

Holographic cusped Wilson loops in q -deformed $AdS_5 \times S^5$ spacetime^{*}

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Abstract: In this paper, a minimal surface in q -deformed $AdS_5 \times S^5$ with a cusp boundary is studied in detail. This minimal surface is dual to a cusped Wilson loop in dual field theory. We find that the area of the minimal surface has both logarithmic squared divergence and logarithmic divergence. The logarithmic squared divergence cannot be removed by either Legendre transformation or the usual geometric subtraction. We further make an analytic continuation to the Minkowski signature, taking the limit such that the two edges of the cusp become light-like, and extract the anomalous dimension from the coefficient of the logarithmic divergence. This anomalous dimension goes back smoothly to the results in the undeformed case when we take the limit that the deformation parameter goes to zero.

Key words: AdS/CFT, Wilson loops

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1 Introduction

Integrability [1] and localization [2] now allow us to compute some important quantities in $\mathcal{N}=4$ super Yang-Mills theory as non-trivial functions of 't Hooft coupling λ and the rank of the gauge group N^4 . These computations lead to results in strong coupling and give a non-trivial test of the famous AdS/CFT correspondence [3–5]. The cusp anomalous dimension $f(\lambda)$ is among these interesting quantities and its value at finite λ in the planar limit can be computed using this powerful integrability method [6, 7]. This function appears as a cusp anomaly of a light-like Wilson loop [8, 9]. It also appears as the coefficient in front of $\log S$ of the anomalous dimension of large spin twist-two operators (here S is the spin of the operator) [10, 11]. The fact that these two approaches give the same function $f(\lambda)$ was proved in perturbative gauge theory in Refs. [12–14].

Both approaches for the cusp anomalous dimension have dual descriptions in the gravity side of gauge/string duality. The twist-two operator is dual to folded spinning strings in AdS_5 found by Gubser–Klebanov–Polyakov (GKP) [15]. The anomalous dimension of the operator can be obtained from the energy of the semi-classical string. The Wilson loop is dual to an open F-string

in AdS_5 , and the contour of the Wilson loop is just the boundary of the F-string worldsheet [16, 17]. The holographical dual of cusped light-like Wilson loops was studied in detail in Ref. [18] (see also Ref. [19]) by performing a nontrivial analytic continuation of the F-string solution dual to cusped Wilson loops in Euclidean space in Ref. [20]. The cusp anomalous dimension obtained from the open F-string solution coincides with the results from the closed string solution obtained in Ref. [15]. In Ref. [18], this was taken as evidence that [15] made the correct identification for string theory dual of the twist-two operators. In Ref. [21], the scaling limits of the above closed string solution and open string solution was shown to be equivalent through an analytic continuation and an AdS_5 isometry rotation. This explained, on the gravity side, why these two approaches give the same results for the anomalous dimension. This can be thought of as a kind of open-closed duality in the AdS background.

It is obviously of great value to search for integrable structures in AdS/CFT correspondence with fewer supersymmetries. Such examples are very rare. Orbifolds [22–24], β - and γ -deformations [25–28], and adding suitable fundamental matters [29–32] are almost the only known examples where the four-dimensional field theo-

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4) Integrability is mainly established in the large N limit.

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ries are still the usual local gauge theories and integrability in the planar limit is preserved [33]. Many other four-dimensional theories and their gravity duals are not integrable. Instead they display chaotic behaviors [34–39]. In the gravity dual of the orbifolds and β (γ)-deformation examples, the AdS_5 part is untouched. For the first case, the five-sphere is replaced by its orbifolds. For the second case, the metric on S^5 is deformed with other background fields turned on. Since the AdS_5 part of the metric is unchanged and the NS-NS field has vanishing components in the AdS_5 part for both cases, the computations for both GKP folded strings in AdS_5 and F-strings dual to cusped Wilson loops are not changed. The above mentioned open-closed duality in AdS space is preserved in a trivial manner. We also notice that this open-closed duality was also found for string theory on $AdS_3 \times S^3 \times M_4$ with both NS-NS and RR three-form fluxes [40].

It is then quite interesting to search for integrable models with a gravity dual involving more complicated geometry replacing the AdS part. One such integrable deformation on the worldsheet theory was constructed in Ref. [41]. Many aspects of such deformation had already been studied in Refs. [42–59]. The background was called q -deformed $AdS_5 \times S^{5(1)}$. The field theory dual of string theory on this background is still unclear. There is hope that the studies of various aspects on the string theory side can give us some hints of the possible dual field theory. Many classical string solutions in this background were studied in detail in Refs. [46, 48, 49, 52, 53, 55]. There are already several interesting features for the classical strings, which are different from the case without deformations. The GKP spinning string solutions found in Refs. [52, 53] cannot be smoothly connected with the original solutions in Ref. [15] when we take the limit that the deformation parameter goes to zero, and the energy E and spin S of these spinning strings will not have the relation $E - S \sim f(\lambda) \log S$ in the large S limit. Another interesting result [55] is that the open F-string solution with a circular boundary has a finite area without performing geometric subtraction or Legendre transformation, which was used for the undeformed case, though there are divergences in the action when the boundary is a straight line [52]. Here the deformation parameter plays the role of UV regularization [55].

