Two-body Spinless Salpeter equation for the Woods-Saxon potential

S. Hassanabadi[1;1)] M. Ghominejad[1] B. H. Yazarloo[2] S. Zarrinkamar[3] H. Hassanabadi[2]

[1] Semnan University, Semnan, Iran
[2] Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran
[3] Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran

Abstract: The two-body Spinless Salpeter equation for the Woods-Saxon potential is solved by using the supersymmetry quantum mechanics (SUSYQM). In our calculations, we have applied an approximation to the centrifugal barrier. Energy eigenvalues and the corresponding eigenfunctions are computed for various values of quantum numbers n, l.

Key words: Spinless-Salpeter equation, SUSY method, Woods-Saxon potential

PACS: 03.65.Ca, 03.65.Pm, 03.65.Nk **DOI:** 10.1088/1674-1137/37/8/083102

1 Introduction

The Bethe-Salpeter equation (BSE) [1–6] can be reduced into the Spinless Salpeter equation (SSE) by neglecting the spin degrees of freedom and applying some approximations [7, 8]. The so-called SSE has two main merits: it generalizes the Schrödinger equation into the relativistic regime and is less complicated than the BSE. Although we have just stated that the SSE is simpler than the BSE, it is more complicated than other wave equations of quantum mechanics, and in particular the Schrödinger equation itself, due to its nonlocal form (we will soon see that the Hamiltonian appears under an inverse square term). Until now, interesting ideas of nonrelativistic quantum mechanics such as operator inequalities, envelope theory and the variational technique have been brilliantly applied to the problem [9–13].

On the other hand, the Woods–Saxon potential is a successful short-range interaction in the potential model of nuclear physics and has had motivating predictions for the nuclear shell model and distribution of nuclear densities [14–25]. It has also been studied in other fields, such as atomic, condensed matter and chemical physics [26]. We first review the two-body SSE. Then, by considering a Pekeris-type approximation as well as some transformations, we bring the problem into a form which can be solved by the analytical SUSYQM technique [27–29].

2 The two-body-Hamiltonian

The SSE for two interacting particles in the center of mass system appears as [30, 31]

$$\left[\sum_{i=1,2}\sqrt{-\Delta+m_i^2}+V(r)-M\right]\chi(\vec{r})=0,\quad \Delta=\nabla^2. \tag{1}$$

In the case of heavy interacting particles, we can write [30, 31]

$$\begin{aligned}&\sum_{i=1,2}\sqrt{-\Delta+m_i^2}\\ &=\sqrt{-\Delta+m_1^2}+\sqrt{-\Delta+m_2^2}=m_1\left(1-\frac{\Delta}{m_1^2}\right)^{\frac{1}{2}}\\ &\quad+m_2\left(1-\frac{\Delta}{m_2^2}\right)^{\frac{1}{2}}=m_1\left(1-\frac{1}{2}\frac{\Delta}{m_1^2}-\frac{1}{8}\frac{\Delta^2}{m_1^4}-\cdots\right)\\ &\quad+m_2\left(1-\frac{1}{2}\frac{\Delta}{m_2^2}-\frac{1}{8}\frac{\Delta^2}{m_2^4}-\cdots\right)\\ &=m_1+m_2-\frac{\Delta}{2}\left(\frac{m_1+m_2}{m_1m_2}\right)-\frac{\Delta^2}{8}\left(\frac{m_1^3+m_2^3}{m_1^3m_2^3}\right)-\cdots,\end{aligned} \tag{2a}$$

where

$$\begin{aligned}\left(\frac{m_1^3+m_2^3}{m_1^3m_2^3}\right)&=\left(\frac{m_1^3+m_2^3}{(m_1+m_2)^3}\right)\frac{(m_1+m_2)^3}{m_1^3m_2^3}\\ &=\frac{1}{\mu^3}\frac{(m_1+m_2)^3-3m_1^2m_2-3m_2^2m_1}{(m_1+m_2)^3}\\ &=\frac{1}{\mu^3}\frac{(m_1+m_2)^3-3m_1m_2(m_1+m_2)}{(m_1+m_2)^3},\end{aligned}$$

