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Finite-Temperature Thermodynamic Potential of Two-loop QED and the Overlapping Divergences in Arbitrary Gauge

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Dimensional regularization at finite temperature is applied to calculate accurately the thermodynamic potential of two-loop QED and the overlapping divergences in arbitrary gauge. The calculation indicates that the overlapping divergences cancel each other and the result is gauge-independent.

Key words: two-loop thermodynamic potential, overlapping divergence, gauge-independent.

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1. INTRODUCTION

Quantum field theory at finite temperature is the theoretical basis for describing the early universe and hot nuclear matter [1,2]. We know that one can learn many fundamental properties of a hot system from its thermodynamic potential. It is therefore of both theoretical and practical importance to study how to accurately calculate the two-loop thermodynamic potential in QED. However, there are still many problems in field theories at finite temperature compared to the zero-temperature case, for instance, overlapping divergence, infrared divergence, gauge dependencies, and so on [3-7]. There has been wide interest in this problem.

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For one-loop Feynman diagrams one can easily separate the divergent part from the finite-temperature part. Since the divergent part is independent of temperature one can directly use the methods in zero-temperature field theory to do the renormalization [5]. However, in higher-order integrals, the overlapping divergence of zero-temperature parts and finite-temperature parts may appear in addition to the zero-temperature divergent part. Besides, the infrared divergences may appear due to Bose distribution factors [3,4].

It is well known that dimensional regularization has been a powerful tool for dealing with the divergences and accurate evaluation of high-order Feynman diagrams in zero-temperature field theories. However, it has hardly been applied to finite-temperature field theories [4]. In Ref. [4], by making use of this method we calculated the temperature-dependent ultraviolet and infrared divergences from three-loop vacuum graphs in massless QED at the Feynman gauge. We found that the divergences canceled one another. We intend to calculate accurately the thermodynamic potential of two-loop QED with a fermion mass in arbitrary gauges and analyze their gauge dependence and how the divergences are canceled.

2. TWO-LOOP THERMODYNAMIC POTENTIAL AND THE OVERLAPPING DIVERGENCES

As we know, renormalization in a 1-loop QED at finite temperature is similar to that at zero temperature [5]. Two-loop thermodynamic potential in QED corresponds to the vacuum graph in Fig. 1(a); the relevant counterterms are depicted in Figs. 1(b) and (c). The symmetry factors for Figs. 1(a), (b) and (c) are 1/2, 1, and 1/2, respectively. And

$$-\!\!\times\!\!- = -\frac{ie^2}{8\pi^2\varepsilon}(4m-\not{p}), \qquad \sim\!\!\times\!\!\sim = -\frac{ie^2}{6\pi^2\varepsilon}(k_\mu k_\nu - g_{\mu\nu}k^2)$$

Using the usual Feynman rules and working in $D = 4 - \varepsilon$, one obtains for Fig. 1(a) in arbitrary covariant gauge:

$$I_a = -\frac{e^2}{2}\iint\frac{d^Dpd^Dk}{(2\pi)^{2D}}\frac{\mathrm{Tr}[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}+\not{k}+m)]}{(p^2-m^2)k^2[(p+k)^2-m^2]}\left(g_{\mu\nu}+\xi\frac{k_\mu k_\nu}{k^2}\right)$$

$$= -\frac{e^2}{2}\times \underline{4}\times(2-D)\iint\frac{d^Dpd^Dk}{(2\pi)^{2D}}\frac{p^2+p\cdot k+Cm^2}{(p^2-m^2)k^2[(p+k)^2-m^2]}$$

$$-\frac{e^2}{2}f(D)\xi\iint\frac{d^Dpd^Dk}{(2\pi)^{2D}}\frac{2(p\cdot k)^2+(p\cdot k)k^2-k^2p^2+m^2k^2}{(p^2-m^2)k^4[(p+k)^2-m^2]}.$$

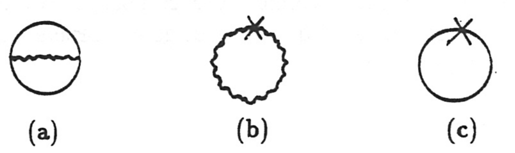

Fig. 1
The two-loop vacuum graphs and their counterterms in QED. The wavy line represents the photon, and the solid line represents the fermion.