The above features led us to the study of the F-string solution with a cusp boundary in q -deformed $AdS_5 \times S^5$. We also consider the case when there is a jump in the deformed S^5 at the cusp. The solution was found by computing the conserved charges from the symmetry of the system. We find the area of the worldsheet has behavior different from both the case with a circle as the boundary and the holographic dual of cusped Wilson loops in the undeformed case. The area has logarithmic

squared divergence, in addition to the logarithmic divergence. The logarithmic squared divergence is softer than the linear divergence in the undeformed case. However, the UV regularization provided by the deformation parameter does not make it finite. We then turn to attempts to renormalize the area. Two commonly used methods, Legendre transformation and geometric subtraction, are considered. We find that neither of these can remove the logarithmic squared divergence, and the two methods are no longer equivalent to each other. Finally, by continuation to the Minkowski signature and subtracting the logarithmic squared divergence by hand, we compute the cusp anomalous dimension for the deformed case. We find that this result can be smoothly connected with the result in the undeformed case when we take the limit that the deformation parameter tends to zero.

The rest of this paper is structured as follows. In the next section, we will find the F-string solution in q -deformed $AdS_5 \times S^5$ dual to cusped Wilson loops. The cusp anomalous dimension will be extracted from the cusped Wilson loops in Section 3. The final section is devoted to discussion and conclusions.

2 F-string solution dual to cusped Wilson loop

2.1 q -deformed $AdS_5 \times S^5$

In Ref. [41], an integrable deformation of type IIB superstring theory on $AdS_5 \times S^5$ was constructed. From this, the string frame metric and B-field for this string background was given in Ref. [42]. Later a new coordinate system was introduced in Ref. [52] which was inspired by studies of GKP (Gubser–Klebanov–Polyakov) strings [15] in q -deformed AdS_5 . A related Poincare-like coordinate system for q -deformed AdS_5 was introduced in Ref. [55]. This will be our starting point. We now list the results of metric and B-field in these Poincare-like coordinates. The metric for the q -deformed AdS part in Poincare coordinates is:

$$ds^2 = \sqrt{1+C^2}R^2 \left[\frac{dy^2+dr^2}{y^2+C^2(y^2+r^2)} + \frac{C^2(ydy+rdr)^2}{y^2(y^2+C^2(y^2+r^2))} + \frac{(y^2+C^2(y^2+r^2))r^2}{(y^2+C^2(y^2+r^2))^2+C^2r^4\sin^2\zeta} (d\zeta^2+\cos\zeta^2 d\phi^2) + \frac{r^2\sin^2\zeta d\psi^2}{y^2+C^2(y^2+r^2)} \right]. \tag{1}$$

The metric for the q -deformed S^5 part is:

$$ds^2 = \sqrt{1+C^2}R^2 \left[\cos^2\gamma d\theta^2 + \frac{d\gamma^2}{1+C^2\cos^2\gamma} \right]$$

1) Some people choose the name η -deformation.

$$\begin{aligned}
 & + \frac{(1+C^2 \cos^2 \gamma) \sin^2 \gamma}{(1+C^2 \cos^2 \gamma)^2 + C^2 \sin^4 \gamma \sin^2 \xi} (d\xi^2 + \cos^2 \xi d\phi_1^2) \\
 & + \left. \frac{\sin^2 \gamma \sin^2 \xi d\phi_2^2}{1+C^2 \cos^2 \gamma} \right]. \quad (2)
 \end{aligned}$$

The action of the Wess-Zumino term for the deformed AdS part is

$$\begin{aligned}
 \mathcal{L}_1^{\text{WZ}} &= \frac{C\sqrt{1+C^2}R^2}{4\pi\alpha'} \epsilon^{\mu\nu} \\
 & \times \frac{r^4 \sin 2\zeta \partial_\mu \phi \partial_\nu \zeta}{[C^2 r^2 + (1+C^2)z^2]^2 + C^4 r^4 \sin^2 \zeta}, \quad (3)
 \end{aligned}$$

and for the deformed S^5 part is

$$\begin{aligned}
 \mathcal{L}_2^{\text{WZ}} &= -\frac{C\sqrt{1+C^2}R^2}{4\pi\alpha'} \epsilon^{\mu\nu} \\
 & \times \frac{\sin^4 \gamma \sin 2\xi}{(1+C^2 \cos^2 \gamma)^2 + C^2 \sin^4 \gamma \sin^2 \xi} \partial_\mu \phi_1 \partial_\nu \xi. \quad (4)
 \end{aligned}$$

It is easy to see that C plays the role of deformation parameter and when we take the limit $C \rightarrow 0$, we will go back to the undeformed case.

