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Scheme-scale ambiguity in analysis of QCD observable

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Abstract The scheme-scale ambiguity that has plagued perturbative analysis in QCD remains on obstacle to making precise tests of the theory. Many attempts have been done to resolve the scale ambiguity. In this regard the BLM, EC, PMS and CORGI approaches are more distinct. We try to employ these methods to fix the scale ambiguity at NLO, NNLO and even in more higher order approximations. By optimizing the renormalization scale, there will be a possibility to predicate higher order terms. We present general results for predicted terms at any order, using different optimization methods. Some observable as specific examples will be used to indicate the validity of scale fixing to predicate the higher order terms.

Key words perturbative QCD, scheme and scale dependence, optimization approach, predicted terms

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1 Introduction

The problem of the renormalization scale (scheme) dependence of fixed order perturbative QCD (PQCD) predictions is always frustrating attempts to obtain reliable results. There are a number of proposals for controlling or avoiding this difficulty. One of them is the principle of minimum sensitivity (PMS) [1]. The PMS approach attempts to resolve the renormalization scheme dependence problem by exploiting the fundamental notion of renormalization group invariance of physical quantities. An other alternative formalism, is called Complete Renormalization Group Improvement (CORGI) [2]. This formalism points out that the renormalization scale dependence of a dimensionless physical QCD observable, depending on a single energy scale Q, can be avoided provided that all ultraviolet logarithms which build the physical energy dependence on Q are resummed. The method of BLM [3] is trying to reduce to the standard criterion that only vacuum polarization insertions contribute to the effective coupling constant. The idea of Effective charge (EC) method is to re-formulate the results of PQCD calculations as renormalization-scheme independent prediction, proposed by G. Grunberg [4].

2 The principle of minimal sensitivity

The PMS applies this reality that the physical quantities are independent of un-physical parameters of renormalization scheme (RS) and also renormalization scale. On this bases for the k-th truncated series $R^{(k)}=a^{(i)}[1+r_1a^{(i)}+\cdots+r_{i-1}(a^{(i)})^{k-1}]$ we will have

$$\frac{\partial R^{(k)}}{\partial(RS)}_{|RS=\text{Optimized }RS}=0.$$

The self consistency principle (SCP) which appears as $\dfrac{\partial R^{(k)}}{\partial(RS)}_{|RS}=O(a^{k+1})$ will help to obtain the invariant quantity of RS. The QCD β-function obeys

$$\frac{\partial a}{\partial\tau}=\hat{\beta}^{(k+1)}=-a^2(1+ca+c2a^2+\cdots+c_ka^k).$$

The SCP demands that

$$\frac{\partial R^{(k+1)}}{\partial(\tau,c_2,\cdots,c_k)}=O(a^{(k+2)}).$$

The dependence of coupling constant a to c_i as scheme parameters is govern by

$$\beta_i=\frac{\partial a}{\partial c_i}=-\hat{\beta}(a)\int_0^a\frac{x^{i+2}}{[\hat{\beta}(a)]^2}.$$

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Using SCP [1]:

$$\frac{\partial r_l}{\partial \tau}=\sum_{m=0}^{l-1}(m+1)r_m c_{l-m-1} \tag{1}$$

$$\frac{\partial r_l}{\partial c_j}=\begin{cases}\dfrac{-1}{j-1}\sum_{m=0}^{l-j} r_m W^j_{l-j-m}, & l\geqslant j\\ 0, & l<j\end{cases}, \tag{2}$$

where $c_0=r_0=W_0^j=1$ and $c_1=c$. The W_n^j are the expansion coefficients of the $\beta_i=\dfrac{\partial a}{\partial c_i}$ as:

$$\beta_i=\frac{1}{i+1}a^{i+1}(1+W_1^i a+W_1^2 a^2+\cdots). \tag{3}$$

Partial derivatives in Eqs. (1), (2) will yield the following invariant quantities

$$\rho_1 = \tau-r_1\,,\rho_2=r_2+c_2-\left(r_1+\frac{c}{2}\right)^2\cdots,$$
$$\rho_k = r_k+\frac{c_k}{k-1}-\Omega^{(k)}$$

where for instance

$$\Omega^{(2)}=(r_1+\frac{c}{2})^2\,,\ \Omega^{(3)}=r_1(c_2+3r_2-2r_1^2-\frac{c}{2}r_1)\cdots. \tag{4}$$

If we rewrite Eq. (4) as $r_k=\Omega^{(k)}-\dfrac{c_k}{k-1}-\rho_k$ and ignore from the invariants quantities ρ_k, the remaining terms can be considered as a predicted term in the required order.