After simple combination and simplification, one obtains

$$
\begin{aligned}
I_a = & A \iint \frac{d^D p d^D k}{(2\pi)^{2D}} \frac{p^2 + p \cdot k + Cm^2}{(p^2 - m^2)k^2[(p+k)^2 - m^2]} \\
& + B\xi \iint \frac{d^D p d^D k}{(2\pi)^{2D}} \left[\frac{p \cdot k}{k^4 (p^2 - m^2)} - \frac{p \cdot k}{k^4 [(p+k)^2 - m^2]} \right. \\
& \left. - \frac{1}{k^2 [(p+k)^2 - m^2]} \right],
\end{aligned} \tag{1}
$$

where ξ is the gauge parameter, and

$$
\begin{aligned}
& A = -\frac{e^2}{2} \times \underline{4} \times (2-D), \quad B = -\frac{e^2}{2} \times f(D), \quad C = \frac{f(D)}{2-D}, \\
& \underline{4} = \mathrm{Tr} I, \quad f(4) = 4.
\end{aligned} \tag{2}
$$

It is easy to prove that the ξ dependent part is canceled, by making use of variable replacement $p + k \to p'$ for Eq. (1), thus

$$
I_a = A \iint \frac{d^D p d^D k}{(2\pi)^{2D}} \frac{p^2 + p \cdot k + Cm^2}{(p^2 - m^2)k^2[(p+k)^2 - m^2]} . \tag{3}
$$

Applying the real-time finite-temperature field theory formalism, the thermal propagators of gauge fields and fermion fields are [2,6]

$$
\begin{aligned}
& i\Delta^{\mu\nu}(k) = \left(-g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right) \left[\frac{i}{k^2 + i\varepsilon} + 2\pi n_B(k) \delta(k^2) \right], \\
& i\Delta(p) = (\not{p} + m) \left[\frac{i}{p^2 - m^2 + i\varepsilon} - 2\pi n_f(p) \delta(p^2 - m^2) \right].
\end{aligned} \tag{4}
$$

where n_B, n_f are the distribution functions of boson and fermion fields, respectively:

$$
n_B(k) = \frac{1}{e^{\beta |k_0|} - 1}, \quad n_f(p) = \frac{1}{e^{\beta |p_0|} + 1} . \tag{5}
$$

Since the temperature-independent part was clearly discussed in zero-temperature field theory, we are only interested in the temperature-dependent parts. We can make use of the following substitution in enumerators of Eq. (3):

$$
\begin{aligned}
& \frac{1}{k^2} \to -2\pi i \delta(k^2) n_B(k), \\
& \frac{1}{p^2 - m^2} \to 2\pi i \delta(p^2 - m^2) n_f(p).
\end{aligned} \tag{6}
$$

The result includes the terms with one, two, or three δ functions. The term with three δ functions vanishes because of

$$\delta(k^2)\delta(p^2-m^2)\delta[(p+k)^2-m^2]=0. \tag{7}$$

The contribution from terms with one δ function includes the following three terms:

$$I_{a_1}^{(\mathrm{I})}=-A\iint\frac{\mathrm{d}^Dp\mathrm{d}^Dk}{(2\pi)^{2D}}\frac{p^2+p\cdot k+Cm^2}{(p^2-m^2)[(p+k)^2-m^2]}2\pi i\delta(k^2)n_{\mathrm{B}}(k)$$

$$=-A\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{(p-kx)^2+(p-kx)\cdot k+Cm^2}{(p^2-R_1^2)^2}$$