2.2 Cusped Wilson loop

2.2.1 Loops without a jump in deformed S^5

We now begin our computation of a minimal surface with a cusped loop boundary. This minimal surface is the worldsheet of an F-string in deformed $AdS_5 \times S^5$ dual to a cusped Wilson loop in the dual field theory. First we study the case with trivial dependence on the coordinates of deformed S^5 , that is to say that the coordinates of deformed S^5 take constant values on the worldsheet.

At the boundary the Wilson loop is put in two lines:

$$r \in [0, \infty), \psi = 0, \zeta = \frac{\pi}{2}, \phi = 0, \quad (5)$$

and

$$r \in [0, \infty), \psi = \Omega, \zeta = \frac{\pi}{2}, \phi = 0. \quad (6)$$

The string worldsheet will extend to the bulk of deformed AdS_5 . Let us choose r and ψ to be the coordinates of string worldsheet and start with the following ansatz:

$$y = y(r, \psi), \zeta = \frac{\pi}{2}, \phi = 0, \quad (7)$$

with the boundary condition

$$y(r, 0) = y(r, \Omega) = 0. \quad (8)$$

Taking into account the invariance of the metric in Eq. (1) under the scaling transformation

$$y \rightarrow \lambda y, r \rightarrow \lambda r, \quad (9)$$

we expect the solution for $y(r, \psi)$ to take the form

$$y(r, \psi) = \frac{r}{f(\psi)}. \quad (10)$$

The boundary condition now gives

$$\lim_{\psi \rightarrow 0} f(\psi) = \lim_{\psi \rightarrow \Omega} f(\psi) = \infty. \quad (11)$$

One can also check that the Wess-Zumino term in the worldsheet action will not affect the equation of motion for the ansatz chosen above¹⁾.

Substituting the ansatz back into the target space metric in Eq. (1), we obtain the induced metric on the worldsheet,

$$\begin{aligned}
 ds_{\text{ind}}^2 &= R^2 \sqrt{1+C^2} \left[\frac{1+f^2}{r^2} dr^2 - \frac{2f'}{rf} dr d\psi \right. \\
 & \left. + \frac{1}{f^2} \frac{(1+C^2)f'^2 + f^4}{1+C^2(1+f^2)} d\psi^2 \right]. \quad (12)
 \end{aligned}$$

Then the area of the surface is

$$A = \sqrt{1+C^2} R^2 \int dr d\psi \frac{1}{r} \sqrt{\frac{f^2 + f^4 + f'^2}{1+C^2 + C^2 f^2}}. \quad (13)$$

So the Nambu-Goto action of the string is

$$S_{\text{NG}} = \frac{1}{2\pi\alpha'} A = \frac{\sqrt{1+C^2} R^2}{2\pi\alpha'} \int dr d\psi \frac{1}{r} \sqrt{\frac{f^2 + f^4 + f'^2}{1+C^2 + C^2 f^2}}. \quad (14)$$

Therefore, finding the minimum surface in the bulk in this case reduces to a one dimensional variational problem with the Lagrangian,

$$\mathcal{L} = \int d\psi \sqrt{\frac{f^2 + f^4 + f'^2}{1+C^2 + C^2 f^2}}. \quad (15)$$

We can solve this extreme value problem by making use of the translation invariance in ψ (L does not depend on ψ explicitly), and the corresponding conserved charge is:

$$E = \frac{1}{\sqrt{1+C^2 + C^2 f^2}} \frac{f^4 + f^2}{\sqrt{f^4 + f^2 + f'^2}}. \quad (16)$$

Due to the symmetry of the system, f will achieve its minimal value f_0 at $\psi = \Omega/2$, then we have $\partial_\psi f|_{\psi = \Omega/2} = 0$. Thus we can also express E in terms of f_0 ,

$$E = \frac{f_0 \sqrt{1+f_0^2}}{\sqrt{1+C^2 + C^2 f_0^2}}. \quad (17)$$

We can work out f' by equating these two expressions of E ,

$$\begin{aligned}
 \left(\frac{df}{d\psi} \right)^2 &= (f^4 + f^2)(f^2 - f_0^2) \\
 & \times \frac{1+f_0^2 + f^2 + C^2(1+f_0^2 + f^2 + f_0^2 f^2)}{f_0^2(1+f_0^2)(1+C^2 + C^2 f^2)}. \quad (18)
 \end{aligned}$$

1) Things will be different if we choose $\zeta=0, \psi=0$ and a worldsheet along the y, r, ϕ directions. In this case, though the WZ term will not contribute to the worldsheet action, it does affect the string equation of motion.