3 Complete RG improvement (CORGI)

An observable $R(Q)$ in a standard approach has a perturbative expansion like:

$$R(Q)=a+r_1a^2+r_2a^3+\cdots+r_na^{n+1}+\cdots. \tag{5}$$

In the CORGI approach [2]:

$$R(Q)=a_0+X_2a_0^3+X_3a_0^4+\cdots+X_na_0^{n+1}+\cdots. \tag{6}$$

In Eq. (5) all terms depend on renormalization scale (μ), while in Eq. (6), $a_0=a_0(Q)$. $X_2,X_3,\cdots$ are constants and scheme invariants. SCP and solving simultaneously the related partial differential equations will yield

$$\begin{aligned}r_2(r_1,c_2) &= {r_1}^2+cr_1+X_2-c_2\\ r_3(r_1,c_2,c_3) &= {r_1}^3+\frac{5}{2}c{r_1}^2+(3X_2-2c_2)r_1+\\ &\quad X_3-\frac{1}{2}c_3\\ &\vdots \quad \vdots\end{aligned} \tag{7}$$

In general the structure is

$$r_n(r_1,c_2,\cdots,c_n)=\hat{r}_n(r_1,c_2,\cdots,c_{n-1})+X_n-c_n/(n-1).$$

The coupling constant a_0 represents a summation over NLO contribution of all terms in Eq. (5) which is an RS independent sum. It is defined as:

$$a_0 \equiv a+r_1a^2+(r_1^2+cr_1-c_2)a^3+\left(r_1^3+\frac{5}{2}cr_1^2-2c_2r_1-\frac{1}{2}c_3\right)a^4+\cdots. \tag{8}$$

A phenomenological application of this approach can be found in Ref. [5].

a) In the NLO approximation we have:

$$R(Q)=a_0. \tag{9}$$

Substituting Eq. (8) in Eq. (9) will give us:

$$R(Q)=a+r_1a^2+(r_1^2+cr_1-c_2)a^3+\cdots. \tag{10}$$

Therefore the predicted term is:

$$r_2(\text{pre})=r_1^2+cr_1-c_2 \text{ or } r_2(\text{pre})=r_1^2+\frac{\beta_1}{\beta_0}r_1-\frac{\beta_2}{\beta_0}.$$

b) In the NNLO approximation, $R(Q)$ has an expansion like :

$$R(Q)=a_0+X_2a_0^3\,. \tag{11}$$

Substituting Eq. (8) in Eq. (11) for a_0 and using Eq. (7) to define X_2 in terms of r_1, r_2, $\cdots$ and rearrange them in terms of a, we will obtain

$$R(Q) = a+r_1a^2+r_2a^3+\left(r1^3+\frac{5}{2}cr_1^2-2c_2r_1-\frac{1}{2}c_3+3(r_2-r_1^2-cr_1+c_2)r_1\right)a^4.$$

The predicted term is:

$$r_3(\text{pre}) = r1^3+\frac{5}{2}\frac{\beta_1}{\beta_0}r_1^2-2\frac{\beta_2}{\beta_0}r_1-\frac{1}{2}\frac{\beta_3}{\beta_0}+3(r_2-r_1^2-\frac{\beta_1}{\beta_0}r_1+\frac{\beta_2}{\beta_0})r_1.$$

4 BLM approach

The idea of BLM has been initiated from the important rule which is allocated to the running coupling constant. The coefficient functions $r_1,r_2,r_3,\cdots$ in a perturbative series of an observable depend on the number of flavor (f) which in fact is arising from fermion loops. Since we have similar fermion loops which are related to electric charge e and finally coupling constant then it will be a good idea to absorb all fermion loops into the coupling constant. This procedure make the series more convergent and consequently the renormalization scale will be fixed. Accordingly we have $R=a_2+r_1a_2^2+r_2a_2^3+\cdots=$