Received 6 August 2012, Revised 8 October 2012

1) Corresponding author's E-mail: s.hassanabadi@yahoo.com

$$=\frac{1}{\mu^3}\frac{\frac{(m_1m_2)^2}{\mu^2}-3m_1m_2}{\frac{(m_1m_2)^2}{\mu^2}}$$

$$=\frac{1}{\mu^3}\frac{m_1m_2-3\mu^2}{m_1m_2}=\frac{1}{\eta^3}. \tag{2b}$$

On the other hand,

$$\sum_{i=1,2}\sqrt{-\Delta+m_i^2}=m_1+m_2-\frac{\Delta}{2\mu}-\frac{\Delta^2}{8\eta^3}-\cdots, \tag{3a}$$

with

$$\mu=\frac{m_1m_2}{m_1+m_2},\quad \eta=\mu\left(\frac{m_1m_2}{m_1m_2-3\mu^2}\right)^{1/3}. \tag{3b}$$

From Eqs. (1) to (3), we have [30, 31]

$$\left[\frac{-\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}+W_{nl}(r)-\frac{W_{nl}^2(r)}{2\tilde{m}}\right]\psi_{nl}(r)=0,\quad W_{nl}(r)=V(r)-E_{nl},\quad \tilde{m}=\eta^3/\mu^2=(m_1m_2\mu)/(m_1m_2-3\mu^2). \tag{4}$$

Here, we have studied the Woods-Saxon potential

$$V(r)=-\frac{V_0}{1+\exp\left(\frac{r-R_0}{a}\right)},$$

where V_0 is the potential depth, the parameters a and R_0 are the thickness of surface and the width of the potential, respectively.

Substituting the potential into Eq. (4), we get

$$\left[\frac{-\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}+\left(-\frac{V_0}{1+\exp\left(\frac{r-R_0}{a}\right)}-E_{n,l}\right)-\frac{1}{2\tilde{m}}\left(-\frac{V_0}{1+\exp\left(\frac{r-R_0}{a}\right)}-E_{n,l}\right)^2\right]\psi_{n,l}(r)=0. \tag{5}$$

A change of variable of the form [32]

$$x=\frac{r-R_0}{R_0},\quad \alpha=\frac{R_0}{a}. \tag{6}$$

Brings Eq. (5) of the form

$$\left[\frac{1}{R_0^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{l(l+1)}{R_0^2(1+x)^2}-\frac{2\mu}{\hbar^2}\left(-\frac{V_0}{1+\exp(\alpha x)}-E_{n,l}\right)+\frac{\mu}{\hbar^2\tilde{m}}\left(-\frac{V_0}{1+\exp(\alpha x)}-E_{n,l}\right)^2\right]\psi_{n,l}(x)=0. \tag{7}$$

The latter, obviously, cannot be exactly solved. Therefore, using an approximation is inevitable.

3 A Pekeris-type approximation and the SUSYQM technique

Here, for the centrifugal term barrier we consider the approximation [32]

$$\frac{1}{r^2}\approx\frac{1}{R_0^2}\frac{1}{(1+x)^2}\approx\frac{1}{R_0^2}\left(C_0+\frac{C_1}{1+\exp(\alpha x)}+\frac{C_2}{(1+\exp(\alpha x))^2}\right). \tag{8a}$$

With

$$C_0=1-\frac{4}{\alpha}+\frac{12}{\alpha^2},\quad C_1=\frac{8}{\alpha}-\frac{48}{\alpha^2},\quad C_2=\frac{48}{\alpha^2}. \tag{8b}$$

This brings Eq. (7) into the form

$$\left\{\frac{1}{R_0^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{l(l+1)}{R_0^2}\left(C_0+\frac{C_1}{1+\exp(\alpha x)}+\frac{C_2}{(1+\exp(\alpha x))^2}\right)-\frac{2\mu}{\hbar^2}\left(-\frac{V_0}{1+\exp(\alpha x)}-E_{n,l}\right)+\frac{\mu}{\hbar^2\tilde{m}}\left(-\frac{V_0}{1+\exp(\alpha x)}-E_{n,l}\right)^2\right\}\psi_{n,l}(x)=0, \tag{9}$$