From this, we get the relation between f_0 and Ω as

$$\frac{\Omega}{2} = f_0 \sqrt{1+f_0^2} \int_{f_0}^{\infty} \frac{df}{f} \sqrt{\frac{1+C^2+C^2 f^2}{(f^2-f_0^2)(1+f^2)(1+f^2+f_0^2+C(1+f_0^2)(1+f^2))}}. \quad (19)$$

By making the transformation $f = \sqrt{z^2+f_0^2}$, we get

$$\frac{\Omega}{2} = f_0 \sqrt{1+f_0^2} \int_0^{\infty} dz \frac{\sqrt{1+C^2(1+z^2+f_0^2)}}{(z^2+f_0^2) \sqrt{(1+z^2+f_0^2)(1+2f_0^2+z^2+C^2(1+f_0^2)(1+f_0^2+z^2))}}. \quad (20)$$

Substituting Eq. (18) back into the Lagrangian, we obtain:

$$\mathcal{L} = 2\sqrt{1+C^2}R^2 \int_{f_0}^{\frac{r}{\epsilon}} df \frac{\sqrt{(1+C^2+f_0^2 C^2)f^2(1+f^2)}}{\sqrt{(f^2-f_0^2)(1+C^2+C^2 f^2)(f_0^2+(1+C^2+f_0^2 C^2)(f^2+1))}}, \quad (21)$$

where we have imposed an infrared cutoff for y , $y > \epsilon$ or $f < r/\epsilon$. We can further make the transformation $f = \sqrt{z^2+f_0^2}$ as above and the integral becomes

$$\mathcal{L} = \mathcal{L}(r, \epsilon) = \frac{2\sqrt{1+C^2}R^2}{C} \int_0^{\sqrt{\frac{r^2}{\epsilon^2}-f_0^2}} dz \frac{\sqrt{z^2+a}}{\sqrt{(z^2+b)(z^2+c)}}, \quad (22)$$

with a, b, c listed below:

$$a = 1+f_0^2, \quad (23)$$

$$b = \frac{1+C^2+C^2 f_0^2}{C^2}, \quad (24)$$

$$c = \frac{1+2f_0^2+C^2(1+2f_0^2+f_0^4)}{1+C^2+f_0^2 C^2}. \quad (25)$$

Here we will give an approximate analysis since a special function solution requires additional constraints for the parameters f_0 and C and will not make the result clearer. By making a change of variable $z = 1/t$ and noticing that

$$\sqrt{\frac{r^2}{\epsilon^2}-f_0^2} \approx \frac{r}{\epsilon}, \quad (26)$$

for small ϵ , we have

$$\mathcal{L}(r, \epsilon) \approx \frac{2\sqrt{1+C^2}R^2}{C} \int_{\epsilon/r}^{\infty} \frac{dt}{t} \sqrt{\frac{1+at^2}{(1+bt^2)(1+ct^2)}}. \quad (27)$$

To extract the divergent part of the integral, we expand the integrand around $t = \epsilon/r$,

$$\mathcal{L}(r, \epsilon) \approx \frac{2\sqrt{1+C^2}R^2}{C} \log \frac{r}{\epsilon} + L_{\text{finite}}. \quad (28)$$

Hence, the area can be evaluated as,

$$S_{\text{NG}} \approx \frac{\sqrt{1+C^2}R^2}{2\pi\alpha' C} \log^2 \frac{L}{\epsilon} - \frac{1}{2\pi} F(\Omega, C) \log \frac{L}{\epsilon}, \quad (29)$$

where L is the cutoff for the length of the two rays of the Wilson loop and the function F comes from the finite part of $\mathcal{L}(r, \epsilon)$. We find that the area is composed of two kinds of divergences - the logarithmic and the logarithmic squared divergence - while for the undeformed case

there is a linear divergence plus a logarithmic one [20]. Unlike the undeformed case where the linear divergence can be removed by means of Legendre transformation, we cannot manage to subtract any divergence in this way in the deformed background, as will be demonstrated in the next subsection.

2.2.2 Loops with a jump in deformed S^5 at the cusp

We continue to study a cusped loop where the points on the two edges correspond to two different points in deformed S^5 . In this work, we will only consider the case where the dual of the two edges have a relative angle of Θ only along the θ direction of deformed S^5 . For the undeformed case, a complete and analytical solution is given in Ref. [60].