$a_2+(A+Bf)a_2^2+(C+Df+Ef^2)a_2^3+\cdots$ Considering the QCD-β function, we will have

$$a_2(\mu')=a(\mu)+\beta_0 la^2(\mu)+(\beta_0^2l^2+\beta_1 l)a^3(\mu)+\cdots \quad (12)$$

where $l=\ln\left(\frac{\mu'}{\mu}\right)$ and $\beta_0=\frac{11}{4}-\frac{1}{6}f$, $\beta_1=\frac{51}{8}-\frac{19}{24}f$. Here we assumed $l=c_1+(c_2+c_3f)\alpha$ [3]. After the required substitution we get

$$\begin{aligned} R \,=\, & a+\left[\left(\frac{11}{4}-\frac{1}{6}f\right)c_1+A+Bf\right]a^2+ \\ & \left[\left(\frac{11}{4}-\frac{1}{6}f\right)(c2+c3f)+\left(\frac{11}{4}-\frac{1}{6}f\right)^2c_1^2+\right. \\ & \left(\frac{51}{8}-\frac{19}{24}f\right)c_1+C+Df+Ef^2+ \\ & \left.2(A+Bf)\left(\frac{11}{4}-\frac{1}{6}f\right)c_1\right]a^3+O(a^4)\,. \end{aligned} \quad (13)$$

The coefficient of a^2 is $\left(\frac{11}{4}c1+A\right)+\left(\frac{-1}{6}c1+B\right)f$. If we choose $c1=6B$ then the f-dependence in coefficient of a^2 is disappeared. New coefficient is now:

$$\left(\frac{11}{4}c1+A\right)=\frac{33}{2}B+A. \quad (14)$$

The coefficient of a^3 is now:

$$\begin{aligned} & \left(\frac{11}{4}c_2+\frac{153}{4}B+33BA+\frac{1089}{4}B^2+C\right)+ \\ & \left(\frac{11}{4}c_3+D-\frac{1}{6}c_2-\frac{19}{4}B-2BA\right)f+ \\ & (-B^2+E-\frac{1}{6}c_3)f^2\,. \end{aligned} \quad (15)$$

Similar technique will remove the f^2 and f dependence in Eq. (15). The l quantity which indicates the change of scale is finally given by:

$$\begin{aligned} l \,=\, & 6B+\left[\left(-99B^2+99E+6D-\frac{57}{2}B-12BA\right)+\right. \\ & \left.(-6B^2+6E)f\right]\alpha. \end{aligned} \quad (16)$$

The relation between coupling constant at two different scales will help us to predicate higher order term in terms of lower ones. The BLM technique leads us to

$$R=a+A'a^2+C'a^3+\cdots \quad (17)$$

where A' and C' are independent of f. If we consider Eq. (12) as the expansion of a with respect to $a_2(\mu')$ and substitute it in Eq. (17), we will get:

$$\begin{aligned} R \,=\, & a_2+(A+Bf)a_2^2+(C+Df+Ef^2)a_3^2+ \\ & \left[-\frac{1}{1024}(-2907B+\cdots)+(-48BC+\cdots)f+\right. \\ & \left.(-76E+\cdots)f^2+(-32EB+\cdots)f^3\right]a_2^4. \end{aligned} \quad (18)$$

The coefficient of a_2^4 is the predicted term which is known completely in our calculations.

5 EC approach

In the BLM approach, we could absorb the f-dependence of coefficient function into the coupling constant. The extension of this idea is to absorb whole coefficients into the coupling constant. Is this possible ? Yes. Consider the perturbative series for R observable as:

$$R=a+r_1a^2+r_2a^3+\cdots\ , \quad (19)$$

By employing CORGI approach, we arrive at:

$$\begin{aligned} R \,&=\, a_0+X_2a_0^3+X_3a_0^4+\cdots\ , \\ a_0 \,&=\, a_0(Q^2)=a(r_1=0;c_2=c_3=\cdots\ 0). \end{aligned} \quad (20)$$