$$\cdot\ 2\pi i\delta(k^2)n_{\mathrm{B}}(k).$$

where $R_1^2=m^2-k^2x(1-x)$ and after integrating over p and x, one obtains

$$I_{a_1}^{(\mathrm{I})}=-Aim^{2-\varepsilon}(4\pi)^{-\frac{D}{2}}\left[-\frac{i\Gamma\left(3-\frac{\varepsilon}{2}\right)}{\Gamma\left(2-\frac{\varepsilon}{2}\right)}\cdot\Gamma\left(-1+\frac{\varepsilon}{2}\right)+iC\Gamma\left(\frac{\varepsilon}{2}\right)\right]I_\beta^{\mathrm{B}}(k), \tag{8}$$

where

$$I_\beta^{\mathrm{B}}(k)=\int\frac{\mathrm{d}^Dk}{(2\pi)^D}2\pi\delta(k^2)n_{\mathrm{B}}(k)\overset{\varepsilon\to0}{=\!=}\frac{T^2}{12}.$$

$$\Gamma\left(-1+\frac{\varepsilon}{2}\right)=-\left[\frac{2}{\varepsilon}-\gamma+1+O(\varepsilon)\right], \tag{9}$$

$$\Gamma\left(\frac{\varepsilon}{2}\right)=\frac{2}{\varepsilon}-\gamma+O(\varepsilon)\quad(\gamma\text{ is the Euler constant})$$

In the limit $\varepsilon\to0$, $C\to-2$, $A\to4e^2$

$$I_{a_1}^{(\mathrm{I})}=-i4e^2m^2(4\pi)^{-2}\left[2i\left(\frac{2}{\varepsilon}-\gamma+1\right)-2i\left(\frac{2}{\varepsilon}-\gamma\right)\right]I_\beta^{\mathrm{B}}=\frac{e^2m^2}{24\pi^2}T^2. \tag{10}$$

Similarly one can write

$$I_{a_2}^{(\mathrm{I})}=A\iint\frac{\mathrm{d}^Dp\mathrm{d}^Dk}{(2\pi)^{2D}}\frac{p^2+p\cdot k+Cm^2}{k^2[(p+k)^2-m^2]}\cdot2\pi i\delta(p^2-m^2)n_{\mathrm{f}}(p)$$

$$=A\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{p^2+p\cdot k-p^2x+Cm^2}{(k^2-R_2^2)^2}\cdot2\pi i\delta(p^2-m^2)n_{\mathrm{f}}(p), \tag{11}$$

where $R_2^2 = m^2x - p^2x(1-x)$ and after integrating over k and x, one obtains

$$I_{a_2}^{(\mathrm{I})} = Aim^{2-\varepsilon}(4\pi)^{-\frac{D}{2}} i\Gamma\left(\frac{\varepsilon}{2}\right)\left(\frac{1}{2}+C\right)I_\beta^{\mathrm{f}}$$
$$\xlongequal{\varepsilon\to 0} 6e^2m^2(4\pi)^{-2}\left(\frac{2}{\varepsilon}-\gamma\right)I_\beta^{\mathrm{f}}, \tag{12}$$

where

$$I_\beta^{\mathrm{f}} = \int \frac{\mathrm{d}^Dp}{(2\pi)^D} \cdot 2\pi\delta(p^2-m^2)n_{\mathrm{f}}(p). \tag{13}$$

$I_{a2}^{(I)}$ includes a finite part and a divergent part

$$I_{a_2}^{(\mathrm{I})\mathrm{f}} = -6e^2m^2(4\pi)^2\gamma I_\beta^{\mathrm{f}} \quad (\gamma \text{ is the Euler constant})$$
$$\mathrm{div}\, I_{a_2}^{(\mathrm{I})} = 6e^2m^2(4\pi)^{-2}\left(\frac{2}{\varepsilon}\right)I_\beta^{\mathrm{f}}. \tag{14}$$

$$I_{a_3}^{(\mathrm{I})} = A\iint \frac{\mathrm{d}^Dp\mathrm{d}^Dk}{(2\pi)^{2D}} \frac{p^2+p\cdot k+Cm^2}{k^2(p^2-m^2)} \cdot 2\pi i n_{\mathrm{f}}(p+k)\delta[(p+k)^2-m^2]$$
$$= I_{a_2}^{(\mathrm{I})}. \tag{15}$$