Since the cusp is still invariant under the rescaling of r , we consider the following ansatz:

$$y(r, \psi) = \frac{r}{f(\psi)}, \quad \zeta = \frac{\pi}{2}, \quad \phi = 0, \\ \gamma = \xi = \phi_1 = \phi_2 = 0, \quad \theta = \theta(\psi). \quad (30)$$

Then the induced metric on the worldsheet turns out to be:

$$h_{rr} = R^2 \sqrt{1+C^2} \cdot \frac{1+f^2}{r^2}, \quad (31)$$

$$h_{r\psi} = R^2 \sqrt{1+C^2} \cdot \frac{-f'}{rf}, \quad (32)$$

$$h_{\psi\psi} = R^2 \sqrt{1+C^2} \left[\frac{(1+C^2)f'^2+f^4}{f^2+C^2 f^2(1+f^2)} + \theta'^2 \right]. \quad (33)$$

The area becomes:

$$A = \sqrt{1+C^2}R^2 \int \frac{dr}{r} \int_0^{\Omega} d\psi \sqrt{\frac{f'^2+f^2+f^4}{1+C^2+C^2 f^2} + (1+f^2)\theta'^2}. \quad (34)$$

We focus on the Lagrangian density,

$$\mathcal{L} = \sqrt{\frac{f'^2+f^2+f^4}{1+C^2+C^2 f^2} + (1+f^2)\theta'^2}. \quad (35)$$

It is then easy to find the two conserved charges of the system, the energy and the canonical momentum conjugate to θ :

$$E = \frac{1}{\sqrt{1+C^2+C^2f^2}} \times \frac{f^4+f^2}{\sqrt{f'^2+f^2+f^4+(1+C^2+C^2f^2)(1+f^2)\theta'^2}}, \quad (36)$$

$$J = \frac{1+f^2}{\mathcal{L}}\theta'. \quad (37)$$

For the convenience of calculation, we introduce two new conserved quantities which are the combinations of J and E ,

$$p = \frac{1}{E}, \quad (38)$$

$$q = \frac{J}{E} = \frac{1+C^2+C^2f^2}{f^2}\theta'. \quad (39)$$

We find immediately $\theta' = qf^2/(1+C^2+C^2f^2)$ and by substituting it into p , f' is easily obtained:

$$\left(\frac{d\psi}{df}\right)^2 = \frac{1}{f^2(1+f^2)} \frac{1+C^2+C^2f^2}{f^2((1+f^2)p^2-q^2-C^2)-C^2-1}. \quad (40)$$

The extreme value f_0 is determined from the condition $\partial_\psi f|_{\psi=\Omega/2}=0$ as follows:

$$\frac{f_0^2(1+f_0^2)p^2-f_0^2q^2}{1+C^2+C^2f_0^2} = 1. \quad (41)$$

The relation between Ω and f_0 is

$$\frac{\Omega}{2} = \int_{f_0}^{\infty} \frac{df\sqrt{1+C^2+C^2f^2}}{f\sqrt{(1+f^2)(f^2((1+f^2)p^2-q^2-C^2)-C^2-1)}}. \quad (42)$$

As in the previous subsection, we make the transformation $f = \sqrt{f_0^2+z^2}$ and obtain

$$\Omega = 2 \int_0^{\infty} \frac{dz\sqrt{1+C^2(1+f_0^2+z^2)}}{(f_0^2+z^2)\sqrt{(1+f_0^2+z^2)(p^2z^2+p^2(1+2f_0^2)-q^2-C^2)}}. \quad (43)$$

The relation between Θ and f_0 is

$$\Theta = \int_0^{\Omega} \frac{qf^2}{1+C^2(1+f^2)} d\psi \quad (44)$$

$$= 2 \int_{f_0}^{\infty} \frac{qf df}{\sqrt{(1+C^2(1+f^2))(1+f^2)(f^2((1+f^2)p^2-q^2-C^2)-C^2-1)}} \quad (45)$$

$$= 2q \int_0^{\infty} \frac{dz}{\sqrt{(1+f_0^2+z^2)(1+C^2(1+f_0^2+z^2))(p^2z^2+p^2(1+2f_0^2)-q^2-C^2)}}. \quad (46)$$