or

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\left(-\frac{A}{(1+\exp(\alpha x))^2}-\frac{B}{(1+\exp(\alpha x))}\right)\right\}=C\psi_{n,l}(x),$$

$$A=-l(l+1)C_2+\frac{\mu V_0^2R_0^2}{\hbar^2\tilde{m}},\quad B=-l(l+1)C_1+\frac{2\mu V_0R_0^2}{\hbar^2}+\frac{2V_0E_{n,l}\mu R_0^2}{\hbar^2\tilde{m}},\quad C=-l(l+1)C_0+\frac{2\mu E_{n,l}R_0^2}{\hbar^2}+\frac{\mu E_{n,l}^2R_0^2}{\hbar^2\tilde{m}}. \tag{10}$$

Eq. (10) can be written as follows

$$-\frac{\mathrm{d}^2\psi_{n,l}(x)}{\mathrm{d}x^2}+V_{\mathrm{eff}}(x)\psi_{n,l}(x)=\tilde{E}_{n,l}\psi_{n,l}(x), \tag{11}$$

where

$$V_{\mathrm{eff}}(x)=-\frac{A}{(1+\exp(\alpha x))^2}-\frac{B}{(1+\exp(\alpha x))},\quad \tilde{E}_{n,l}=C. \tag{12}$$

Bearing in mind Eq. (A1), we search for the solution of the Riccati equation [33, 34]

$$\varphi^2(x)-\varphi'(x)=V_{\mathrm{eff}}(x)-\tilde{E}_{0,l}, \tag{13}$$

which is

$$\varphi(x)=\frac{\gamma}{1+\exp(\alpha x)}+\xi. \tag{14}$$

Substituting Eq. (14) into Eq. (13) and comparing similar terms, we can find

$$\gamma = \frac{\alpha\pm\sqrt{\alpha^2-4A}}{2}, \tag{15a}$$

$$\xi = \frac{1}{2\gamma}(-B-\alpha\gamma), \tag{15b}$$

$$\tilde{E}_{0,l} = -\xi^2. \tag{15c}$$

Therefore, our partner potentials are

$$V^{+}_{\text{eff}}(x) = \phi^2+\frac{\mathrm{d}\phi}{\mathrm{d}r}=\frac{-\gamma(\gamma+\alpha)\exp(\alpha x)}{(1+\exp(\alpha x))^2} +\frac{2\gamma\xi+\gamma^2}{1+\exp(\alpha x)}+\xi^2, \tag{16a}$$

$$V^{-}_{\text{eff}}(x) = \phi^2-\frac{\mathrm{d}\phi}{\mathrm{d}r}=\frac{-\gamma(\gamma-\alpha)\exp(\alpha x)}{(1+\exp(\alpha x))^2} +\frac{2\gamma\xi+\gamma^2}{1+\exp(\alpha x)}+\xi^2. \tag{16b}$$

Which are the shape invariants via the mapping $\gamma\to\gamma+\alpha$. Thus, from Eq. (A2),

$$R(a_1) = \left(\frac{-B-\alpha\gamma}{2\gamma}\right)^2-\left(\frac{-B-\alpha(\gamma+\alpha)}{2(\gamma+\alpha)}\right)^2,$$

$$\vdots$$

$$R(a_n) = \left(\frac{-B-\alpha(\gamma+(n-1)\alpha)}{2(\gamma+(n-1)\alpha)}\right)^2-\left(\frac{-B-\alpha(\gamma+n\alpha)}{2(\gamma+n\alpha)}\right)^2,$$

$$\tilde{E}^{-}_{n,l} = \sum_{k=1}^{n}R(a_k)=\left(\frac{-B-\alpha a_0}{2a_0}\right)^2-\left(\frac{-B-\alpha a_n}{2a_n}\right)^2. \tag{17}$$

And the energy is

$$\tilde{E}^{-}_{n,l}=\sum_{k=1}^{n}R(a_k)=\left(\frac{-B-\alpha a_0}{2a_0}\right)^2-\left(\frac{-B-\alpha a_n}{2a_n}\right)^2. \tag{18a}$$

