tr}u_p+\mathrm{tr}u_p^\dagger)=\mathrm{const} \tag{2.16}$$

In this paper, we only do the calculation for $M = 2$ and 3.

3. UNIMODULAR CONDITION FOR $SU(3)$ LGT

The high order graphs produced by $[E,R_n][E,R_{n'}]$ are not entirely independent each other, because there is a unimodular condition for special unitary group elements. It is necessary to replace those non-independent graphs with a group of independent graphs when solving Eq. (2.16).

The unimodular condition for $SU(3)$ group is

$$U_{ii}U_{kl}U_{mn}\varepsilon_{jln}=\varepsilon_{ikm} \tag{3.1}$$

where U is any element in the group. Multiplying Eq. (3.1) by $U_{pi}^\dagger$ and summing it over i, we have

$$U_{kl}U_{mn}\varepsilon_{pln}=U_{pi}^\dagger\varepsilon_{ikm} \tag{3.2}$$

Multiplying Eq. (3.2) by ε_{pqr}, summing the product over p, we obtain

$$\begin{aligned}U_{il}U_{ki}=U_{ii}U_{kl}-\mathrm{tr}U^\dagger(\delta_{ii}\delta_{lk}-\delta_{ki}\delta_{il})-U_{il}^\dagger\delta_{ki}\\+U_{ii}^\dagger\delta_{kl}-U_{ki}^\dagger\delta_{il}+U_{kl}^\dagger\delta_{ii}\end{aligned} \tag{3.3}$$

Here we have used $\varepsilon_{jln}\varepsilon_{ikp}=\delta_{ji}\delta_{lk}\delta_{np}-\delta_{ji}\delta_{lp}\delta_{nk}+\delta_{jk}\delta_{lp}\delta_{ni}-\delta_{jk}\delta_{li}\delta_{np}+\delta_{jp}\delta_{li}\delta_{nk}-\delta_{jp}\delta_{lk}\delta_{ni}$. In the above reasoning: every step is reversible, so Eq. (3.3) is equivalent to (3.1). Eq. (3.1) may be changed into a more convenient form. Multiplying by $V_{lk}\cdot V_{ji}$, we get

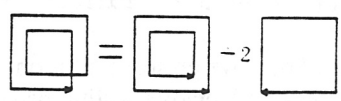
$$\begin{aligned}\mathrm{tr}UVUV'=\mathrm{tr}UV\mathrm{tr}UV'-\mathrm{tr}U^\dagger(\mathrm{tr}V'\mathrm{tr}V-\mathrm{tr}VV')-\mathrm{tr}U^\dagger V'V\\-\mathrm{tr}U^\dagger VV'+\mathrm{tr}V\mathrm{tr}U^\dagger V'+\mathrm{tr}V'\mathrm{tr}U^\dagger V\end{aligned} \tag{3.4}$$

where U, V, and V' are any elements in the group. Because the gauge freedom is not physical freedom, in the LGT Hamiltonian formalism, the plaquette U which is invariant under a gauge transformation is a fundamental field quantity. Hence, U, V, and V' in Eq. (3.4) can be understood as any plaquette or product of plaquettes. Some special case of Eq. (3.4) are given in the following.

1) Let $V = V' = I$ in Eq. (3.4), then

$$\mathrm{tr}U^2=(\mathrm{tr}U)^2-2\mathrm{tr}U^\dagger \tag{3.5}$$

For example

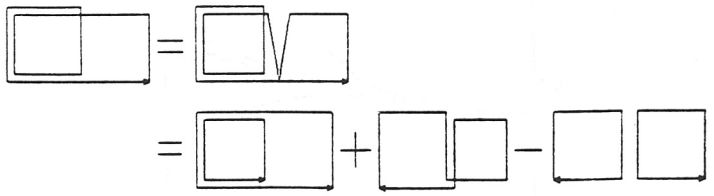


(3.5a)

2) Suppose $V' = I$, then Eq. (3.4) leads to

$$\mathrm{tr}U^2V = \mathrm{tr}UV\,\mathrm{tr}U + \mathrm{tr}U^\dagger\mathrm{tr}V - \mathrm{tr}U^\dagger\mathrm{tr}V \tag{3.6}$$