The area becomes:

$$\begin{aligned} A &= \sqrt{1+C^2}R^2 \int_r \int_0^{\Omega} d\psi \mathcal{L} = \sqrt{1+C^2}R^2 \int_r \int_0^{\Omega} d\psi \frac{pf^2(1+f^2)}{1+C^2+C^2f^2} \\ &= 2\sqrt{1+C^2}R^2 \int_r \int_{f_0}^{r/\epsilon} df \frac{pf\sqrt{1+f^2}}{\sqrt{1+C^2+C^2f^2}} \frac{1}{\sqrt{f^2((1+f^2)p^2-q^2-C^2)-C^2-1}} \\ &= \frac{2\sqrt{1+C^2}R^2}{C} \int_r \int_0^{\sqrt{\frac{r^2}{\epsilon^2}-f_0^2}} dz \frac{\sqrt{z^2+k_1}}{\sqrt{z^2+k_2}\sqrt{z^2+k_3}}, \end{aligned} \quad (47)$$

where

$$k_1 = f_0^2+1, \quad (48)$$

$$k_2 = \frac{1+C^2+C^2f_0^2}{C^2}, \quad (49)$$

$$k_3 = \frac{p^2(2f_0^2+1)-q^2-C^2}{p^2}. \quad (50)$$

We can analyze the integral approximately by using a

new variable $t=1/z$,

$$\begin{aligned} &\int_0^{\sqrt{\frac{r^2}{\epsilon^2}-f_0^2}} dz \frac{\sqrt{z^2+k_1}}{\sqrt{z^2+k_2}\sqrt{z^2+k_3}} \\ &\approx \int_{\frac{\epsilon}{r}}^{\infty} dt \frac{1}{t} \frac{\sqrt{1+k_1t^2}}{\sqrt{(1+k_2t^2)(1+k_3t^2)}} \\ &\approx \log \frac{r}{\epsilon} + \text{finite terms}. \end{aligned} \quad (51)$$

Therefore the area is

$$A \approx \frac{\sqrt{1+C^2}R^2}{C} \log^2 \frac{L}{\epsilon} - \frac{1}{2\pi} F(\Omega, \Theta, C) \log \frac{L}{\epsilon}. \quad (52)$$

We find that the structure of the divergences is the same as the no-jump case.

2.3 Renormalization of the area

Let us first recall the story in the undeformed case. When the contour of the Wilson loops is smooth, the bare area of the F-string worldsheet diverges universally as L/ϵ [62] where L is the length of the loop and ϵ is the cut-off as introduced in this paper. This divergence can be removed either via a Legendre transformation [20] or by a geometric subtraction [17], and these two methods are equivalent to each other. For the case with a cusp, beside this L/ϵ term, there is a subleading divergence term growing as $\log(L/\epsilon)$. The leading divergence can be removed by either of the two methods, and the subleading $\log(L/\epsilon)$ term will remain. This is consistent with the perturbative computations from the field theory side [20].

2.3.1 Legendre transformation

Firstly, we consider the loop with no dependence on the deformed S^5 , where the only coordinate that needs to be replaced by its conjugate momentum is the radial coordinate y . From the Nambu–Goto action (29), it can be easily obtained as

$$P_y = \frac{\sqrt{1+C^2}R^2}{2\pi\alpha' r^2} \frac{-f' f^2}{\sqrt{1+C^2+C^2 f^2}} \frac{1}{\sqrt{f^4+f^2+f'^2}}. \quad (53)$$

Near the boundary $y = \epsilon$ or $f = r/\epsilon$, we can evaluate f' approximately from (18),

$$\left(\frac{df}{d\psi}\right)^2 \approx \frac{r^6}{\epsilon^6} \frac{1+C^2(1+f_0^2)}{C^2 f_0^2(1+f_0^2)}, \quad (54)$$

which indicates $f'^2 \gg f^4 \gg f^2$. Thus, we obtain from Eq. (53),

$$P_y \approx \frac{\sqrt{1+C^2}R^2}{2\pi\alpha' C r \epsilon}. \quad (55)$$

So the boundary term is

$$-2 \int_{\epsilon}^L dr (P_y y)|_{y=\epsilon} \approx -\frac{\sqrt{1+C^2}R^2}{\pi\alpha' C} \log \frac{L}{\epsilon}. \quad (56)$$

Notice this cannot be used to cancel the leading \log^2 divergence found in the previous section. The computation of the Legendre transformation for the case with a jump in deformed S^5 is similar and we arrive at the same conclusion.

2.3.2 Geometric subtraction

We may consider a geometric subtraction scheme which is performed by discarding two ‘flat’ planes in the deformed AdS space with the metric

$$ds^2 = R^2 \sqrt{1+C^2} \left[\frac{dy^2 + dr^2}{y^2 + C^2(y^2 + r^2)} + \frac{C^2(y^2 dy^2 + r^2 dr^2 + 2y r dy dr)}{y^2(y^2 + C^2(y^2 + r^2))} \right]. \quad (57)$$