The contribution from terms with two δ functions includes

$$I_{a_1}^{(\mathrm{II})} = A\iint \frac{\mathrm{d}^Dp\mathrm{d}^Dk}{(2\pi)^{2D}} \frac{p^2+p\cdot k+Cm^2}{[(p+k)^2-m^2]} \cdot 2\pi\delta(p^2-m^2)n_{\mathrm{f}}(p) \cdot 2\pi\delta(k^2)n_{\mathrm{B}}(k)$$
$$\xlongequal{\varepsilon\to 0} -\frac{e^2}{6}T^2\int \frac{\mathrm{d}^4p}{(2\pi)^4} \cdot 2\pi n_{\mathrm{f}}(p)\delta(p^2-m^2)$$
$$= -\frac{e^2}{6}T^2 I_\beta^{\mathrm{f}}. \tag{16}$$

After some variable replacements, one obtains

$$I_{a_2}^{(\mathrm{II})} = A\iint \frac{\mathrm{d}^Dp\mathrm{d}^Dk}{(2\pi)^{2D}} \frac{p^2+p\cdot k+Cm^2}{(p^2-m^2)} 2\pi\delta[(p+k)^2-m^2]n_{\mathrm{f}}(p+k)2\pi\delta(k^2)n_{\mathrm{B}}(k)$$
$$= I_{a_1}^{(\mathrm{II})} = -\frac{e^2}{6}T^2 I_\beta^{\mathrm{f}}. \tag{17}$$

$$
\begin{aligned}
I_{a_3}^{(\mathrm{II})} &= -A\iint \frac{\mathrm{d}^D p \mathrm{d}^D k}{(2\pi)^{2D}} \frac{p^2 + p\cdot k + Cm^2}{k^2} 2\pi\delta(p^2 - m^2) n_{\mathrm{f}}(p) \\
&\quad \cdot 2\pi\delta[(p+k)^2 - m^2] n_{\mathrm{f}}(p+k) \\
&= -2e^2 \int \frac{\mathrm{d}^4 p \mathrm{d}^4 k}{(2\pi)^{2\times 4}} \left(1 + \frac{2m^2}{(p-k)^2}\right) \\
&\quad \cdot 2\pi\delta(p^2 - m^2) n_{\mathrm{f}}(p) 2\pi\delta(k^2 - m^2) n_{\mathrm{f}}(k).
\end{aligned} \tag{18}
$$

It is easy to see that the results of Eqs. (16), (17), and (18) are finite.

For Fig. 1(b) one obtains

$$
\begin{aligned}
I_b &= \frac{ie^2}{8\pi^2\varepsilon} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{Tr}[(4m - \not{p})(\not{p} + m)]}{p^2 - m^2} \\
&= \frac{ie^2}{8\pi^2\varepsilon} \times 4 \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{4m^2 - p^2}{p^2 - m^2} .
\end{aligned} \tag{19}
$$

The temperature-dependent part reads

$$
\begin{aligned}
I_b^{(\mathrm{I})} &= -\frac{e^2}{8\pi^2\varepsilon} \times 4 \int \frac{\mathrm{d}^D p}{(2\pi)^D} (4m^2 - p^2) 2\pi n_{\mathrm{f}}(p) \delta(p^2 - m^2) \\
&= -12 m^2 e^2 \frac{\frac{2}{\varepsilon}}{(4\pi)^2} I_\beta^{\mathrm{f}} .
\end{aligned} \tag{20}
$$