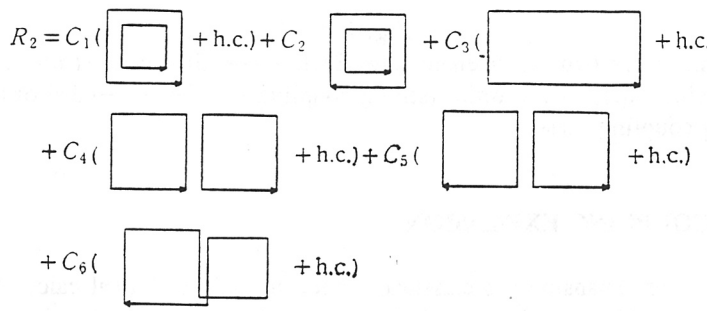
For example



(3.6a)

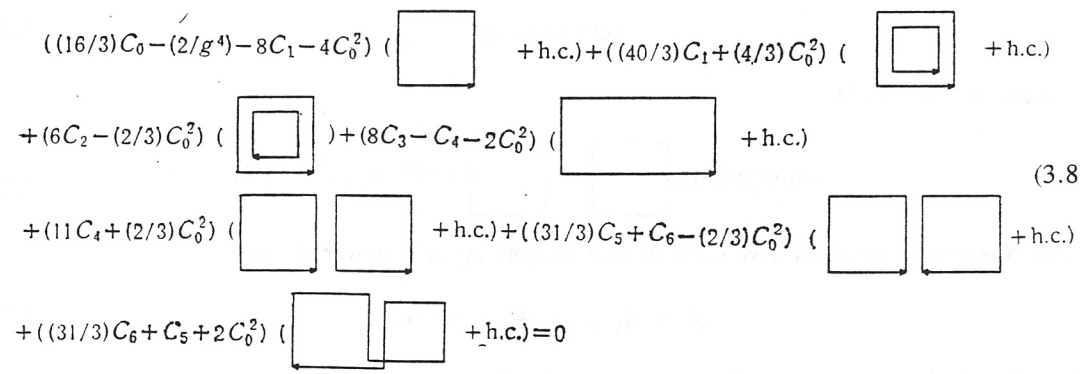
Eqs.(3.5) and (3.6) are all possible relations among the graphs up to the order $M = 3$.

When $M = 2$, the term $[E,R_1][E,R_1]$ in Eq. (2.16) produces seven new graphs. Using Eq. (3.5a), one of them is replaced by others, we have



(3.7)

where h.c. represents the graph obtained from a graph under C transformation $(U \xrightarrow{C} U^\dagger)$. Eq. (2.16) leads to



(3.8)

Every graph in (3.8) is independent from the others. So, its coefficient must be zero. From this, we determine C_0, C_1, C_2, C_3, C_4, C_5, and C_6 and compute μ_0 and μ_2. The results are shown in Fig.1.

At 3rd order, there are 36 independent graphs totally. It is too many to list them. The results are also plotted in Fig.1.

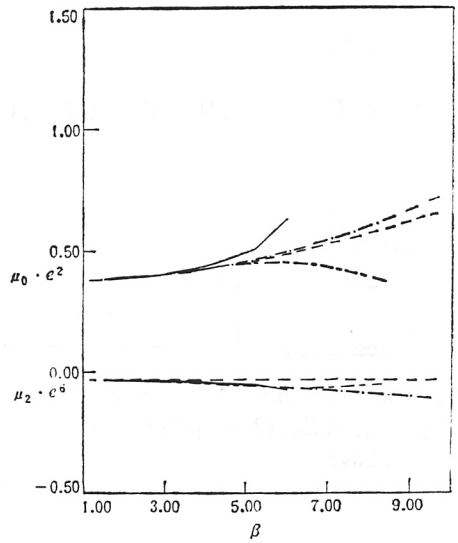


Fig. 1

—— 2nd order (truncated eigenvalue Eq.); ——— 3rd order (truncated eigenvalue Eq.); --- 2rd order (strong coupling series); —·—· 3rd order (strong coupling series).

4. THE STRONG COUPLING EXPANSION

The strong coupling expansion is a classical method in LGT analytical calculation. Any other analytical calculation should be consistent with its results in the strong coupling region. Its defaults is that the calculation ceases to be effective in the crossover and weak region. In this section, we calculate the quantities in Sec.3 using the strong-coupling expansion.