So the area to be subtracted is:

$$\begin{aligned} A_s &= 2 \int dy dr \sqrt{G_{yy} G_{rr} - G_{ry}^2} \\ &= 2R^2 \sqrt{1+C^2} \int_{y_1}^{y_2} dy \int_{r_1}^{r_2} dr \frac{1}{y \sqrt{y^2 + C^2(y^2 + r^2)}} \\ &\approx \frac{2R^2 \sqrt{1+C^2}}{C} \left(\log \frac{2C}{\sqrt{1+C^2}} \log \frac{y_2}{y_1} \right. \\ &\quad \left. + \log r_2 \log \frac{y_2}{y_1} - \frac{1}{2} \log^2 y_2 + \frac{1}{2} \log^2 y_1 \right), \quad (58) \end{aligned}$$

where $y_1, r_1 \equiv \epsilon$ and $y_2, r_2 \equiv L$ are the IR and UV cutoffs respectively. In other words, we have

$$\begin{aligned} A_s &\approx \frac{2R^2 \sqrt{1+C^2}}{C} \left(\log \frac{2C}{\sqrt{1+C^2}} \log \frac{L}{\epsilon} + \log L \log \frac{L}{\epsilon} \right. \\ &\quad \left. - \frac{1}{2} \log^2 L + \frac{1}{2} \log^2 \epsilon \right). \quad (59) \end{aligned}$$

From this result, one can see that the leading \log^2 divergence cannot be canceled using this geometric subtraction. One can also see that the Legendre transformation is not equivalent to the geometric subtraction, as we indicated earlier.

3 Anomalous dimension from cusped Wilson loop

The anomalous dimension can be obtained by the vacuum expectation value of a light-like Wilson loop with a cusp [18]. We will only consider the case without a jump in deformed S^5 at the cusp. The light-like system can be reached from the solution we have found by analytically continuing $f_0 \rightarrow if_0$ and taking f_0 to a fixed value which will be given later. Thus, the cusp angle Ω becomes $\pi+i\gamma$, with¹⁾

$$\gamma = P.P. \int_{-\infty}^{+\infty} dz \frac{f_0 \sqrt{1-f_0^2} \sqrt{1+C^2-C^2 f_0^2 + C^2 z^2}}{(z^2 - f_0^2) \sqrt{1-f_0^2+z^2} \sqrt{z^2 - 2f_0^2 + 1 + C^2(1-f_0^2)(1-f_0^2+z^2)}}. \quad (60)$$

1) The real part of Ω , which equals π , comes from the residual at $z=f_0$.

The (renormalized) area A now becomes

$$A = \sqrt{(1+C^2)(1+C^2-f_0^2 C^2)} R^2 \log \frac{L}{\epsilon} \int_{-\infty}^{+\infty} dz \times \left\{ \frac{\sqrt{1+z^2-f_0^2}}{\sqrt{1+C^2-C^2 f_0^2+C^2 z^2} \sqrt{z^2-2f_0^2+1+C^2(1-f_0^2)(1-f_0^2+z^2)}} - \frac{1}{zC\sqrt{1+C^2-C^2 f_0^2}} \right\}. \quad (61)$$

As discussed in the previous section, neither Legendre transformation nor geometric subtraction can cancel the leading \log^2 divergence, and to extract the anomalous dimension which comes from the coefficient of the logarithmic divergence, we subtract the leading divergence by hand in the above expression. In order to make the above two integrals real when $z \rightarrow 0$, we can choose f_0^2 to satisfy

$$f_0^2 \leq \frac{C^2+1-\sqrt{C^2+1}}{C^2}. \quad (62)$$

We then make the transformation

$$f_0^2 = \frac{C^2+1-\sqrt{C^2+1+C^2\delta}}{C^2}, \quad (63)$$

which gives

$$\delta = 1 - 2f_0^2 + C^2(1-f_0^2)^2, \quad (64)$$

and the integral can be expressed in terms of δ as

$$\gamma = \int_{-\infty}^{+\infty} dz \frac{\sqrt{\frac{C^2+1-\sqrt{C^2+1+C^2\delta}}{C^2}} \sqrt{\frac{\sqrt{1+C^2+C^2\delta}-1}{C^2}} \sqrt{\sqrt{1+C^2+C^2\delta}+C^2 z^2}}{\frac{C^2 z^2 - C^2 - 1 + \sqrt{C^2+1+C^2\delta}}{C^2} \sqrt{\frac{\sqrt{1+C^2+C^2\delta}-1}{C^2} + z^2 \sqrt{\sqrt{C^2+1+C^2\delta} z^2 + \delta}}}. \quad (65)$$