For Fig. 1(c) one can show that the temperature-dependent part turns out to be zero, because $k^2\delta(k^2) = 0$, namely,

$$
\begin{aligned}
I_c^{(\mathrm{I})} &= \frac{e^2}{6\pi^2\varepsilon} \int \frac{\mathrm{d}^D k}{(2\pi)^D} (k_\mu k_\nu - g_{\mu\nu} k^2) \left(g_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2} \right) 2\pi\delta(k^2) n_{\mathrm{B}}(k) \\
&= 0.
\end{aligned} \tag{21}
$$

Now we can give the temperature-dependent contributions of two-loop vacuum graphs in QED. The divergent part is

$$
\begin{aligned}
\mathrm{div} I &= \mathrm{div} I_a + \mathrm{div} I_b + \mathrm{div} I_c \\
&= 2 \times \mathrm{div} I_{a_2}^{(\mathrm{I})} + I_b^{(\mathrm{I})} \\
&= 12 m^2 e^2 \frac{\frac{2}{\varepsilon}}{(4\pi)^2} I_\beta^{\mathrm{f}} - 12 m^2 e^2 \frac{\frac{2}{\varepsilon}}{(4\pi)^2} I_\beta^{\mathrm{f}} = 0.
\end{aligned} \tag{22}
$$

and the finite part is

$$
\begin{aligned}
I^{\mathrm{f}} &= I_{a_1}^{(\mathrm{I})} + 2\times I_{a_2}^{(\mathrm{I})\mathrm{f}} + 2\times I_{a_1}^{(\mathrm{II})} + I_{a_3}^{(\mathrm{II})} \\
&= \frac{e^2m^2T^2}{24\pi^2} - \frac{3m^2e^2}{4\pi^2}\gamma I_\beta^{\mathrm{f}}(p) - \frac{e^2}{3}T^2 I_\beta^{\mathrm{f}} \\
&\quad -2e^2\iint \frac{\mathrm{d}^4p\mathrm{d}^4k}{(2\pi)^{2\times4}}\left(1+\frac{2m^2}{(p-k)^2}\right) \\
&\quad \cdot 2\pi\delta(p^2-m^2)n_{\mathrm{f}}(p)\times 2\pi\delta(k^2-m^2)n_{\mathrm{f}}(k).
\end{aligned}
\tag{23}
$$

One thing we should point out is that in Ref. [1] the two-loop thermodynamic potential was discussed by the imaginary time finite-temperature field theory in Feynman gauge, but the divergent part was not separated explicitly and their cancellations were not proved. In addition, the finite part from the terms with one δ function was not included, for instance, the first two terms in Eq. (23).

For the massless QED, $m = 0$, one obtains from Eqs. (14) and (20)

$$\mathrm{div}I_a=0,\ \ \mathrm{div}I_b=0,\ \ \mathrm{div}I_c=0. \tag{24}$$

$$I^{\mathrm{f}} = -\frac{5e^2}{288}T^4. \tag{25}$$

This indicates that the temperature-dependent divergences vanish in the effective potential of two-loop massless QED.

On the other hand, although there appears $n_{\mathrm{B}}(k)$, the preceding result is still convergent at $k \to 0$. This is because in the two-loop case, there is no such type of integration as $\int \mathrm{d}k\frac{1}{k}n_{\mathrm{B}}(k)$; therefore, the infrared divergence does not appear. However, for higher-order diagrams the infrared divergence may appear [4].

3. CONCLUSION

We employ the dimensional regularization at finite temperature to caloulate accurately the thermodynamic potential of two-loop massive QED and the overlapping divergences in arbitrary gauge in the $D = 4 - \varepsilon$ dimension. We compared our results with previous ones and pointed out the missing part in previous results. Our results indicate explicitly that the overlapping divergences cancel each other. In addition, our analysis shows that the inferred divergence does not appear at two loops but can only appear in diagrams with more loops.

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