X_is are RG-invariant. Changing the RS parameters to $\tilde{r}_1=r_1-\bar{r}_1$, $\tilde{c}_2=c_2-\bar{c}_2$, $\cdots$ $\tilde{c}_n=c_n-\bar{c}_n$, will yield:

$$R=\bar{a}_0+\bar{r}_1\bar{a}_0^2+\tilde{X}_2\bar{a}_0^3+\tilde{X}_3\bar{a}_0^4+\cdots\ , \quad (21)$$

where $\bar{a}_0=\bar{a}_0(Q^2)=a(r_1=\bar{r}_1;c_2=\bar{c}_2,\cdots)$. We can choose the constant $\bar{c}_2,\bar{c}_3,\cdots$ so as $\tilde{X}_2,\tilde{X}_3,\cdots$ are equal zero. Therefore $R=\bar{a}_0+\bar{r}_1\bar{a}_0^2$ and this is the desired result. The NLO $\bar{r}_1\bar{a}_0^2$ term can be absorbed to $\bar{a}_0$ by changing the scale. So we achieve to the appropriate result in EC approach.

The quantity l which determines the change of scale is determined in a way that at NLO approximation, r_1 is absorbed into the coupling constant. So using Eq. (12), $R=a+r_1a^2+\cdots$ converts to $R=a_2+(-\beta_0 l+r_1)a_2^2+\cdots$. EC approach demands that $\beta_0 l+r_1=0$ so $l=\frac{r_1}{\beta_0}$ Consequently $a_{\rm EC}=a_2+r_2'a_2^3$ where $r_2'=r_2-r1^2+\frac{\beta_1}{\beta_0}r_1$ Using Eq. (12) and definition of l we will arrive at $R=a_{\rm EC}=a+r_1a^2+r_2a^3+\left(-3r_1^3-3\frac{\beta_1}{\beta_0}r_1^2+3r_1r_2\right)a^4+\cdots$. The last term is the predicted, $r_3^{\rm Pre}$, term.

6 Examples

I) The R-ratio [6] of $\dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \text{leptons})}$:

$$R_{e^-e^+} = 3\sum_f Q_f^2\left[1+\frac{\alpha_R(Q)}{\pi}+\cdots\right]$$

where

$$\frac{\alpha_R(Q)}{\pi} = a(s)+1.4092a(s)^2-12.7673a(s)^3-79.9795a(s)^4+\cdots .$$

II)

$$R_\tau = 3S_{EW}(|V_{ud}|^2+|V_{us}|^2)\times(1+\delta'_{EW}+\delta'_{Pert}+\delta'_{Non-Pert}),$$

where [6]

$$\delta'_{Pert} = a(s)+5.2023a(s)^2+26.366a(s)^3$$

$127.08a(s)^4+\cdots$ Where $a(s)=\dfrac{\alpha_s(m_\tau)^2}{\pi}$.

The result for predicted terms, using different approaches are tabulated in Table 1.

Table 1. Predicted term in different approaches.

	$r_3^{ex}(R_{ratio})$	$r_3^{pre}(R_{ratio})$	$r_3^{ex}(R_\tau)$	$r_3^{pre}(R_\tau)$
PMS	−79.9795	−54.7861	127.08	120.5124
CORGI	−79.9795	−63.6515	127.08	115.606
BLM	−79.9795	−63.8952	127.08	120.73
EC	−79.9795	−69.8820	127.08	113.2646

References

1 Stevenson P M. Phys. Rev. D, 1981, **23**: 2916
2 Maxwell C J, Mirjalili A. Nucl. Phys. B, 2000, **577**: 209
3 Brodsky S J, Gabadadze G T, Kataev A L, LU H J. Phys. Lett. B, 1996, **372**: 133
4 Grunberg G. Phys. Rev. D, 1992, **46**: 2228
5 Keshavarzian K, Mirjalili A, Yazdanpanah M M. Int. J. Mod. Phys. A, 2008, **23**: 5037
6 Baikov P A, Chetyrkin K G, Kuhn G H. Phys. Rev. Lett., 2008, **101**: 012002; S.Menke hep-ph/0904.1796