Where n=0, 1, 2,$\cdots$ and

$$a_n=a_0+n\alpha,\quad a_0=\gamma. \tag{18b}$$

From Eqs. (15c) and (17) the eigenvalues are

$$\tilde{E}_{n,l} = \tilde{E}^{-}_{n,l}+\tilde{E}_{0,l}-l(l+1)C_0+\frac{2\mu E_{n,l}R_0^2}{\hbar^2}+\frac{\mu E^2_{n,l}R_0^2}{\hbar^2\tilde{m}} = -\left(\frac{1}{2a_n}(-l(l+1)C_1+\frac{2\mu V_0R_0^2}{\hbar^2}+\frac{2V_0E_{n,l}\mu R_0^2}{\hbar^2\tilde{m}}-\frac{\alpha}{2}\right)^2. \tag{19}$$

From the above equation, one can obtain the energy eigenvalues of the system. For obtaining the wavefunction of the system we start from Eq. (10), i.e.

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{A}{(1+\exp(\alpha x))^2}+\frac{B}{1+\exp(\alpha x)}+C\right\}\psi_{n,l}(x)=0. \tag{20}$$

By a change of variable of the form

$$z=\frac{1}{1+\exp(\alpha x)}. \tag{21}$$

We arrive at

$$\left\{z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(1-2z)\frac{\mathrm{d}}{\mathrm{d}z}+\frac{A}{\alpha^2}\frac{z}{1-z}+\frac{B}{\alpha^2}\frac{1}{1-z} +\frac{C}{\alpha^2}\frac{1}{z(1-z)}\right\}\psi_{n,l}(z)=0. \tag{22}$$

To obtain the solution of the above equation, we consider $\psi_{n,l}(z)$ as below

$$\psi_{n,l}(z)=z^{\upsilon}(1-z)^{\beta}f_{n,l}(z). \tag{23}$$

By substituting of Eq. (23) in Eq. (22), we have

$$\left\{z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(c'-(1+a'+b')z)\frac{\mathrm{d}}{\mathrm{d}z} -\left(\upsilon(\upsilon+1)+\beta(\beta+1)+2\beta\upsilon+\frac{A}{\alpha^2}\right)\right\}f_{n,l}(z)=0, \tag{24}$$

where

$$\begin{aligned} a'&=\frac{1}{2}\left[(1+2\upsilon)+2\beta+\sqrt{1-\frac{A}{\alpha^2}}\right],\\ b'&=\frac{1}{2}\left[(1+2\upsilon)+2\beta-\sqrt{1-\frac{A}{\alpha^2}}\right],\\ c'&=1+2\upsilon \end{aligned} \tag{25}$$

with

$$\begin{aligned} \upsilon^2&=-\frac{C}{\alpha^2},\\ \beta^2&=-\frac{A+B+C}{\alpha^2}. \end{aligned} \tag{26}$$

Equation (24) is just a hypergeometric equation, and its solution is the hypergeometric function

$$f_{n,l}(z)={}_2F_1(a',b',c';z). \tag{27}$$

So we have

$$\psi_{n,l}(z) = z^{\upsilon}(1-z)^{\beta}f_{n,l}(z) = z^{\upsilon}(1-z)^{\beta}{}_2F_1(a',b',c';z), \tag{28}$$

or equivalently

$$\psi_{n,l}(x) = (1+\exp(\alpha x))^{-\upsilon}\left(1-\frac{1}{1+\exp(\alpha x)}\right)^{\beta} \times {}_2F_1(a',b',c';(1+\exp(\alpha x))^{-1}). \qquad (29)$$

4 Conclusion

The successful description of many phenomena in particle and nuclear physics by the Woods-Saxon potential on the one hand, and the high number of two-body systems on the other hand, motivated us to solve the two-body SSE under this interaction. To go through the problem, we followed the analytical footprint due to its clarity and comprehensibility. We observed that instead of the cumbersome numerical programming, the equation can be simply solved via a Pekeris-type approximation and the SUSYQM technique. We hope our work will motivate further studies on the mesonic systems.

We wish to express our sincere gratitude to the referee for his suggestions which improved the manuscript.