According to Ref. [2], the vacuum state is written

$$|\Omega\rangle = e^{A}|0\rangle \tag{4.1}$$

Eigenvalue equation is

$$A = \mathrm{Inv}[(2/g^4)\ (\square + \square) - [E_l^a, A_l]\ [E_l^a, A_l]] \tag{4.2}$$

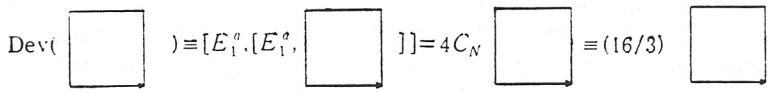
Here. Summation symbols have been omitted. Expanding A in power of $1/g^4$

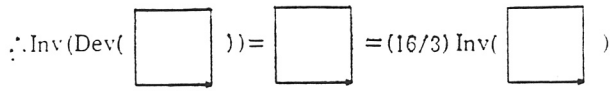
$$A = A_1 + A_2 + A_3 + \cdots \tag{4.3}$$

From Eq. (4.2), we obtain the 1st power approximation

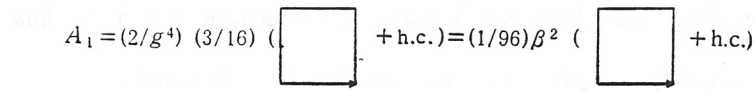
$$A_1 = \mathrm{Inv}\left(\frac{2}{g^4}\left(\square + \text{h.c.}\right)\right)$$

where  + h.c. = $\mathrm{tr}U_p + \mathrm{tr}U_p^+$, so





This leads to



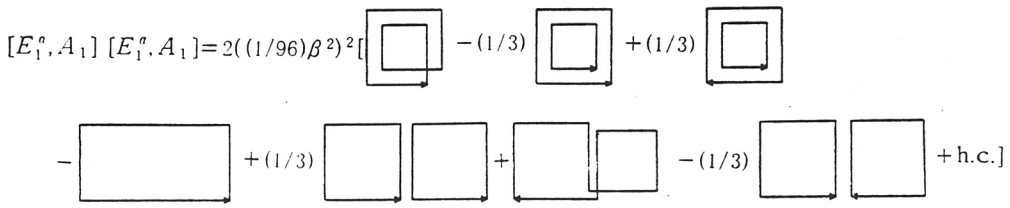
(4.4)

with $\beta = (2N/g^2) = 6/g^2$.

The second term A_2 is

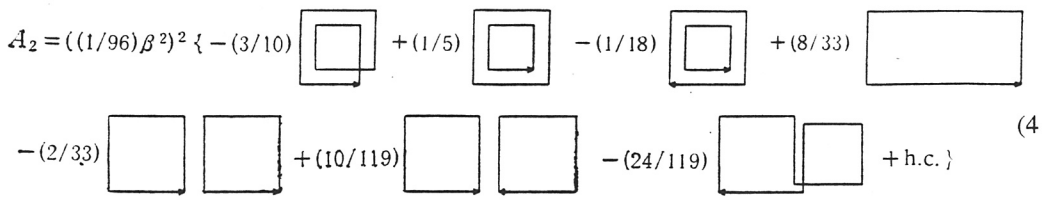
$$A_2 = -\mathrm{Inv}[[E_l^a, A_1][E_l^a, A_1]] \tag{4.5}$$

whereas



(4.6)

Similar with the calculation in the 1st order case, we have



(4.7)

As for A_3, the calculation is much complicated. We don't exhibit the detail here. The total number of graphs up to the 3rd order is 36. Results for μ_0 and μ_2 are plotted in Fig.1.

5. CONCLUSIONS AND DISCUSSIONS

In Fig. 1, we observe that, in strong coupling region $1/g^2 < 0.8$, the results from the truncated eigenvalue equation and the strong coupling series are agreement with each other. However, from crossover region to weak coupling region, the former shows better scaling behavior than the later. Therefore, the method of truncating an eigenvalue equation according to the continuum limit is an effective analytical one.

In the strong coupling expansion, the coefficient of every graph is determined directly by the expanding itself, while truncating an eigenvalue equation, the coefficients of independent graphs are determined by algebraic equations. This is the main difference between these two methods.

Although the results from truncated eigenvalue equations at 3rd order are the best in Fig.1, they exhibit the tendency to descend slightly in weak coupling region. This suggest that higher order calculation be needed, i.e. more complicated vacuum state should be considered, to obtain better scaling behavior.

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