In order for the two edges of the cusped Wilson loops to be light-like, we need to take a limit such that $\gamma \rightarrow \infty$. This limit is given by $\delta \rightarrow 0$ (which obviously corresponds to $f_0^2 \rightarrow \frac{C^2+1-\sqrt{C^2+1}}{C^2}$), and one can see the largest contribution stems from the term $\sqrt{\sqrt{C^2+1+C^2\delta} z^2 + \delta}$ around $z \approx 0$, i. e. $z \in (-\epsilon, \epsilon)$. When $\delta \ll \epsilon \ll 1$, we get

$$\gamma \approx \frac{C}{\sqrt{C^2+1-\sqrt{C^2+1}}} \log \delta. \quad (66)$$

The same method can be applied to compute the area, which gives

$$A \approx -\frac{R^2(C^2+1)^{1/4} \sqrt{\sqrt{C^2+1}-1}}{C} \log \delta \log \frac{L}{\epsilon}. \quad (67)$$

So the cusp anomaly is

$$\bar{\Gamma}_{\text{cusp}} = -\frac{A}{2\pi\alpha' |\gamma| \log \frac{L}{\epsilon}} = -\frac{R^2(1+C^2-\sqrt{1+C^2})}{2\pi\alpha' C^2}. \quad (68)$$

In the $C \rightarrow 0$ limit, we have

$$\bar{\Gamma}_{\text{cusp}} = -\frac{R^2}{4\pi\alpha'}. \quad (69)$$

By using the relation $R^2 = \alpha' \sqrt{\lambda}$ in the undeformed case with λ the 't Hooft of the $\mathcal{N}=4$ super Yang–Mills theory,

we get

$$\bar{\Gamma}_{\text{cusp}} = -\frac{\sqrt{\lambda}}{4\pi}, \quad (70)$$

which coincides with the results obtained in Ref. [18].

4 Discussion and conclusions

q -deformed $AdS_5 \times S^5$ is quite an interesting background of type IIB string theory. It is integrable, and its dual field theory is still unclear; it is probably dual to certain non-local field theories. We hope various computations on the string theory side could give some hints on the possible dual field theory. The study in this paper adds one more example of such computations. We studied the F-string theory solution dual to cusped Wilson loops on the field theory side. Both the case with a jump in deformed S^5 at the cusp and the case without such a jump were studied.

The first interesting aspect we find for the cusped Wilson loops is the divergence behavior of the area of the F-string worldsheet before renormalization. In Ref. [55], it is shown that the F-string dual to a circular Wilson loop has finite area, before any renormalization or Legendre transformation though the action of F-string dual to straight line is divergent [52]. The first result was explained by the deformation parameter C providing UV

regularization [55]. For the Wilson loops with a cusp, the bare area goes like

$$\frac{\sqrt{1+C^2}R^2}{C} \log^2 \frac{L}{\epsilon} - \frac{1}{2\pi} F(\Omega, \Theta, C) \log \frac{L}{\epsilon}.$$

Now, the leading divergence scales as $\log^2 \frac{L}{\epsilon}$ which is not made finite due to the deformation, however it is less divergent than L/ϵ . In other words, q -deformation softens the divergence although it does not soften it into a finite term. This divergence can be removed neither by the Legendre transformation nor by geometric subtraction, and these two methods are no longer equivalent to each other. In general, q -deformation will spoil the asymptotic AdS geometry of the spacetime and the dual field theory will probably be a non-local one. Therefore the usual subtraction schemes suitable for the undeformed case may not be appropriate here. The results of our calculations strongly support the above analysis.

Another feature of our solution is that the cusped anomalous dimension obtained from the solution without a jump in deformed S^5 can be smoothly connected with the result in the undeformed case when we take the limit that the deformation parameter C tends to zero¹⁾. This is quite different from the case for the spinning folded GKP-like string [52, 53]. For the GKP string, the relation $E-S \sim f(\lambda) \log S$ for large S was destroyed by the deformation. This makes us unable to extract the anomalous dimension from the GKP string side and compare it with the results given here from the cusped Wilson loops. The equivalent of these two approaches for the undeformed case is broken down by the deformation, partly because the background has a much smaller

isometry group after deformation.

For the undeformed case, the solution dual to the cusped Wilson loop with two light-like edges in Ref. [18] was found to actually have four cusps using the embedding coordinations [61]. This observation made this solution play a key role in the holographic computations of four-gluon planar amplitudes at strong coupling. It should be interesting to try to embed the deformed AdS_5 into a higher-dimensional spacetime and study the geometry of the minimal surface dual to the cusped Wilson loop from this point of view.

It should be interesting to compute the holographic entanglement entropy [63–65] from this background to investigate whether the area law [66, 67] of the entanglement entropy is lost or not, since the dual field theory is probably a non-local one. However, to perform this computation, we need to know the metric in the Einstein frame. Since the metric in the string frame is known, we need to know the dilaton field. Some progress has been made in Refs. [54, 57], but a complete solution is still unknown. It should be valuable to find a consistent solution, including dilaton and Ramond–Ramond fields, and compute the holographic entanglement entropy. We hope to work on this point in the near future.

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