Appendix A

Supersymmetry quantum mechanics

Within this appendix, a thorough introduction to SUSY quantum mechanics is included. These few lines form. Our first goal in SUSYQM mechanics is finding the solution of the Riccati equation

$$V_{\mp}=\Phi^2\mp\Phi', \qquad (A1)$$

with V being the potential of Schrödinger equation. If

$$V_{+}(a_0,x)=V_{-}(a_1,x)+R(a_1), \qquad (A2)$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F:a_0\mapsto a_1=F(a_0)$ and the residual term $R(a_1)$ does not include x, the partner potentials are shape invariant and the necessary information of the system is obtained via

$$E_n = \sum_{s=1}^{n} R(a_s), \qquad (A3)$$

$$\phi_n^-(a_0,x) = \prod_{s=0}^{n-1}\left(\frac{A^{\dagger}(a_s)}{[E_n-E_s]^{1/2}}\right)\phi_0^-(a_n,x), \qquad (A4)$$

$$\phi_0^-(a_n,x) = C\exp\left\{-\int_0^x \mathrm{d}z\Phi(a_n,z)\right\}. \qquad (A5)$$

$$A_s^{\dagger} = -\frac{\partial}{\partial x}+\Phi(a_s,x). \qquad (A6)$$

References

1 Salpeter E E, Bethe H A. Phys. Rev., 1951, **84**: 1232
2 Wick G C. Phys. Rev., 1954, **96**: 1124
3 Maris P, Roberts C D. Int. J. Mod. Phys. E, 2003, **12**: 297
4 Chang L, Roberts C D. Phys. Rev. Lett., 2009, **103**: 081601
5 Maris P, Roberts C D. Phys. Rev. C, 1997, **56**: 3369
6 Nakanishi N. Prog. Theor. Phys. Suppl., 1969, **43:** 1
7 Lucha W, Schoberl F F. Int. J. Mod. Phys. A, 1999, **14**: 2309
8 Lucha W, Schoberl F F. Fiz. B, 1999, **8**: 193
9 Lucha W, Schoberl F F. Int. J. Mod. Phys. A, 2002, **17**: 2233
10 Lucha W, Schoberl F F. Phys. Rev. A, 1996, **54**: 3790
11 Lucha W, Schoberl F F. Int. J. Mod. Phys. A, 2000, **15**: 3221
12 Lucha W, Schoberl F F. Phys. Rev. D, 1994, **50**: 5443
13 Hassanabadi S et al. Mod. Phys. Lett. A, 2012, **27**: 1250057
14 Erkol H, Demiralp E. Phys. Lett. A, 2007, **365**: 55
15 Woods R D, Saxon D S. Phys. Rev., 1954, **95**: 577
16 Williams W S C. Nuclear and Particle Physics. Oxford: Clarendon, 1996
17 Walz M et al. J. Phys. G: Nucl. Phys., 1988, **14**: L91
18 Garcia F et al. Eur. Phys. J. A, 1999, **6**: 49
19 Bespalova V et al. J. Phys. G: Nucl. Part. Phys., 2003, **29**: 1193
20 Dasgupta M et al. Prog. Theor. Phys. Suppl., 2004, **154**: 209
21 Sadeghi J, Pahlavani M R. Afr. J. Math. Phys., 2004, **1**: 195
22 Khounfais K et al. Chech. J. Phys., 2004, **54**: 697
23 Newton J O et al. Phys. Rev. C, 2004, **70**: 024605
24 GUO J Y, SHENG Q. Phys. Lett. A, 2005, **338**: 90
25 Diaz-Torres A, Scheid W. Nucl. Phys. A, 2005, **757**: 373
26 Badalov V H et al. arXiv:0905.2731v1 [math-ph]
27 Cooper F et al. Phys. Rep., 1995, **251**: 267
28 Junker G. Supersymmetric Methods in Quantum and Statistical Physics. New York: Springer-Verlag, 1996
29 Bagchi B. Supersymmetry in quantum and classical mechanics, Chapman and Hall/CRC 2000
30 Zarrinkamar S et al. Phys. Scr., 2011, **84**: 065008
31 Zarrinkamar S et al. Few-Body Sys., 2011, **52**: 165
32 Berkdemir C. Nuclear Physics A, 2006, **770**: 32
33 LU L L et al. Few-Body Syst., 2012, **53**: 573
34 Zarrinkamar S et al. Mod. Phys. Lett. A, 2011, **26**: 1621